Further calculations for Israeli options

E. J. Baurdoux and A. E. Kyprianou

Department of Mathematics, University of Utrecht, P.O.Box 80.010, 3500TA Utrecht, The Netherlands. (e-mail: baurdoux,kyprianou@math.uu.nl).

Abstract. Recently Kifer (2000) introduced the concept of an Israeli (or Game) option. That is a general American-type option with the added possibility that the writer may terminate the contract early inducing a payment not less than the holder's claim had they exercised at that moment. Kifer shows that pricing and hedging of these options reduces to evaluating a stochastic saddle point problem associated with Dynkin games. Kyprianou (2004) gives two examples of perpetual Israeli options where the value function and optimal strategies may be calculated explicity. In this article we give a third example of a perpetual Israeli option where the contingent claim is based on the integral of the price process. This time the value function is shown to be the unique solution to a (two sided) free boundary value problem on $(0,\infty)$ which is solved by taking an appropriately rescaled linear combination of Kummer functions. The probabilistic methods we appeal to in this paper centre around the interaction between the analytic boundary conditions in the free boundary problem, Itô's formula with local time and the martingale, supermartingle and submartingale properties associated with the solution to the stochastic saddle point problem.

Date: 26th August 2004

Key words: Stochastic games, option pricing, free boundary value problem, Integral options, Israeli options, local time.

Mathematics Subject Classification (1991): 34B99, 60G44, 60J65, 60J55.

1 Israeli options

Consider the Black-Scholes market. That is, a market with a risky asset S and a riskless bond, B. The bond evolves according to the dynamic

$$dB_t = rB_t dt \text{ where } r, t \geq 0.$$

The value of the risky asset is written as the process $S = \{S_t : t \geq 0\}$ where

$$S_t = s\exp\{\sigma W_t + \mu t\},$$

$s > 0$ is the initial value of S and $W = \{W_t : t \geq 0\}$ is a Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}, P)$ satisfying the usual conditions.

Let $0 < T \leq \infty$. Suppose that $X = \{X_t : t \in [0,T]\}$ and $Y = \{Y_t : t \in [0,T]\}$ are two continuous stochastic processes defined on $(\Omega, \mathcal{F}, \mathbb{F}, P)$ such that with probability one $Y_t \geq X_t$ for all $t \in [0,T]$. The Israeli option, introduced by Kifer (2000), is a contract between a writer and a holder at time $t = 0$ such that both have the right to exercise at any $\mathbb{F}$-stopping time before the expiry date T. If the holder exercises, then (s)he may claim the value of X at the exercise date and if the writer exercises, (s)he is obliged to pay to the holder the value of Y at the time of exercise. If neither have exercised at time T and $T < \infty$ then the writer pays the holder the value X_T. If both decide to claim at the same time then the lesser of the two claims is paid. (Note that the assumption that X and Y are continuous processes is not the most generic case but will suffice for the following discussion). In short, if the holder will exercise with strategy σ and the writer with strategy τ we can conclude that the holder will receive $Z_{\sigma,\tau}$ where

$$Z_{s,t} = X_s\mathbf{1}_{(s\leq t)} + Y_t\mathbf{1}_{(t<s)}.$$

Suppose now that $\mathbb{P}_s$ is the risk-neutral measure for S under the assumption that $S_0 = s$. [Standard Black-Scholes theory dictates that this measure exists and is uniquely defined via a Girsanov change of measure]. We shall denote $\mathbb{E}_s$ to be expectation under $\mathbb{P}_s$. Note that the process S under $\mathbb{P}_s$ is equal to the process

$$\{se^{\sigma W_t - (\sigma^2/2 - r)t} : t \geq 0\}$$

under law P.

The following Theorem is Kifer's pricing result.

Theorem 1 (Kifer) *Suppose that for all $s > 0$*

$$\mathbb{E}_s\left(\sup_{0\leq t\leq T} e^{-rt}Y_t\right) < \infty \tag{1}$$

and if $T = \infty$ that

$$\mathbb{P}_s\left(\lim_{t\uparrow\infty} e^{-rt}Y_t = 0\right) = 1. \tag{2}$$

Let $\mathcal{T}_{t,T}$ be the class of $\mathbb{F}$-stopping times valued in $[t,T]$. The value of the Israeli option under the Black-Scholes framework is given by $V = \{V_t : t \in [0,T]\}$ where

$$\begin{aligned} V_t &= \text{ess-inf}_{\tau\in\mathcal{T}_{t,T}}\text{ess-sup}_{\sigma\in\mathcal{T}_{t,T}}\mathbb{E}_s\left(e^{-r(\sigma\wedge\tau-t)}Z_{\sigma,\tau}\middle|\mathcal{F}_t\right) &(3)\\ &= \text{ess-sup}_{\sigma\in\mathcal{T}_{t,T}}\text{ess-inf}_{\tau\in\mathcal{T}_{t,T}}\mathbb{E}_s\left(e^{-r(\sigma\wedge\tau-t)}Z_{\sigma,\tau}\middle|\mathcal{F}_t\right) &(4)\end{aligned}$$

(meaning implicitly that a saddle point exists). Further, the optimal stopping strategies for the holder and writer respectively are

$$\sigma^* = \inf\{t \in [0,T] : V_t \le X_t\} \wedge T \text{ and } \tau^* = \inf\{t \ge 0 : V_t \ge Y_t\} \wedge T.$$

The formulae given in this theorem reflect the fact that the essence of this option contract is based on the older theory of Dynkin games or stochastic games; see Friedman (1976) or Dynkin (1969) for example.

Two cases of perpetual ($T = \infty$) Israeli options were considered in the past by Kyprianou (2004) for which an explicit solution the pricing saddle point problem were obtained. They are given as follows.

Israeli δ-penalty put options. In this case, the holder may claim as a normal American put,

$$X_t = (K - S_t)^+.$$

The writer on the other hand will be assumed to payout the holders claim plus a constant,

$$Y_t = (K - S_t)^+ + \delta \text{ for } \delta > 0.$$

Israeli δ-penalty Russian options. The holder may exercise to take a normal Russian claim,

$$X_t = e^{-\alpha t} \max\left\{ m, \sup_{u\in[0,t]} S_u \right\} \text{ for } \alpha \ge 0, m > s$$

and the writer is punished by an amount $e^{-\alpha t}\delta S_t$ for annulling the contract early,

$$Y_t = e^{-\alpha t}\left(\max\left\{ m, \sup_{u\in[0,t]} S_u \right\} + \delta S_t \right) \text{ for } \delta > 0.$$

In this paper we shall perform calculations showing that for a specific choice of X and Y which are based on the processes $\{\int_0^t S_u du : t \ge 0\}$ and S, strategies σ^* and τ^* can be obtained together with a semi-explicit characterization of the process V when $T = \infty$. Specifically we consider the following option which is based on the integral option of Kramkov and Mordecky (1994).

Israeli δ-penalty integral options. The holder may exercise to take an integral claim,

$$X_t = e^{-\lambda t}\left(\int_0^t S_u du + s\psi \right) \text{ for } \lambda, \psi > 0$$

and the writer is punished by an amount $e^{-\lambda t}\delta S_t$ for annulling the contract early,

$$Y_t = e^{-\lambda t}\left(\int_0^t S_u du + s\psi + \delta S_t \right) \text{ for } \delta > 0.$$

Our method of analysis is both similar and different to that of the first two cases. The similarity lies in the fact that we identify the saddle point as being characterized by a free boundary problem of an elliptic type. Specifially, taking the solution to a special free boundary value problem, we may construct a stochastic process which, when stopped on exiting the boundaries, will form a supermartingale, a martingale and a submartingale. These three properties together with upper and lower bounds on the solution to the free boundary value problem and path continuity of the aforementioned stochastic process are enough to establish that the two stopping times lead to the saddle point in Kifer's formula. The difference compared to the analysis in Kyprianou (2004) lies with the problem of establishing that the proposed free boundary value problem has a solution. In the case of the put and Russian type payouts this followed simply by manual computation using polynomials. In this case, there is no explicit solution available, however, it is likely that an analytical proof of existence can be established. Instead we prefer to establish existence using probabilistic techniques. The reason for this being that the methodology has potential to be adapted to the the case of finite expiry and indeed other optimal stopping problems with finite horizon for which the associated free boundary problem is of a parabolic type. Recent probabilistic studies suggest that in certain cases, probabilistic methods can reach further than known analytic methods with regard to free boundary problems. Compare for example Peskir (2002, 2003) and Duistermaat *et al.* (2003).

The probabilistic methods we appeal to in this paper centre around the interaction between the analytic boundary conditions in the free boundary problem, Itô's formula with local time and the martingale, supermartingle and submartingale properties proven, for example, in Kühn and Kyprianou (2003) in the following form.

