Examples of optimal stopping via measure transformation for processes with one-sided jumps

E. J. Baurdoux
Universiteit Utrecht
baurdoux@math.uu.nl

Abstract

In this short note we show that the method introduced by Beibel and Lerche in [1] for solving certain optimal stopping problems for Brownian motion can be applied as well to some optimal stopping problems involving processes with one-sided jumps.

Keywords: Optimal stopping problems, spectrally negative Lévy processes, stable processses, generalised Ornstein-Uhlenbeck processes.

1 Introduction

In [1] Beibel and Lerche proposed a method for solving certain optimal stopping problems for a Brownian motion B. They used a change of measure to reduce the optimal stopping problem to the problem of finding the maximum of a (deterministic) function. One example solved in [1] is

$$\sup_{\tau} \mathbb{E}\left[\frac{B_\tau}{\tau+1}\right]. \tag{1}$$

This problem was first solved in ([5], Theorem 1) and, independently, in ([6], Example 2). In section 10 of [5] it was suggested that it is of interest to replace B in (1) by a stable process of index $\alpha \in (1,2)$. In this note we show that in some cases, the method proposed in [1] can be used as well for processes with one-sided jumps. In particular, for a spectrally negative strictly stable process of index $\alpha \in (1,2)$ we solve the problem (1) in two ways : firstly by a change of measure similar to the one used in Problem 3 in [1] and secondly by using results from [3] about generalised Ornstein-Uhlenbeck processes.

2 Alphabolic boundaries

Denote by $\{X_t\}_{t\geq 0}$ a spectrally negative strictly stable process of index $\alpha \in (1,2)$ defined on $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, a filtered probability space which satisfies the usual conditions. We denote by $\mathbb{P}_x$ the translation of $\mathbb{P}$ under which $X_0 = x$. Without loss of generality we assume that the Laplace exponent of X is given by $\psi(\lambda) = \lambda^\alpha$. We refer to Chapter VIII in [2] and Chapter 3 in[4] for further details about stable processes. Let $\beta > 0$ and define the (finite) function

$$H(x) = \int_0^\infty e^{ux-u^\alpha} u^{\alpha\beta-1} du.$$

Suppose h is a function on $\mathbb{R}$ such that there exists some x^* satisfying

$$x^* = \arg\max_x \frac{h(x)}{H(x)}. \tag{2}$$

Denote by $\mathcal{T}$ the set of stopping times with respect to $\{\mathcal{F}_t\}_{t\geq 0}$. The aim of this section is to find the optimal stopping time in

$$V(x) := \sup_{\tau\in\mathcal{T}} \mathbb{E}_x \left[\frac{h\left((\tau+1)^{-1/\alpha} X_\tau\right)}{(\tau+1)^\beta} 1_{\{\tau<\infty\}} \right]. \tag{3}$$

We have the following result.

Theorem 1. *Let h be a function on $\mathbb{R}$ such that x^* in (2) exists. Suppose $x < x^*$. The optimal stopping time in (3) is given by*

$$\tau^* = \inf\{t \geq 0 : X_t = (t+1)^{1/\alpha} x^*\}.$$

Furthermore

$$V(x) = \frac{h(x^*)}{H(x^*)} H(x).$$

Proof. By changing variables $y = u(t+1)^{-1/\alpha}$ we find that

$$\begin{aligned} H((t+1)^{-1/\alpha} X_t) &= \int_0^\infty e^{u(t+1)^{-1/\alpha}X_t - u^\alpha} u^{\alpha\beta-1} du \\ &= (t+1)^\beta \int_0^\infty e^{yX_t - y^\alpha t - y^\alpha} y^{\alpha\beta-1} dy. \end{aligned}$$

