AN INTRODUCTION TO MARKOV PROCESSES AND THEIR APPLICATIONS IN MATHEMATICAL ECONOMICS

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In this section we will focus on one-dimensional continuous Markov processes on real line. Our aim is to better understand their extended generators, transition functions, and to construct diffusion process from a Brownian motion by a change of time and space.

We will deal with a Markov process, $X$, whose state space, $E$, is an interval $(l, r)$ in $\mathbb{R}$, which may be closed, open, semi-open, bounded or unbounded. The life-time is denoted with $\zeta$ as usual. The following is the standing assumptions on $X$ throughout the section:

1. $X$ is continuous on $[0, \zeta]$;
2. $X$ has strong Markov property;
3. $X$ is regular; i.e. if $T_x := \inf\{t > 0 : X_t = x\}$, then $P^x(T_y < \infty) > 0$ for any $x \in \text{int}(E)$ and $y \in E$.

The last assumption on the regularity of $X$ is very much like the concept of irreducibility for Markov chains. Indeed, when $X$ is regular, its state space cannot be decomposed into smaller sets from which $X$ cannot exit.

For any interval $I = [a, b]$ such that $[a, b] \subset E$, we denote by $\sigma_I$ the exit time of $I$. Note that for $x \in I$, $\sigma_I = T_a \wedge T_b$, $P^x$-a.s., and for $x \notin I$, $\sigma_I = 0$, $P^x$-a.s.. We also put $m_I(x) := E^x[\sigma_I]$.

**Proposition 6.1.** $m_I$ is bounded on $I$. In particular $\sigma_I$ is finite almost surely.

**Proof.** Let $y$ be a fixed point in $I$. Since $X$ is regular, we can find $\alpha < 1$ and $t > 0$ such that

\[
\max\{P^y(T_a > t), P^y(T_b > t)\} = \alpha.
\]

If $y < x < b$, then

\[
P^x(\sigma_I > t) \leq P^x(T_b > t) \leq P^y(T_b > t) \leq \alpha.
\]

Applying the same reasoning to $a < x < y$, we thus get

\[
\sup_{x \in I} P^x(\sigma_I > t) \leq \alpha < 1.
\]

Now, since $\sigma_I = u + \sigma_I \circ \theta_u$ on $[\sigma_I > u]$, we have

\[
P^x(\sigma_I > nt) = P^x\left([\sigma_I > (n-1)t] \bigcap (\cap_{1}[(n-1)t + \sigma_I \circ \theta_{(n-1)t} > nt]\right).
\]
Thus, in view of the Markov property

\[ P^x(\sigma_I > nt) = E^x \left[ 1_{[\sigma_I > (n-1)t]} E^{X(n-1)t} \left[ 1_{[\sigma_I > t]} \right] \right]. \]

However, on \([\sigma_I > (n-1)t], X(n-1)t \in I\), thus, \( E^{X(n-1)t} \left[ 1_{[\sigma_I > t]} \right] \leq \alpha \) so that

\[ P^x(\sigma_I > nt) \leq \alpha P^x(\sigma_I > (n-1)t), \]

and consequently \( P^x(\sigma_I > nt) \leq \alpha^n \) for every \( x \in I \). Therefore,

\[
\begin{align*}
\sup_{x \in I} E^x[\sigma_I] &= \sup_{x \in I} \int_0^\infty P^x(\sigma_I > s)ds = \sup_{x \in I} \sum_{n=0}^\infty \int_{nt}^{(n+1)t} P^x(\sigma_I > s)ds \\
&\leq \sum_{n=0}^\infty tP^x(\sigma_I > nt) \leq t \frac{1}{1 - \alpha}.
\end{align*}
\]

\[ \Box \]

In view of the above proposition, for any \( l \leq a < x < b \leq r \), we have

\[ P^x[T_a < T_b] + P^x[T_b < T^a] = 1. \]

**Theorem 6.1.** There exists a continuous and strictly increasing function \( s \) on \( E \) such that for any \( a, b, x \) in \( E \) with \( l \leq a < x < b \leq r \)

\[ P^x(T_b < T_a) = \frac{s(x) - s(a)}{s(b) - s(a)}. \]

Moreover, if \( \tilde{s} \) is another function with the same properties, then \( \tilde{s} = s + \beta \) for some \( \alpha > 0 \).

**Proof.** We will only give the proof of the formula when \( E \) is a closed and bounded interval. For the rest of the proof and the extension to the general case see the proof of Proposition 3.2 in Chap. VII of Revuz & Yor (or better try it yourself!).

First, observe that

\[ [T_r < T_I] = [T_r < T_I, T_a < T_b] \cap [T_r < T_I, T_b < T_a], \]

and that \( T_I = T_a + T_I \circ \theta_{T_a} \) and \( T_r = T_a + T_r \circ \theta_{T_a} \) on \([T_a < T_b]\). Thus,

\[ P^x(T_r < T_I, T_a < T_b) = E^x \left[ 1_{[T_a < T_b]} 1_{[T_r < T_I]} \circ \theta_{T_a} \right]. \]

Using the strong Markov property at \( T_a \) and that \( X_{T_a} = a \) when \( T_a < \infty \), we obtain

\[ P^x(T_r < T_I, T_a < T_b) = P_x(T_a < T_b)P_a(T_r < T_I). \]

Similarly,

\[ P^x(T_r < T_I, T_b < T_a) = P_x(T_b < T_a)P_b(T_r < T_I), \]

so that

\[ P^x(T_r < T_I) = P_x(T_a < T_b)P_a(T_r < T_I) + P_x(T_b < T_a)P_b(T_r < T_I). \]

Setting \( s(x) = P^x(T_r < T_I) \)

and solving for \( P_x(T_b < T_a) \), we get the formula in the statement. \( \Box \)

**Definition 6.1.** The function \( s \) in the above result is called the scale function of \( X \).

Observe that \( s \) is unique only upto an affine transformation. We will say that \( X \) is on **natural scale** when \( s(x) = x \). Moreover, if we apply the scale function to \( X \), we get the following
**Proposition 6.2.** Let $\tilde{X} := s(X)$. Then, $\tilde{X}$ satisfies the standing assumptions of this section and it is on natural scale.