Theorem 2 *For the value process V given in Theorem 1 we have that*

$$\{e^{-r(t\wedge\sigma^*)}V_{t\wedge\sigma^*} : t \geq 0\} \text{ is a } \mathbb{P}\text{-submartingale,}$$
$$\{e^{-r(t\wedge\tau^*)}V_{t\wedge\tau^*} : t \geq 0\} \text{ is a } \mathbb{P}\text{-supermartingale and}$$
$$\{e^{-r(t\wedge\tau^*)}V_{t\wedge\tau^*\wedge\sigma^*} : t \geq 0\} \text{ is a } \mathbb{P}\text{-martingale.}$$

In particular, we show that in proving that the solution to the saddle point problem characterizes the unique solution to the free boundary value problem, the boundary conditions must be observed otherwise there is an inconsistency in the martingale statements of Theorem 2 on account of terms that would appear in Itô's formula with local time.

2 Review of the integral option

It will be helpful to recall the mathematical structure of the perpetual integral option given in Kramkov and Mordecky (1994). Recall that it is an American type option whose payout is given by

$$e^{-\lambda t}\left(\int_0^t S_u\,du + s\psi\right) \text{ for } \lambda, \psi > 0.$$

Standard theory of American option pricing (cf. Shiryaev *et. al.* (1995)) tells us that in the Black-Scholes market, the value process of the integral option is given by

$$U_t := \text{ess-sup}_{\sigma\in\mathcal{T}_{t,\infty}}\mathbb{E}_s\left(e^{-(\sigma-t)r}e^{-\lambda t}\left(\int_0^t S_u\,du + s\psi\right)\Big|\,\mathcal{F}_t\right)$$

Using a second change of measure proposed in Shiryaev *et al.* (1995) given by

$$\left.\frac{d\widetilde{\mathbb{P}}_s}{d\mathbb{P}_s}\right|_{\mathcal{F}_t} = \frac{e^{-rt}S_t}{s} \tag{5}$$

the expression U_t reduces to

$$U_t = e^{-\lambda t}S_t \sup_{\sigma\in\mathcal{T}_{0,\infty}}\widetilde{\mathbb{E}}_s\left(e^{-(r+\lambda)\sigma}S_{t+\sigma}^{-1}\left(\int_0^{t+\sigma} S_u\,du + s\psi\right)\Big|\,\mathcal{F}_t\right).$$

If we now define the process

$$\Psi = \{\Psi_t := S_t^{-1}\int_0^t S_u\,du : t\geq 0\},$$

which can easily be checked to be Markovian (see Kramkov and Mordecky (1994)), and denote its law $\mathbf{P}_\psi$ when $\Psi_0 = \psi$, it is easily confirmed that Ψ under $\mathbf{P}_\psi$ has the same law as $(\int_0^\cdot S_u\,du + s\psi)/S_\cdot$ under $\mathbb{P}_s$. We shall henceforth use $\mathbf{E}_\psi$ to mean expectation with respect to $\mathbf{P}_\psi$ We may now identify more conveniently

$$U_t = e^{-\lambda t}S_t\widehat{h}(\Psi_t)$$

where

$$\widehat{h}(\psi) := \sup_{\sigma\in\mathcal{T}_{0,\infty}}\mathbf{E}_\psi(e^{-(r+\lambda)\sigma}\Psi_\sigma). \tag{6}$$

The effect of the change of measure has been to reduce the number of stochastic processes in the optimal stopping problem from two, S and $\int_0^\cdot S_u\,du$, to just one, Ψ.

The function $\widehat{h}$ comes in the following form. Denote by $\gamma_1 < \gamma_2$ the two roots of the quadratic equation

$$\frac{\sigma^2}{2}\gamma^2 - (\frac{\sigma^2}{2} + r)\gamma - \lambda = 0, \tag{7}$$

and define the function u by

$$u(\psi) = \int_0^\infty e^{-2y/\sigma^2} y^{-(\gamma_1+1)} (1+\psi y)^{\gamma_2} dy \quad \text{for } \psi \geq 0. \tag{8}$$

It is easily verified that u is a smooth convex function, strictly increasing with limiting behaviour $\lim_{\psi\uparrow\infty} u(\psi)/\psi = \infty$. Therefore there exists a unique $\widehat{c} > 0$ and $\widehat{\psi} > 0$ such that

$$\begin{aligned} \widehat{c}u(\widehat{\psi}) &= \widehat{\psi}, \\ \widehat{c}u'(\widehat{\psi}) &= 1. \end{aligned}$$

The function $\widehat{h}$ is now defined by

$$\widehat{h}(\psi) = \begin{cases} \widehat{c}u(\psi) & \text{if} \quad 0 < \psi \leq \widehat{\psi}, \\ \psi & \text{if} \quad \psi > \widehat{\psi}. \end{cases} \tag{9}$$

Furthermore the optimal stopping strategy of (6) is given by

$$\widehat{\sigma} = \inf\{t \geq 0 : \Psi_t \geq \widehat{\psi}\}. \tag{10}$$

3 The Israeli δ-penalty integral option

Let us suppose that X and Y are those of the Israeli δ-penalty integral option. Before we may use Kifer's Theorem to tell us that the latter option has a value, we must check the two conditions (1) and (2).

For (1), we know that

$$\mathbb{E}_s\left(\sup_{t\geq 0} e^{-rt} Y_t\right) \leq \mathbb{E}_s\left(\sup_{t\geq 0} e^{-(r+\lambda)t}\left(\int_0^t S_u\, du + s\psi\right)\right) + \mathbb{E}_s\left(\sup_{t\geq 0} e^{-(r+\lambda)t}\delta S_t\right).$$

Firstly note that by Fubini's theorem, the fact that $\lambda > 0$ and that $e^{-rt}S_t$ is a $\mathbb{P}$-martingale, we know that

$$\begin{aligned} \mathbb{E}_s\left(\sup_{t\geq 0} e^{-(r+\lambda)t}\left(\int_0^t S_u\, du\right)\right) &\leq \int_0^\infty e^{-\lambda u}\mathbb{E}_s\left(e^{-ru}S_u\right) du \\ &= \int_0^\infty e^{-\lambda u} s\, du < \infty. \end{aligned}$$

Secondly let $M = \sup_{t\geq 0} e^{-(r+\lambda)t} S_t$. Using standard distributional properties of Brownian motion we get

$$\begin{aligned} \mathbb{P}_s(M > x) &= P(\sup_{t\geq 0} s e^{\sigma W_t - (\sigma^2/2+\lambda)t} \geq x) \\ &= P(\sup_{t\geq 0} W_t^{-(\sigma/2+\lambda/\sigma)} \geq \sigma^{-1}\log(x/s)) \\ &= \left(\frac{x}{s}\right)^{-1-2\lambda/\sigma^2}. \end{aligned}$$

Now since $\mathbb{E}_s(M) = \int_s^\infty \mathbb{P}_s(M > x)dx$ and $\lambda > 0$ we have that $\mathbb{E}_s(M) < \infty$ and hence (1) is proven.

To check that condition (2) holds we will use the law of the iterated logarithm for Brownian motion at large times. Specifically, if $W = \{W_t : t \geq 0\}$ is a P-Brownian motion, then for each $\varepsilon > 0$ we may pick a $T(\omega) > 0$ such that

$$W_u \leq \sqrt{(2+\varepsilon)u} \quad \text{for all } u \geq T \tag{11}$$

and hence

$$S_u = se^{\sigma W_u + (r-\sigma^2/2)u} \leq se^{\sigma\sqrt{(2+\varepsilon)u}+(r-\sigma^2/2)u} \quad \text{for all } u \geq T.$$

Let

$$F(y) = \int_0^y e^{u^2} du \quad \text{for } y \geq 0,$$

$a = r - \sigma^2/2$ and $b = \sigma\sqrt{2+\varepsilon}$. We have that $\lim_{t\uparrow\infty} e^{-(r+\lambda)t} S_t = 0$ and so

$$\begin{aligned}
\lim_{t\uparrow\infty} e^{-rt} Y_t &\leq \lim_{t\uparrow\infty} e^{-(r+\lambda)t} s \int_T^t e^{au+b\sqrt{u}} du \\
&= \lim_{t\uparrow\infty} \left\{ \frac{s}{a} e^{b\sqrt{t}+(-\sigma^2/2-\lambda)t} - \frac{be^{-b^2/(4a)}}{a^{3/2}} F\left(\frac{b+2a\sqrt{t}}{2\sqrt{a}}\right) e^{-(r+\lambda)t} \right\} \\
&\leq \lim_{t\uparrow\infty} \left\{ \frac{s}{a} e^{b\sqrt{t}+(-\sigma^2/2-\lambda)t} - \frac{b}{a^{3/2}} \frac{b+2a\sqrt{t}}{2\sqrt{a}} e^{(-\sigma^2/2-\lambda)t+b\sqrt{t}} \right\}
\end{aligned}$$

where in the last line we have used that $F(y) \leq ye^{y^2}$ for any $y \geq 0$. This last expression tends to zero as t tends to infinity and (2) is proven.