Since $\{e^{yX_t - y^\alpha t}\}_{t\geq 0}$ is a martingale, it follows that $\{M_t\}_{t\geq 0}$ defined by

$$M_t = \frac{H((t+1)^{-1/\alpha} X_t)}{H(x)(t+1)^\beta}$$

is a mean 1 martingale under $\mathbb{P}_x$. Hence for any $\mathbb{P}_x$ stopping time τ we have that

$$\begin{aligned} \mathbb{E}_x \left[\frac{h((\tau+1)^{-1/\alpha} X_\tau)}{(\tau+1)^\beta} 1_{\{\tau<\infty\}} \right] &= \mathbb{E}_x \left[H(x) \frac{h((\tau+1)^{-1/\alpha} X_\tau)}{H((\tau+1)^{-1/\alpha} X_\tau)} M_\tau 1_{\{\tau<\infty\}} \right] \\ &\leq H(x) \frac{h(x^*)}{H(x^*)} \mathbb{E}_x[M_\tau 1_{\{\tau<\infty\}}] \\ &\leq H(x) \frac{h(x^*)}{H(x^*)}, \end{aligned}$$

and thus

$$\tau^* := \inf\{t \geq 0 : (t+1)^{-1/\alpha} X_t = x^*\}$$

is the optimal stopping time if we can show that $\mathbb{P}_x(\tau^* < \infty) = 1$ and that $\mathbb{E}_x[M_{\tau^*}] = 1$. By the law of iterated logarithm for spectrally negative stable processes (see Theorem 5 (ii) in [2]) we deduce that for any $x < x^*$

$$\mathbb{P}_x(\tau^* < \infty) = 1.$$

Also, since H is an increasing function and since $(\tau^* + 1)^{-1/\alpha} X_{\tau^*} \leq x^*$ we deduce that for $x < x^*$ and any $n \in \mathbb{N}$

$$M_{\tau^* \wedge n} \leq \frac{H(x^*)}{H(x)} \quad \text{under } \mathbb{P}_x.$$

We use the optional sampling theorem and bounded convergence to conclude that

$$\begin{aligned} 1 &= \lim_{n\to\infty} \mathbb{E}_x[M_{\tau^* \wedge n}] \\ &= \mathbb{E}_x[M_{\tau^*}]. \end{aligned}$$

This completes the proof. □

3 Generalised Ornstein-Uhlenbeck process

Let Z be a spectrally negative Lévy Process defined on a filtered probability space $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions. The Laplace exponent ψ of Z is given by

$$\psi(\lambda) = \frac{\sigma^2}{2}\lambda^2 + a\lambda + \int_{-\infty}^{0} \left(e^{\lambda x} - 1 - \lambda x 1_{\{x \geq -1\}}\right) \Pi(dx), \quad \lambda \geq 0.$$

Again we refer to [2] for further details. The Generalised Ornstein-Uhlenbeck process $\{Y_t\}_{t\geq 0}$ is the solution to

$$dY_t = -\lambda Y_t dt + dZ_t, \quad Y_0 = y \quad \text{under } \mathbb{P}_y.$$

Let $r > 0$. In this section we consider optimal stopping problems of the form

$$U(y) := \sup_{\tau \in \mathcal{T}} \mathbb{E}_y[e^{-r\tau} g(Y_\tau) 1_{\{\tau < \infty\}}],. \tag{4}$$

where g belongs to a class of functions which is yet to be specified. Assume that

$$\sigma > 0 \quad \text{or} \quad a - \int_{-1}^{0} z\, \Pi(dz) > \lambda y, \tag{5}$$

since otherwise the Generalised Ornstein-Uhlenbeck process never hits points $b > y$ with probability one (see Remark 1 in [3]). Clearly (5) is satisfied when