Now that we know we are within the scope of Kifer's pricing theorem, we may appeal to the same change of measure (5) as in the previous section and deduce that the value of the Israeli δ-penalty integral option is given by

$$V_t = e^{-\lambda t} S_t h(\Psi_t), \ t \geq 0$$

where $h : (0,\infty) \to [0,\infty)$ is the solution to the stochastic saddle point problem

$$\begin{aligned}
h(\psi) &= \inf_{\tau\in\mathcal{T}_{0,\infty}} \sup_{\sigma\in\mathcal{T}_{0,\infty}} \mathbf{E}_\psi[e^{-\lambda(\sigma\wedge\tau)} \left((\Psi_\tau + \delta)\mathbf{1}_{\{\tau<\sigma\}} + \Psi_\sigma \mathbf{1}_{\{\tau\geq\sigma\}}\right)] \quad (12) \\
&= \sup_{\sigma\in\mathcal{T}_{0,\infty}} \inf_{\tau\in\mathcal{T}_{0,\infty}} \mathbf{E}_\psi[e^{-\lambda(\sigma\wedge\tau)} \left((\Psi_\tau + \delta)\mathbf{1}_{\{\tau<\sigma\}} + \Psi_\sigma \mathbf{1}_{\{\tau\geq\sigma\}}\right)] \quad (13)
\end{aligned}$$

and we implicitly understand (following from Kifer's theorem) that a solution to the above stochastic saddle point problem exists.

Remark 3 It can also be checked that under the change of measure (5), the optimal stopping strategies when written in terms of the process Ψ reduce to

$$\sigma^* = \inf\{t \geq 0 : h(\Psi_t) \leq \Psi_t\} \text{ and } \tau^* = \inf\{t \geq 0 : h(\Psi_t) \geq \Psi_t + \delta\}.$$

Theorem 2 now simplifies to the following statement. We have that

$$\{e^{-\lambda(t\wedge\sigma^*)}h(\Psi_{t\wedge\sigma^*}) : t \geq 0\} \text{ is a } \mathbf{P}\text{-submartingale,}$$
$$\{e^{-\lambda(t\wedge\tau^*)}h(\Psi_{t\wedge\tau^*}) : t \geq 0\} \text{ is a } \mathbf{P}\text{-supermartingale and}$$
$$\{e^{-\lambda(t\wedge\tau^*)}h(\Psi_{t\wedge\tau^*\wedge\sigma^*}) : t \geq 0\} \text{ is a } \mathbf{P}\text{-martingale.}$$

4 Large and small δ

As both intuition and the experience in Kyprianou (2004) suggests, the stochastic saddle point problem in Kifer's theorem for the option at hand should reduce to an optimal stopping problem for the holder when δ is too big. The reasoning being that large values of δ would imply that the writer must pay out far more than they would ever pay for a regular integral option. The threshold turns out to be

$$\delta^* := \frac{\widehat{\psi}u(0)}{u(\widehat{\psi})} \tag{14}$$

where the function u and the value $\widehat{\psi}$ is defined in Section 2. Formally speaking we have the following result.

Theorem 4 *When $\delta \geq \delta^*$ the Israeli δ-penalty integral option is identical to an integral option with the same parameters. That is to say $h = \widehat{h}$, defined in (9).*

Proof. We have that

$$\widehat{h}(0) = \widehat{c}u(0) = \frac{\widehat{\psi}u(0)}{u(\widehat{\psi})} \leq \delta.$$

For any $0 \leq \psi \leq \widehat{\psi}$, there exists some $\xi \in (0, \widehat{\psi})$ such that $\widehat{h}(\psi) = \widehat{h}'(\xi)\psi + \widehat{h}(0)$ and by convexity of u we deduce that

$$\widehat{h}(\psi) = \widehat{h}'(\xi)\psi + \widehat{h}(0) \leq \widehat{h}'(\widehat{\psi})\psi + \delta = \psi + \delta.$$

Therefore

$$\psi \leq \widehat{h}(\psi) \leq \psi + \delta \tag{15}$$

for all $0 \leq \psi \leq \widehat{\psi}$ and hence for all $\psi \geq 0$.

By virtue of the fact that $\widehat{h}$ solves the optimal stopping problem (6) with optimal stopping time σ^*, given by (10), we have that

$$\{e^{-\lambda(t\wedge\sigma^*)}\widehat{h}\,(\Psi_{t\wedge\sigma^*}) : t \geq 0\}$$

is a martingale and

$$\{e^{-\lambda t}\widehat{h}\,(\Psi_t) : t \geq 0\}$$

is a supermartingale (cf. Kramkov and Mordecky (1994)). Using these latter two facts together with (15) it follows that

$$
\begin{aligned}
\widehat{h}(\psi) &= \inf_{\tau\in\mathcal{T}_{0,\infty}} \mathbf{E}_\psi[e^{-\lambda(\tau\wedge\sigma^*)}\widehat{h}(\Psi_{\tau\wedge\sigma^*})] \\
&= \inf_{\tau\in\mathcal{T}_{0,\infty}} \mathbf{E}_\psi[e^{-\lambda(\tau\wedge\sigma^*)}(\widehat{h}(\Psi_\tau)\mathbf{1}_{\{\tau<\sigma^*\}} + \Psi_{\sigma^*}\mathbf{1}_{\{\tau\geq\sigma^*\}})] \\
&\leq \inf_{\tau\in\mathcal{T}_{0,\infty}} \mathbf{E}_\psi[e^{-\lambda(\tau\wedge\sigma^*)}((\Psi_\tau+\delta)\mathbf{1}_{\{\tau<\sigma^*\}} + \Psi_{\sigma^*}\mathbf{1}_{\{\tau\geq\sigma^*\}})] \\
&\leq \sup_{\sigma\in\mathcal{T}_{0,\infty}} \inf_{\tau\in\mathcal{T}_{0,\infty}} \mathbf{E}_\psi[e^{-\lambda(\tau\wedge\sigma)}((\Psi_\tau+\delta)\mathbf{1}_{\{\tau<\sigma\}} + \Psi_\sigma\mathbf{1}_{\{\tau\geq\sigma\}})] \\
&\leq \sup_{\sigma\in\mathcal{T}_{0,\infty}} \mathbf{E}_\psi[e^{-\lambda\sigma}\Psi_\sigma] \\
&\leq \sup_{\sigma\in\mathcal{T}_{0,\infty}} \mathbf{E}_\psi[e^{-\lambda\sigma}\widehat{h}(\Psi_\sigma)] \\
&\leq \widehat{h}(\psi).
\end{aligned}
$$

Note that the infimum and supremum may be reversed in the fourth line by arguing backwards. Hence $\widehat{h}$ is the solution to the saddle point problem given in (12, 13). ■

For the case that $\delta < \delta^*$ our objective, which will consume the majority of the remainder of the paper, is to prove the following theorem which characterizes the value of the Israeli δ-penalty integral option in terms of the unique solution to a free boundary problem.

Theorem 5 (Free boundary problem) *When $\delta < \delta^*$, there exist a unique triple (a, b, w) such that*

$$0 < a \leq \frac{1-\lambda\delta}{r+\lambda} \text{ and } \frac{1}{r+\lambda} \leq b < \infty$$

and w is a C^2-finction on (a, b) which satisfies

$$\left(\frac{1}{2}\sigma^2\psi^2\frac{d^2}{d\psi^2} + (1-r\psi)\frac{d}{d\psi} - \lambda\right) w(\psi) = 0 \tag{16}$$

such that

$$w(\psi) = \begin{cases} \psi+\delta & \text{if } \psi \leq a, \\ \psi & \text{if } \psi \geq b. \end{cases} \tag{17}$$

Furthermore $\psi \leq w(\psi) \leq \psi + \delta$ for all $\psi > 0$ and w is a C^1 function on $(0, \infty)$. Moreover the solution to the stochastic saddle point problem (12, 13) is characterized by $h = w$, and

$$\sigma^* = \inf\{t \geq 0 : \Psi_t \geq b\}$$

and

$$\tau^* = \inf\{t \geq 0 : \Psi_t \leq a\}$$

respectively.

This theorem will be proven by combining the conclusions of Theorems 10 and 11 in the following sections.

Remark 6 Clearly we need two boundary conditions to solve (16) on (a,b) and these are given by $w(a) = a + \delta$ and $w(b) = b$. Since the two boundaries a and b are free, we need two more conditions to pin them down. These are given by the requirement that w as a function on $(0,\infty)$ is C^1. Specifically, $w'(a+) = w'(a-) = 1$ and $w'(b+) = w'(b-) = 1$.

Remark 7 Note that it is implicit in the above theorem that for any $\delta < \delta^*$ we have that $\delta < 1/\lambda$ and hence

$$\delta^* \leq \frac{1}{\lambda}.$$

A fact which is not necessarily obvious from the expression given by (14).

Remark 8 Later on we shall see that the solution to the above free boundary value problem is essentially a linear combination of two linearly independent special functions which are rescaled versions of the Kummer functions.

5 Properties of the diffusion Ψ

As earlier mentioned, the process Ψ is Markovian. In the sequel, we shall make heavy use of some elementary facts concerning this diffusion and hence we shall devote some time in this section discussing them.

It is straightforward to confirm that

$$dS_t = S_t(r + \sigma^2)dt + \sigma S_t dW_t^{\mathbf{P}}$$

and

$$d\Psi_t = (1 - r\Psi_t)dt - \sigma\Psi_t dW_t^{\mathbf{P}}$$

where $W^{\mathbf{P}} = \{W_t^{\mathbf{P}} : t \geq 0\}$ is a standard Brownian motion with respect to $\mathbf{P}$. A simple calculation using the Itô formula thus shows that for any $f \in C^2$ we have

$$d\left(f\left(\Psi_t\right)\right) = \mathcal{L}f\left(\Psi_t\right)dt - \sigma\Psi_t f'\left(\Psi_t\right)dW_t^{\mathbf{P}}$$

where

$$\mathcal{L}f\left(\psi\right) = \frac{1}{2}\sigma^2\psi^2 f''\left(\psi\right) + (1 - r\psi)f'(\psi).$$

The operator $\mathcal{L}$ is the generator of Ψ and its harmonic functions are given by taking linear combinations of the two linear independent functions

$$G(x) = x^{\gamma_2} M(-\gamma_2, 1 - \gamma_2 + \gamma_1, 2/(x\sigma^2)) \tag{18}$$

and

$$H(x) = x^{\gamma_2} U(-\gamma_2, 1-\gamma_2+\gamma_1, 2/(x\sigma^2)) \tag{19}$$

(recall that γ_1 and γ_2 were given in (7)). Here the functions U and M are known as the Kummer functions. Using an integral representation, it can be shown that H is proportional to the function u in Section 2 (see Lebedev (1972)). In order that $1-\gamma_2+\gamma_1$ is within the parameter range where the functions M is well defined, we also need to make the assmption that

$$1-\gamma_2+\gamma_1 = 1 - \frac{2\sqrt{(\sigma^2/2+r)^2+2\lambda\sigma^2}}{\sigma^2} \notin \mathbb{Z}_{\leq 0} \tag{20}$$

(see the Appendix for a detailed description of these functions).

Define for $a > 0$ the hitting times by

$$\tau_a = \inf\{t \geq 0 : \Psi_t \leq a\},$$

and

$$\sigma_a = \inf\{t \geq 0 : \Psi_t \geq a\}.$$

The following theorem confirms that Ψ has range $(0,\infty)$ almost surely. Further, when started at $\psi > a$ (resp. $\psi < a$) and stopped at σ_a (resp. τ_a), Ψ has range (a,∞) (resp. (0,a)) with positive probability. Part of the theorem and indeed the methodology comes from Kramkov and Mordecky (1994) but we we include all proofs for completeness.

Theorem 9 *Let $a > 0$ and $\psi > 0$. Then both τ_a and σ_a are $\mathbf{P}_\psi$-almost surely finite stopping times, i.e.*

$$\mathbf{P}_\psi(\tau_a < \infty) = 1 \quad \textit{and} \quad \mathbf{P}_\psi(\sigma_a < \infty) = 1.$$

Furthermore when $0 < a < \psi < b$, then

$$\mathbf{P}_\psi(\tau_a < \sigma_b) > 0 \quad \textit{and} \quad \mathbf{P}_\psi(\sigma_b < \tau_a) > 0.$$

Proof. For $x \geq 1$, define the function

$$K(x) = \int_1^x y^{2r/\sigma^2} e^{2/(y\sigma^2)} dy,$$

and when $0 < x < 1$, let

$$K(x) = -\int_x^1 y^{2r/\sigma^2} e^{2/(y\sigma^2)} dy.$$

Let $a > 0$. The fact that K is a strictly increasing continuous function on $(0,\infty)$ allows us to deduce that

$$\tau_a = \inf\{t \geq 0 : \Psi_t \leq a\} = \inf\{t \geq 0 : K(\Psi_t) \geq K(a)\}$$

and

$$\sigma_a = \inf\{t \geq 0 : \Psi_t \geq a\} = \inf\{t \leq 0 : K(\Psi_t) \leq K(a)\}$$

Let $\psi > 0$. Since $P_\psi(\Psi_t > 0 \text{ for all } t \geq 0) = 1$, we can use Itô's formula to verify that the process $\{K(\Psi_t) : t \geq 0\}$ is a local martingale. It can also be easily shown that $c := \inf_{\psi > 0} \sigma\psi K'(\psi) > 0$ and hence the quadratic variation $\langle K(\psi)\rangle$ of $K(\Psi_t)$ satisfies

$$\langle K(\psi)\rangle_t = \int_0^t (\sigma\psi K'(\psi_s))^2 ds \geq c^2 t.$$

Therefore we can conclude that

$$\mathbf{P}_\psi\left(\lim_{t\to\infty}\langle K(\psi)\rangle_t = \infty\right) = 1. \tag{21}$$

Let $t \geq 0$ and define the stopping time

$$\chi(t) = \inf\{s \geq 0 : \langle K(\psi)\rangle_s \geq t\},$$

which is finite $\mathbf{P}_\psi$-a.s. Now Theorem (4.6) of chapter 3 in Karatzas and Shreve (1988) allows us to conclude that $\{K(\psi_{\chi(t)})\}_{t\geq 0}$ is a Brownian motion with respect to the filtration $\mathcal{F}_{\chi(t)}$. Because a Brownian motion reaches any level in an almost surely finite time and because $\chi(t)$ is almost surely finite, $\{K(\Psi_t)\}_{t\geq 0}$ must also hit any level almost surely, thus

$$\mathbf{P}_\psi(\tau_a < \infty) = \mathbf{P}_\psi(\sigma_a < \infty) = 1.$$

Next define for any $a \in \mathbb{R}$ the stopping time

$$T_a = \inf\{t \geq 0 : W_t^{\mathbf{P}} = a\}.$$

For any $a < 0 < b$ we know that $\mathbf{P}(T_a < T_b)$ and $\mathbf{P}(T_b < T_a)$ are strictly positive. Now let $0 < a < \psi < b$. As $\{K(\psi_{\chi(t)})\}_{t\geq 0}$ is a $\mathbf{P}$-Brownian motion it reaches $K(a)$ before it reaches $K(b)$ with positive probability and vice versa. As $K(\Psi_t)$ hits $K(c)$ if and only if Ψ_t hits c, the required result follows. ■

Armed with these facts concerning the diffusion Ψ we are now ready to move on to the proof of Theorem 5.

6 At most one solution to the free boundary problem

In this section we shall prove one of the directions in Theorem 5. Namely the following.

Theorem 10 *Suppose that $\delta < \delta^*$. Given a solution (a, b, w) to the free boundary value problem given in Theorem 5, it characterizes the solution to the stochastic saddle point problem (12, 13).*

Proof. Recall the definitions of τ_a and σ_b from Section 5 and for each $x \in \mathbb{R}$, let $L^x = \{L_t^x : t \geq 0\}$ be the local time of Ψ at point x. The process

$$e^{-\lambda(t\wedge\sigma_b)} w(\Psi_{t\wedge\sigma_b}),\ t \geq 0$$

is a submartingale. To see this we use a modern version of Itô's formula which allows for the discontinuity of first derivatives at points at the cost of an extra local time term (see for example Peskir (2002)) to deduce that on $t < \sigma_b$

$$\begin{aligned} d[e^{-\lambda t} w(\Psi_t)] &= e^{-\lambda t}(\mathcal{L} - \lambda) w(\Psi_t) dt - e^{-\lambda t} \sigma \Psi_t w'(\Psi_t) dW_t^{\mathbf{P}} \\ &\quad + e^{-\lambda t}(w'(a^+) - w'(a^-)) dL_t^a \\ &= e^{-\lambda t}(1 - \lambda\delta - (r+\lambda)\Psi_t) \mathbf{1}_{\{\Psi_t \leq a\}} dt \\ &\quad - e^{-\lambda t} \sigma \Psi_t w'(\Psi_t) dW_t^{\mathbf{P}}, \end{aligned}$$

where we also used that $w'(a^+) = 1 = w'(a^-)$. Since for $\psi \leq a$

$$(1 - \lambda\delta - (r+\lambda)\psi) \geq (1 - \lambda\delta - (r+\lambda)a) \geq 0$$

it follows that the process $\{M_t\}_{t\geq 0}$ is indeed a submartingale.

A similar calculation using the fact that $w'(b^-) = 1 = w'(b^+)$ and $1 - (r+\lambda)\psi \leq 0$ when $\psi \geq b$ shows that

$$\{e^{-\lambda(t\wedge\tau_a)} w(\Psi_{t\wedge\tau_a}) : t \geq 0\}$$

is a supermartingale.