Z is of unbounded variation.
To simplify we also assume that

$$\mathbb{E}[\log(1 + (-Z_1)^+)] < \infty. \tag{6}$$

Denote

$$\phi(u) = \frac{1}{\lambda} \int_0^u \frac{\psi(v)}{v} dv.$$

Introduce for $r > 0$

$$G(x) = \int_0^\infty e^{ux - \phi(u)} u^{-1+r/\lambda} du$$

and

$$N_t = e^{-rt} G(Y_t). \tag{7}$$

Theorem 1 in [3] states that under the assumptions (5) and (6) the process $\{N_t\}_{t\geq 0}$ is a martingale for any $r > 0$. Introduce the locally equivalent measure $\mathbb{Q}$ by

$$\left.\frac{d\mathbb{Q}_y}{d\mathbb{P}_y}\right|_{\mathcal{F}_t} = \frac{N_t}{G(y)}.$$

We see that (4) can be written as

$$U(y) = G(y) \sup_{\tau \in \mathcal{T}} \mathbb{E}_y^{\mathbb{Q}} \left[\frac{g(Y_\tau)}{G(Y_\tau)} 1_{\{\tau < \infty\}} \right].$$

Theorem 2. *Suppose g is a function on $\mathbb{R}$ such that g/G attains its maximum at y^* and suppose that $\{Z_t\}_{t\geq 0}$ is a spectrally negative Lévy process satisfying (6) and*

$$\sigma > 0 \quad or \quad a - \int_{-1}^0 z\, \Pi(dz) > \lambda y^*.$$

Then for any $Y_0 = y < y^*$ the optimal stopping time in (4) is given by

$$\sigma^* = \inf\{t \geq 0 : Y_t = y^*\}.$$

Furthermore

$$U(y) = \frac{g(y^*)}{G(y^*)} G(y).$$

Proof. Let $y < y^*$. It suffices to prove that σ^* is almost surely finite under $\mathbb{P}_y$ and $\mathbb{Q}_y$. The first statement is contained in Theorem 2 in [3]. The proof of the second statement is similar to the end of the proof of Theorem 1. □

Denote by $Y^{(\alpha)}$ the generalised Ornstein-Uhlenbeck process which has a spectrally negative strictly stable process $X^{(\alpha)}$ with index $\alpha \in (1, 2)$ as driving Lévy process and for which $\lambda = 1/\alpha$ and $Y_0^{(\alpha)} = 0$. It is not difficult to show that $e^{-t/\alpha}(X^{(\alpha)}(e^t-1))$ is equal in distribution to $Y_t^{(\alpha)}$ (they have the same Laplace exponent). We deduce that

$$\sup_\tau \mathbb{E}\left[\frac{X_\tau^{(\alpha)}}{\tau+1}\right] = \sup_\tau \mathbb{E}\left[e^{-\tau} X^{(\alpha)}(e^\tau - 1)\right] = \sup_\tau \mathbb{E}\left[e^{-(1-\alpha^{-1})\tau} Y_\tau^{(\alpha)}\right].$$

Hence for a spectrally negative strictly stable process we can also solve (1) by applying Theorem 2 to the case $g(x) = x$ and $r = (\alpha - 1)/\alpha$.
Acknowledgement This note was written during a stay at Heriot-Watt University, Edinburgh and I would like to express my gratitude for their hospitality and support. I also thank the referee for his/her valuable comments.

References

[1] Beibel, M. and Lerche, H.R. (1997) New look at optimal stopping problems related to mathematical finance. *Statist. Sinica* **7**, 93-108.

[2] Bertoin, J. (1996) *Lévy processes.* Cambridge University Press.

[3] Novikov, A. (2003) Martingales and first-passage times for Ornstein-Uhlenbeck processes with a jump component. *Theory Probab. Appl.* **48**, 288-303.

[4] Sato, K. (1999) *Lévy processes and infinitely divisible distributions.* Cambridge University Press.

[5] Shepp, L.A. (1969) Explicit solutions to some problems of optimal stopping. *Ann. Math. Stat.* **40**, 993-1010.

[6] Taylor, H.M. (1968) Optimal stopping in a Markov process. *Ann. Math. Stat.* **39**, 1333-1344.