Now using the fact that $x \leq w(x) \leq x + \delta$ for all $x > 0$, we get for any $\psi > 0$

$$\begin{aligned} w(\psi) &\leq \inf_{\tau \in \mathcal{T}_{0,\infty}} \mathbf{E}_\psi[e^{-\lambda(\tau\wedge\sigma_b)} w(\Psi_{\tau\wedge\sigma_b})] \\ &= \inf_{\tau \in \mathcal{T}_{0,\infty}} \mathbf{E}_\psi[e^{-\lambda(\tau\wedge\sigma_b)}(w(\Psi_\tau)\mathbf{1}_{\{\tau<\sigma_b\}} + \Psi_{\sigma_b}\mathbf{1}_{\{\tau\geq\sigma_b\}})] \\ &\leq \inf_{\tau \in \mathcal{T}_{0,\infty}} \mathbf{E}_\psi[e^{-\lambda(\tau\wedge\sigma_b)}((\Psi_\tau + \delta)\mathbf{1}_{\{\tau<\sigma_b\}} + \Psi_{\sigma_b}\mathbf{1}_{\{\tau\geq\sigma_b\}})] \\ &\leq \sup_{\sigma \in \mathcal{T}_{0,\infty}} \inf_{\tau \in \mathcal{T}_{0,\infty}} \mathbf{E}_\psi[e^{-\lambda(\tau\wedge\sigma)}((\Psi_\tau + \delta)\mathbf{1}_{\{\tau<\sigma\}} + \Psi_\sigma\mathbf{1}_{\{\tau\geq\sigma\}})] \\ &\leq \sup_{\sigma \in \mathcal{T}_{0,\infty}} \mathbf{E}_\psi[e^{-\lambda(\tau_a\wedge\sigma)}((\Psi_{\tau_a} + \delta)\mathbf{1}_{\{\tau_a<\sigma\}} + \Psi_\sigma\mathbf{1}_{\{\tau_a\geq\sigma\}})] \\ &= \sup_{\sigma \in \mathcal{T}_{0,\infty}} \mathbf{E}_\psi[e^{-\lambda(\tau_a\wedge\sigma)}(w(\Psi_{\tau_a})\mathbf{1}_{\{\tau_a<\sigma\}} + \Psi_\sigma\mathbf{1}_{\{\tau_a\geq\sigma\}})] \\ &\leq \sup_{\sigma \in \mathcal{T}_{0,\infty}} \mathbf{E}_\psi[e^{-\lambda(\tau_a\wedge\sigma)} w(\Psi_{\tau_a\wedge\sigma})] \\ &\leq w(\psi). \end{aligned}$$

We have used the submartingale property in the first inequality and the supermartingale property in the last one. Noting that the role of supremum and infimum can be interchanged on the fourth line of the calculation by reversing the argument, we see that that $w(\psi) = h(\psi)$ for any $\psi > 0$, $\tau^* = \tau_b$ and $\sigma^* = \sigma_a$. ∎

7 At least one solution to the free boundary value problem

In this section we shall show the converse to Theorem 10. That is to say, we shall prove the following theorem.

Theorem 11 *Suppose that $\delta < \delta^*$. There is at least one solution (a, b, w) to the free boundary value problem given in Theorem 5. This solution is identified by $w = h$.*

As stated in the introduction, this fact could likely be established by analytical techniques. However, prefering to shed light directly on the relation between stochastic saddle point problems and free boundary value problems, we opt for a probabilistic proof. The proof is long and we break it into smaller components. Throughout this section we assume that $\delta < \delta^*$.

7.1 The continuation region $\mathcal{C}$ is open

Our goal here is to show that $t < \tau^* \wedge \sigma^*$ if and only if $\Psi_t \in \mathcal{C}$ where $\mathcal{C}$ is some non-empty open set in $(0, \infty)$. The first step is establishing continuity of the function h.

Lemma 12 *The function h is continuous on $(0, \infty)$.*

Proof. For this proof we shall need to slightly adjust our notation. For any $\psi \in (0, \infty)$ we shall denote $\sigma^*(\psi)$ and $\tau^*(\psi)$ for the optimal stopping times in the stochastic saddle point problem (12) and (13) when the initial position $\Psi_0 = \psi$. Note then that for $0 < x, y < \infty$,

$$\begin{aligned} h(x) &= \inf_{\tau \in \mathcal{T}_{0,\infty}} \mathbf{E}_x[e^{-\lambda(\sigma^*(x)\wedge\tau)}((\Psi_\tau + \delta)\mathbf{1}_{(\tau<\sigma^*(x))}) + \Psi_{\sigma^*(x)}\mathbf{1}_{(\tau\geq\sigma^*(x))}] \\ &\leq \mathbf{E}_x[e^{-\lambda(\sigma^*(x)\wedge\tau^*(y))}((\Psi_{\tau^*(y)} + \delta)\mathbf{1}_{(\tau^*(y)<\sigma^*(x))} + \Psi_{\sigma^*(x)}\mathbf{1}_{(\tau^*(y)\geq\sigma^*(x))})] \end{aligned}$$

and similarly

$$h(y) \geq \mathbf{E}_y(e^{-\lambda(\sigma^*(x)\wedge\tau^*(y))}((\Psi_{\tau^*(y)} + \delta)\mathbf{1}_{(\tau^*(y)<\sigma^*(x))} + \Psi_{\sigma^*(x)}\mathbf{1}_{(\tau^*(y)\geq\sigma^*(x))})).$$

By choosing $\psi_1 > \psi_2$ and interchanging the roles of ψ_1 and ψ_2 in the above inequalities, we find that

$$\begin{aligned} h(\psi_1) - h(\psi_2) &\leq \mathbb{E}_1(e^{-(r+\lambda)(\tau^*(\psi_2)\wedge\sigma^*(\psi_1))}) \times (\psi_1 - \psi_2) \\ &\leq (\psi_1 - \psi_2) \end{aligned}$$

and

$$\begin{aligned} h(\psi_1) - h(\psi_2) &\geq \mathbb{E}_1(e^{-(r+\lambda)(\tau^*(\psi_1)\wedge\sigma^*(\psi_2))}) \times (\psi_1 - \psi_2) \\ &\geq 0 \end{aligned}$$

where we have also changed measure back to $\mathbb{P}$. Continuity of h follows immidiately. ■

The continuity of h now implies that

$$\mathcal{C} := \{\psi \in (0,\infty) : \psi < h(\psi) < \psi + \delta\}$$

is an open subset of $(0,\infty)$. In addition we observe directly from the proof of continuity the following corollary.

Corollary 13 *The function $h(\psi) - \psi$ is decreasing.*

Lemma 14 $\mathcal{C} \neq \emptyset$.

Proof. Suppose on the contrary that $\mathcal{C} = \emptyset$. By continuity of h we then have $h(\psi) = \psi$ for all $\psi > 0$ or $h(\psi) = \psi + \delta$ for all $\psi > 0$.

Let us consider the case then that $h(\psi) = \psi$. Since $\sigma^* = 0$ and $\tau^* = \infty$, we deduce from Remark 3 that the process

$$\{e^{-\lambda t} h(\Psi_t) : t \geq 0\} = \{e^{-\lambda(t\wedge\tau^*)} h(\Psi_{t\wedge\tau^*}) : t \geq 0\}$$

is a supermartingale. It can be checked using Itô's formula that

$$d(e^{-\lambda t} h(\Psi_t)) = d(e^{-\lambda t}\Psi_t) = e^{-\lambda t}(1 - (r+\lambda)\Psi_t)dt - e^{-\lambda t}\sigma\Psi_t dW_t^{\mathbf{P}},$$

showing that

$$\int_s^t 1 - (r+\lambda)\Psi_u du \leq 0 \quad \text{for any } 0 \leq s \leq t < \infty. \tag{22}$$

But from Theorem 9 we know that the process $\{\Psi_t\}_{t\geq 0}$ hits $1/(2(r+\lambda))$ after an almost surely finite time, say ζ. As $1-(r+\lambda)\Psi_\zeta = 1/2$, we use the continuity of the paths of the process $\{\Psi_t\}_{t\geq 0}$ to reach a contradiction with (22).

The case that $h(\psi) = \psi + \delta$ is handled using a similar argument by contradiction. ■

7.2 h is a C^2 function on $\mathcal{C}$

Recall that

$$\mathcal{C} = \{\psi \in (0,\infty) : \psi < h(\psi) < \psi + \delta\}.$$

From Remark 3, the fact that the stopped process

$$\{e^{-\lambda(t\wedge\sigma^*\wedge\tau^*)} h(\Psi_{t\wedge\sigma^*\wedge\tau^*}) : t \geq 0\}$$

is a martingale will allow us to deduce that in fact h is a C^2 function on $\mathcal{C}$.

Lemma 15 *The function h is C^2 on $\mathcal{C}$ and satisfies $(\mathcal{L} - \lambda)h(\psi) = 0$ for any $\psi \in \mathcal{C}$.*

Proof. Let $\psi \in \mathcal{C}$. Because $\mathcal{C}$ is open, there exist $x_1 < \psi < x_2$ such that $[x_1, x_2] \subseteq \mathcal{C}$. Let f be a C^2 function on $[x_1, x_2]$ defined by

$$\begin{aligned} (\mathcal{L} - \lambda)f &= 0 \quad \text{on} \quad (x_1, x_2), \\ f(x_1) &= h(x_1), \\ f(x_2) &= h(x_2). \end{aligned}$$

We know such a function exists and can be written as a linear combination of G and H. Let

$$\tau = \inf\{t : \Psi_t \in \{x_1, x_2\}\},$$

which is almost surely finite because of Theorem 9. Using the the definition of f above and the fact that f' is uniformly bounded, an application of Itô's Lemma shows that

$$\{e^{-\lambda(t\wedge\tau)} f(\Psi_{t\wedge\tau}) : t \geq 0\}$$

is a $\mathbf{P}_\psi$-martingale. Using Remark 3 in conjunction with the fact that $\tau \leq (\tau^* \wedge \sigma^*)$, thus have for every $\psi \in (x_1, x_2)$

$$f(\psi) = \mathbf{E}_\psi[e^{-\lambda\tau} f(\Psi_\tau)] = \mathbf{E}_\psi[e^{-\lambda\tau} h(\Psi_\tau)] = h(\psi).$$

Since the point ψ was arbitrarily chosen in $\mathcal{C}$ the Lemma is proven. ∎

7.3 $\mathcal{C}$ is an interval

Lemma 16 *There exist some $\alpha > 0$ and $\beta > \alpha$ such that $h(\psi) = \psi$ for $\psi \in (\beta, \infty)$ and $h(\alpha) = \alpha + \delta$.*

Proof. Suppose that

$$\forall n \in \mathbb{N}\ \exists x_n \in (n, \infty) \cap \mathcal{C}. \tag{23}$$

Then from Lemma 15 we deduce that $(\mathcal{L} - \lambda)h(x_n) = 0$. We know there exist $A, B \in \mathbb{R}$ such that

$$h(x_n) = AG(x_n) + BH(x_n).$$

Using known asymptotics of the functions U and M at zero (see Lebedev (1972)), we have that

$$h(x_n) \sim x_n^{\gamma_2} \left(A + B \left(\frac{\Gamma(\gamma_2 - \gamma_1)}{\Gamma(-\gamma_1)} + \frac{\Gamma(\gamma_1 - \gamma_2)}{\Gamma(-\gamma_2)} \left(\frac{2}{x_n \sigma^2} \right)^{\gamma_2 - \gamma_1} \right) \right) \quad \text{as } n \to \infty$$

where $\Gamma(\cdot)$ is the analytic extention of the Gamma function. Now since $\gamma_2 > 1$ and $\gamma_1 - \gamma_2 < 0$, we deduce that $\lim_{n\to\infty} |h(x_n)|/x_n$ is equal to 0 or ∞ accordingly as $A + B\Gamma(\gamma_2 - \gamma_1)/\Gamma(-\gamma_1)$ is zero or non-zero valued. In both cases we get a contradiction with the fact that $\psi \leq h(\psi) \leq \psi + \delta$ for all $\psi > 0$ and thus (23) is false. Therefore we can choose $\beta > 0$ such that $(\beta, \infty) \subset \mathcal{C}^c$. Since h is continuous we deduce that either $h(\psi) = \psi$ for all $\psi \in (\beta, \infty)$ or $h(\psi) = \psi + \delta$ for all $\psi \in (\beta, \infty)$.

Suppose the latter case holds. Choose the initial position

$$\psi > \xi := \frac{1}{r + \lambda} \vee \beta$$

and let

$$\zeta(\xi) := \inf\{t \geq 0 : \Psi_t = \xi\}.$$

Since $\xi \geq \beta$, the process Ψ has to hit ξ before it can hit the line $y = \psi$, hence $\zeta(\xi) < \sigma^*$. Therefore

$$\{e^{-\lambda(t\wedge\sigma^*\wedge\zeta(\xi))} h(\Psi_{t\wedge\sigma^*\wedge\zeta(\xi)}) : t \geq 0\} = \{e^{-\lambda(t\wedge\zeta(\xi))}(\Psi_{t\wedge\zeta(\xi)} + \delta) : t \geq 0\}$$

is a submartingale (from Remark 3) such that for $t < \zeta(\xi)$

$$\begin{aligned} d(e^{-\lambda t} h(\Psi_t)) &= e^{-\lambda t}(1 - \lambda\delta - (r + \lambda)\Psi_t) dt \\ & \quad -e^{-\lambda t}\sigma\Psi_t dW_t^{\mathbf{P}}. \end{aligned} \tag{24}$$

As $\mathbf{P}_\psi$-almost surely

$$\Psi_{t\wedge\zeta(\xi)} \geq \Psi_{\zeta(\xi)} = \xi \geq \frac{1}{r + \lambda}$$

we deduce that

$$1 - \lambda\delta - (r + \lambda)\Psi_{t\wedge\zeta(\xi)} < 0,$$

which, becasue of (24) leads to a contradiction with the fact that $e^{-\lambda(t\wedge\zeta(\xi))}(\Psi_{t\wedge\zeta(\xi)} + \delta) : t \geq 0\}$ is a submartingale. This shows that $h(\psi) = \psi$ for all $x \in (\beta, \infty)$.

Next suppose that $h(\psi) < \psi + \delta$ for every $\psi > 0$. This implies that $\tau^* = \infty$ and hence the Israeli δ penalty integral option is nothing more than a regular integral option. However since $\delta < \delta^*$ this (together with continuity of h) would imply that $\delta^* = h(0+) \leq \delta$; a contradiction. ■

Corollary 17 *It now follows as an immediate consequence of the previous Lemma and Corollary 13 that there exist two points* $0 < \psi^{(1)} < \psi^{(2)} < \infty$ *such that* $\mathcal{C} = (\psi^{(1)}, \psi^{(2)})$.

7.4 Properties at $\psi^{(1)}$ and $\psi^{(2)}$

In this part we give two more properties of the end points $\psi^{(1)}$ and $\psi^{(2)}$. Specifically, bounds on their values and that h observes the smooth pasting principle at these two points.

Lemma 18 *The following inequalities hold:*

$$\psi^{(1)} \leq \frac{1-\lambda\delta}{r+\lambda} \text{ and } \frac{1}{r+\lambda} \leq \psi^{(2)}.$$

Proof. Assume that

$$\psi^{(1)} > \frac{1-\lambda\delta}{r+\lambda}.$$

Then, together with the conclusion of the previous Corollary, we can say that there exists some $\varepsilon > 0$ such that $\psi^{(1)} - \varepsilon > 0$,

$$1 - \lambda\delta - (r+\lambda)\psi < 0, \quad h(\psi) = \psi + \delta \text{ for all } \psi \in (\psi^{(1)} - \varepsilon, \psi^{(1)}).$$

Define

$$\tau^\varepsilon = \inf\{t \geq 0 : \Psi_t = \psi^{(1)} - \varepsilon\}.$$

From Remark 3 we know that

$$\{e^{-\lambda(t\wedge\tau^\varepsilon\wedge\sigma^*)} h(\Psi_{t\wedge\tau^\varepsilon\wedge\sigma^*}) : t \geq 0\}$$

is a submartingale and using an extended version of Itô's formula we have that on $t < \tau^\varepsilon \wedge \sigma^*$

$$\begin{aligned} & d[e^{-\lambda t} h(\Psi_t)] \\ = \;& e^{-\lambda t}(1 - \lambda\delta - (r+\lambda)\Psi_t)\mathbf{1}_{(\Psi_t \in (\psi^{(1)}-\varepsilon, \psi^{(1)}))} dt \\ & -e^{-\lambda t}(\sigma\Psi_t h'(\Psi_t) dW_t^{\mathbf{P}} \\ & +e^{-\lambda t}(h'(\psi^{(1)}+) - 1) dL_t^{\psi^{(1)}}. \end{aligned}$$

Since $h(\psi) \leq \psi + \delta$ implies that $h'(\psi^{(1)}+) - 1 \leq 0$ and as in the previous Lemma we may deduce from Theorem 9 that $\mathbf{P}_\psi(L^{\psi^{(1)}}_{t\wedge\tau^\varepsilon\wedge\sigma^*} > 0) > 0$, we get a contradiction in previous calculation with the aforementioned submartingale property.

The second part of the Lemma is proven in a similar manner using the previous Lemma and the supermartingale property associated with h. ■

Remark 19 As earlier remarked upon, the first statement in the above Lemma implies that $\delta^* \leq 1/\lambda$.

Now we establish smooth pasting at $\psi^{(1)}$ and $\psi^{(2)}$. Our proof is very much guided by the structure of the proof of Theorem 3.16 In Shiryaev (1968).

Lemma 20 *The function $h(\psi)$ has continuous first derivatives at $\psi^{(1)}$ and $\psi^{(2)}$,*

$$h'(\psi^{(1)}+) = h'(\psi^{(1)}-) = 1 \text{ and } h'(\psi^{(2)}+) = h'(\psi^{(2)}-) = 1$$

Proof. We shall only give the proof of $h'(\psi^{(2)}+) = h'(\psi^{(2)}-) = 1$. The proof at $\psi^{(1)}$ is essentially the same. Define the function $f(\psi) = h(\psi) - \psi$ for $\psi > 0$. Then it suffices to show that $f'(\psi^{(2)}+) = f'(\psi^{(2)}-) = 0$. Note that since $f(\psi) = 0$ for $\psi \geq \psi^{(2)}$ we in fact only need to deduce that $f'(\psi^{(2)}-) = 0$.

Define for sufficiently small $0 < \rho < \psi^{(2)} - \psi^{(1)}$ the two-sided stopping time

$$T_\rho = \inf\{t \geq 0 : \Psi_t \notin (\psi^{(2)} - \rho, \psi^{(2)} + \rho)\}.$$

From Lemma 7.4 in Karatzas and Shreve (1988) we know that for any $\psi \in (\psi^{(2)} - \rho, \psi^{(2)} + \rho)$

$$\mathbf{E}_\psi[T_\rho] < \infty. \tag{25}$$

Our first objective is to show that

$$\mathbf{E}_{\psi^{(2)}}[e^{-\lambda T_\rho}\Psi_{T_\rho}] = \psi^{(2)} + o(\rho) \quad \text{as } \rho \downarrow 0.$$

From the definition of T_ρ we deduce that

$$\begin{aligned}
\mathbf{E}_{\psi^{(2)}}[e^{-\lambda T_\rho}\Psi_{T_\rho}] &= \mathbf{E}_{\psi^{(2)}}[e^{-\lambda T_\rho}\mathbf{1}_{\{\Psi_{T_\rho} = \psi^{(2)} - \rho\}}](\psi^{(2)} - \rho) \\
&\quad + \mathbf{E}_{\psi^{(2)}}[e^{-\lambda T_\rho}\mathbf{1}_{\{\Psi_{T_\rho} = \psi^{(2)} + \rho\}}](\psi^{(2)} + \rho) \\
&= \psi^{(2)}\mathbf{E}_{\psi^{(2)}}[e^{-\lambda T_\rho}] \\
&\quad + \rho(\mathbf{E}_{\psi^{(2)}}[e^{-\lambda T_\rho}] - 2\mathbf{E}_{\psi^{(2)}}[e^{-\lambda T_\rho}\mathbf{1}_{\{\Psi_{T_\rho} = \psi^{(2)} - \rho\}}]).
\end{aligned} \tag{26}$$

Define the function

$$g_\rho(\psi) = \mathbf{E}_\psi[e^{-\lambda T_\rho}\mathbf{1}_{\{\Psi_{T_\rho} = \psi^{(2)} - \rho\}}]$$

Because condition (25) is satisfied, we know by Proposition 7.2 in Karatzas and Shreve (1988) that g_ρ satisfies $(\mathcal{L} - \lambda)g_\rho = 0$ in $(\psi^{(2)} - \rho, \psi^{(2)} + \rho)$. Hence there exist A_ρ, $B_\rho \in \mathbb{R}$ such that

$$g_\rho(\psi) = A_\rho G(\psi) + B_\rho H(\psi) \quad \text{for all } \psi \in (\psi^{(2)} - \rho, \psi^{(2)} + \rho).$$

Obviously $g_\rho(\psi^{(2)} - \rho) = 1$ and $g_\rho(\psi^{(2)} + \rho) = 0$. From this we have after some algebra that

$$g_\rho(\psi^{(2)}) = \frac{H(\psi^{(2)})G(\psi^{(2)} + \rho) - H(\psi^{(2)} + \rho)G(\psi^{(2)})}{H(\psi^{(2)} - \rho)G(\psi^{(2)} + \rho) - H(\psi^{(2)} + \rho)G(\psi^{(2)} - \rho)}.$$

We can use de l'Hôpital's rule to conclude

$$\lim_{\rho \downarrow 0} g_\rho(y) = \frac{1}{2}. \tag{27}$$

(Note when taking limits, one will also use the linear independence of G and H). The regularity of the paths of Ψ and the Dominated Convergence Theorem allow us to deduce that

$$\lim_{\rho \downarrow 0} \mathbf{E}_{\psi^{(2)}}[e^{-\lambda T_\rho}] = 1. \tag{28}$$

If we combine (27) and (28) with (26) we get

$$\mathbf{E}_{\psi^{(2)}}[e^{-\lambda T_\rho}\Psi_{T_\rho}] = \psi^{(2)}\mathbf{E}_{\psi^{(2)}}[e^{-\lambda T_\rho}] + o(\rho) \quad \text{as } \rho \downarrow 0. \tag{29}$$

Next define the function

$$h_\rho(\psi) = \mathbf{E}_\psi[e^{-\lambda T_\rho}].$$

Again by Proposition 7.2 in Karatzas and Shreve (1988) we know that $h_\rho(\psi)$ satisfies $(\mathcal{L}-\lambda)h_\rho(\psi) = 0$ for any $\psi \in (\psi^{(2)}-\rho, \psi^{(2)}+\rho)$. As $h_\rho(\psi^{(2)}-\rho) = h_\rho(\psi^{(2)}+\rho) = 1$ we find that

$$h_\rho(\psi^{(2)}) \quad = \quad \frac{(G(\psi^{(2)}+\rho)-G(\psi^{(2)}-\rho))H(\psi^{(2)}) + (H(\psi^{(2)}-\rho)-H(\psi^{(2)}+\rho))G(\psi^{(2)})}{G(\psi^{(2)}+\rho)H(\psi^{(2)}-\rho)-G(\psi^{(2)}-\rho)H(\psi^{(2)}+\rho)}.$$

Again by using de l'Hôpital's rule (this time twice) together with the linear independence of G and H we find

$$\lim_{\rho\downarrow 0}\frac{h_\rho(\psi^{(2)})-1}{\rho} = 0$$

We deduce that

$$h_\rho(\psi^{(2)}) = 1 + o(\rho) \quad \text{as } \rho \downarrow 0.$$

If we combine this with (29) we conclude that

$$\mathbf{E}_{\psi^{(2)}}[e^{-\lambda T_\rho}\Psi_{T_\rho}] = \psi^{(2)} + o(\rho) \quad \text{as } \rho \downarrow 0. \tag{30}$$

The fact that $0 < \rho < \psi^{(2)} - \psi^{(1)}$ implies that $T_\rho \leq \tau^*$. This means that the process

$$\{e^{-\lambda(T_\rho\wedge t)}h(\Psi_{T_\rho\wedge t}) : t \geq 0\}$$

is a supermartingale, hence

$$\mathbf{E}_{\psi^{(2)}}[e^{-\lambda T_\rho}h(\Psi_{T_\rho})] \leq \psi^{(2)}.$$

By combining this result with (30) and the observation that $f \geq 0$ it follows that

$$\begin{aligned} 0 &\leq \mathbf{E}_{\psi^{(2)}}[e^{-\lambda T_\rho}f(\Psi_{T_\rho})] \\ &= \mathbf{E}_{\psi^{(2)}}[e^{-\lambda T_\rho}h(\Psi_{T_\rho})] - \mathbf{E}_{\psi^{(2)}}[e^{-\lambda T_\rho}\Psi_{T_\rho}] \\ &\leq \psi^{(2)} - \mathbf{E}_{\psi^{(2)}}[e^{-\lambda T_\rho}\Psi_{T_\rho}] = o(\rho) \quad \text{as } \rho \downarrow 0, \end{aligned}$$

which shows that

$$\mathbf{E}_{\psi^{(2)}}[e^{-\lambda T_\rho}f(\Psi_{T_\rho})] = o(\rho) \quad \text{as } \rho \downarrow 0. \tag{31}$$

As f is a C^2-function on $(\psi^{(1)}, \psi^{(2)})$ and $f(\psi^{(2)}) = 0$ we know that as $\rho \downarrow 0$

$$f(\psi^{(2)} - \rho) = -\rho f'(\psi^{(2)}-) + o(\rho).$$

Now as $f(\psi) = 0$ when $\psi > \psi^{(2)}$ we find

$$\begin{aligned}\mathbf{E}_{\psi^{(2)}}[e^{-\lambda T_\rho} f(\Psi_{T_\rho})] &= \mathbf{E}_{\psi^{(2)}}[e^{-\lambda T_\rho} f(\psi^{(2)} - \rho)\mathbf{1}_{\{\Psi_{T_\rho} = \psi^{(2)}-\rho\}}] \\ &= (-\rho f'(\psi^{(2)}-) + o(\rho))\mathbf{E}_{\psi^{(2)}}[e^{-\lambda T_\rho}\mathbf{1}_{\{\Psi_{T_\rho} = \psi^{(2)}-\rho\}}].\end{aligned}$$

According to (27) and (31) we know that

$$\begin{aligned}0 &= 2\lim_{\rho\downarrow 0} \frac{\mathbf{E}_{\psi^{(2)}}[e^{-\lambda T_\rho} f(\Psi_{T_\rho})]}{\rho} \\ &= 2\lim_{\rho\downarrow 0} \frac{-\rho f'(\psi^{(2)}-) + o(\rho)}{\rho}\mathbf{E}_{\psi^{(2)}}[e^{-\lambda T_\rho}\mathbf{1}_{\{\Psi_{T_\rho} = \psi^{(2)}-\rho\}}] \\ &= 2\lim_{\rho\downarrow 0} \frac{-\rho f'(\psi^{(2)}-)}{2\rho} \\ &= f'(\psi^{(2)}-)\end{aligned}$$

and the proof is complete. ■

7.5 Proof of Theorem 11

Combining the conclusions of all the Lemmas in this section together, we thus have the proof of Theorem 11.

8 Plots of the functions $\widehat{h}$ and h

We offer a plot of the value function of the Israeli δ-penalty integral option next to the value function of the regular integral option with the same parameters. By choosing the parameters $\lambda > 0$ and $r > 0$ appropriately, we can attain a choice of γ_1 and γ_2 satisfying

$$\gamma_1 < 0 \text{ and } \gamma_1 + \gamma_2 > 1.$$

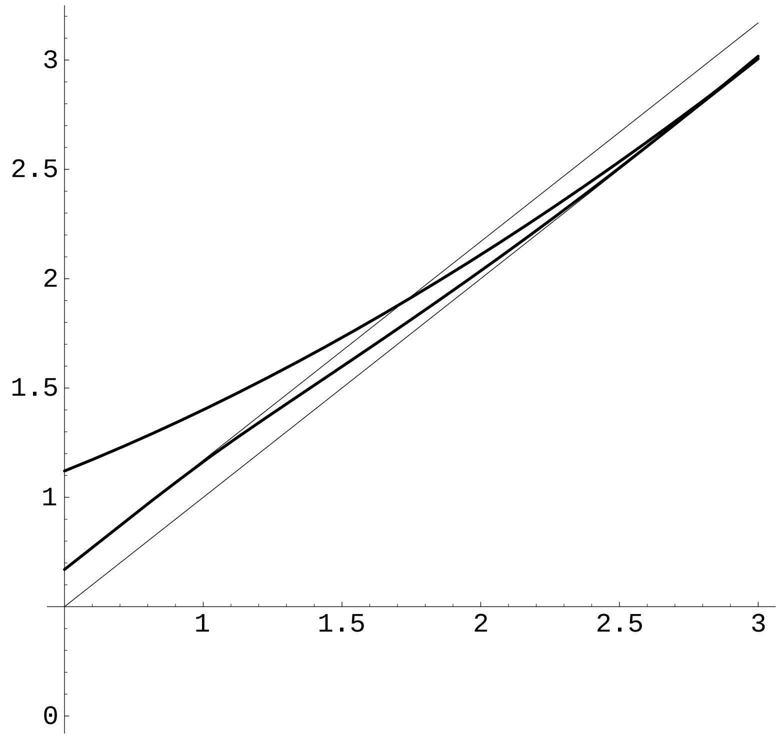
We take $\sigma^2 = 1$, $\gamma_1 = -0.49$ and $\gamma_2 = 1.8$. Recall the definition of u from (8). Solving the equation

$$u(\psi) - u'(\psi)\psi = 0$$

numerically leads to $\widehat{\psi} \approx 3$ and

$$\delta^* = \frac{\widehat{\psi}u(0)}{u(\widehat{\psi})} \approx 0.89.$$

Of course, interesting cases only arise by choosing a relatively small δ and therefore we take $\delta = 0.17$. According to Theorem 5 the problem of finding the value funtion of the Israeli Integral option reduces to finding a linear combination of the functions $\psi^{1.8}U(-1.8, -1.29, 2/\psi)$ and $\psi^{1.8}M(-1.8, -1.29, 2/\psi)$ which satisfies the free boundary value problem. The definition of $\widehat{h}$ given in (9) is also given in terms of the function $\psi^{1.8}U(-1.8, -1.29, 2/\psi)$. Using the computer programme Mathematica, for particular parameter choices mentioned above, one may use the above functions to establish numerical curves for the value functions $h(\psi)$ and $\widehat{h}(\psi)$. These are plotted below on the same graph. On the same graph the diagonal lines represent the curves ψ and $\psi + \delta$. Note that the domination of h by $\widehat{h}$ is clearly apparent in the diagram.



Appendix

Kummer's equation takes the form

$$x\frac{d^2 f}{dx^2} + (b - x)\frac{df}{dx} - af = 0, \tag{32}$$

where $a \in \mathbb{R}$ and $b \in \mathbb{R}\backslash\mathbb{Z}_{\leq 0}$. One solution to this differential equation takes the form

$$M(a, b, x) = \sum_{k=0}^{\infty} \frac{\Gamma(a + k)\Gamma(b)}{\Gamma(a)\Gamma(b + k)k!} x^k$$

Remark 21 Note that in the above expression we understand the Γ function in its analytically extended form given by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \quad \text{when } \Re z > 0.$$

To define $\Gamma(z)$ in the rest of the complex plane we can use the idea of analytic continuation to find that

$$\Gamma(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{1}{z+k} + \int_1^\infty e^{-t} t^{z-1} dt \quad \text{for } z \notin \mathbb{Z}_{\leq 0}.$$

The function Γ has simple poles at the points $z \in \mathbb{Z}_{\leq 0}$.

A second, linearly independent, solution of (32) by

$$U(a,b,x) := \frac{\Gamma(1-b)}{\Gamma(1+a-b)} M(a,b,x) + \frac{\Gamma(b-1)}{\Gamma(a)} x^{1-b} M(1+a-b, 2-b, x). \tag{33}$$

When presented with the equation

$$\mathcal{L}f(\psi) = \frac{1}{2}\sigma^2 \psi^2 f''(\psi) + (1 - r\psi) f'(\psi) = 0.$$

it is a matter of checking to confirm that

$$G(x) = x^{\gamma_2} M(-\gamma_2, 1 - \gamma_2 + \gamma_1, 2/(x\sigma^2))$$

and

$$H(x) = x^{\gamma_2} U(-\gamma_2, 1 - \gamma_2 + \gamma_1, 2/(x\sigma^2))$$

provide linearly independent solution where as before $\gamma_1 < \gamma_2$ are the two roots of

$$\frac{\sigma^2}{2}\gamma^2 - (\frac{\sigma^2}{2} + r)\gamma - \lambda = 0.$$

The requirement that $b \in \mathbb{R} \backslash \mathbb{Z}_{\leq 0}$ in the definition of $M(a,b,x)$ now requires us to impose that

$$1 - \gamma_2 + \gamma_1 = 1 - \frac{2\sqrt{(\sigma^2/2 + r)^2 + 2\lambda\sigma^2}}{\sigma^2} \notin \mathbb{Z}_{\leq 0}$$

which was condition (20). For a full account of the Kummer functions, the reader is referred to Lebedev (1972).

References

[1] Duistermaat, J.J., Kyprianou, A.E. and van Schaik, K. (2003) Finite expiry Russian options. *To appear in stochastic processes and their applications.*

[2] Dynkin, E.B. (1969) Game variant problem on optimal stopping. *Soviet Math, Dokl.* **10**, 270–274.

[3] Friedman, A. (1976) *Stochastic differential equations and applications. Vol II.* Academic Press.

[4] Karatzas, I. and Shreve, S. (1988) *Brownian motion and stochastic calculus. 2nd Edition.* Springer New York.

[5] Kifer, Yu. (2000) Game options. *Fin. Stoch.* **4**, 443–463.

[6] Kramkov, D.O. and Mordecky, E. (1994) An Integral option *Theory Prob. Appl.* **1**, 162–172.

[7] Kühn, C. and Kyprianou, A.E. (2003) Pricing Israeli options: a duality approach. *Preprint.*

[8] Kyprianou, A.E. (2004) Some calculations for Israeli Options. *Fin. Stoch.* **8**, 73 - 86.

[9] Lebedev, N.N. (1972) *Special functions and their applications.* Dover publications.

[10] Peskir, G. (2002) On the American option problem. *To appear in Mathematical Finance.*

[11] Peskir, G. (2003) The Russian Option: Finite Horizon. *To appear in Finance and Stochastics.*

[12] Shiryaev, A.N. (1978) *Optimal stopping Rules.* Springer Verlag, Berlin

[13] Shiryaev, A.N. and Kabanov, Yu. and Kramkov, D.O. and Melnikov, A.V. (1994) Toward the theory of options of both European and American types.II. Continuous time. *Theory Probab. Appl.* **39**, 61–102.