Let’s put $R = T_r \land T_l$.

**Theorem 6.2.** Let $f$ be a locally bounded and increasing Borel function. Then, $f$ is a scale function if and only if $f(X)^R$ is a local martingale.

**Proof.** We refer to the proof of Proposition 3.5 in Chap. VII of Revuz & Yor for the proof of that $s(X)^R$ is a local martingale. We will now give the proof of the converse.

Suppose that $f(X)^R$ is a local martingale and let $[a, b]$ be a closed interval in the interior of $E$. Then, $f(X)_{T_a \land T_b}$ is a bounded martingale. Thus, an application of the Optional Stopping Theorem yields

$$f(x) = f(a) P^x(T_a < T_b) + f(b) P^x(T_b < T_a).$$

Since $P^x(T_a < T_b) + P^x(T_b < T_a) = 1$, solving for $P^x(T_b < T_a)$ shows that $f$ is a scale function. \qed

**Exercise 6.1.** Suppose that $X$ is a diffusion with the infinitesimal generator

$$L = \frac{1}{2} \sigma^2(x) \frac{d^2}{dx^2} + b(x) \frac{d}{dx},$$

where $\sigma$ and $b$ are locally bounded functions on $E$ and $\sigma > 0$. Prove that the scale function is given by

$$s(x) = \int_c^x \exp \left( - \int_c^y 2b(z) \sigma^{-2}(z) dz \right) dy$$

where $c$ is an arbitrary point in the interior of $E$.

**Exercise 6.2.** A 3-dimensional Bessel process is a Markov process on $(0, \infty)$ satisfying

$$X_t = x + \int_0^t \frac{1}{X_s} \, ds + B_t, \quad x > 0.$$ 

Compute its scale function.

Since $X$ has continuous paths, $s(X)$ is a continuous local martingale. Thus, it can be viewed as a time-changed Brownian motion. This time change can be explicitly written in terms of a so-called speed measure.

Recall that $m_I(x) = E^x[\sigma_I]$. Let now that $J = ]c, d[\, ]$ be a subinterval of $I$. By the strong Markov property, for any $a < c < x < d < b$ one has

$$m_I(x) = E^x[\sigma_J + \sigma_I \circ \theta_{\sigma_J}]$$

$$= m_I(x) + \frac{s(d) - s(x)}{s(d) - s(c)} m_I(c) + \frac{s(x) - s(c)}{s(d) - s(c)} m_I(d)$$

$$> \frac{s(d) - s(x)}{s(d) - s(c)} m_I(c) + \frac{s(x) - s(c)}{s(d) - s(c)} m_I(d),$$

which says that $m_I$ is an $s$-concave function. Thus, one can define a Green’s function $G_I$ on $E \times E$ by

$$G_I(x, y) = \begin{cases} 
\frac{(s(x) - s(a))(s(b) - s(y))}{s(b) - s(a)}, & \text{if } a \leq x \leq y \leq b, \\
\frac{(s(y) - s(a))(s(b) - s(x))}{s(b) - s(a)}, & \text{if } a \leq y \leq x \leq b, \\
0, & \text{otherwise;}
\end{cases}$$

where $a < c < x < d < b$. 

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so that there exists a measure \( \mu_I \) on the interior of \( E \) such that
\[
m_I(x) = \int G_I(x, y) \mu_I(dy).
\]

In fact, we have more. The measure \( \mu_I \) does not depend on \( I \) (see Theorem 3.6 in Chap. VII of Revuz & Yor) and one can write
\[
m_I(x) = \int G_I(x, y) m(dy).
\]
The measure in the above representation is called the \textit{speed measure} of \( X \). The next two results state some of the basic properties of the speed measure.

**Proposition 6.3.** For any positive Borel function \( f \)
\[
E^x \left[ \int_0^{T_a \wedge T_b} f(X_s) ds \right] = \int G_I(x, y) f(y) m(dy).
\]

**Proof.** See the proof of Corollary 3.8 in Chap. VII of Revuz & Yor. \( \Box \)

**Theorem 6.3.** For a bounded \( f \in D_A \) and \( x \in \text{int}(E) \)
\[
Af(x) = \frac{d}{dm} \frac{d}{ds} f(x)
\]
in the sense that
i) the \( s \)-derivative \( \frac{df}{ds} \) exists except possibly on the set \( \{x : m(\{x\}) > 0\} \),
ii) if \( x_1 \) and \( x_2 \) are two points for which this \( s \)-derivative exists, then
\[
\frac{df}{ds}(x_2) - \frac{df}{ds}(x_1) = \int_{x_1}^{x_2} Af(y) m(dy).
\]

**Proof.** If \( f \in \), then \( M^f \) is a martingale where
\[
M^f_t = f(X_t) - f(X_0) - \int_0^t Af(X_s) ds.
\]
Moreover, for any stopping time \( T \) with \( E^x[T] < \infty \), \( (M^f_{t \wedge T}) \) is a uniformly integrable martingale since \( |M^f_{t \wedge T}| \leq 2 \| f \| + T \| Af \| \). Thus, if we choose \( T = T_a \wedge T_b \), by the Optional Stopping Theorem
\[
E^x[f(X_T)] = f(x) + E^x \left[ \int_0^T Af(X_s) ds \right].
\]
Thus, it follows from the preceding result that
\[
f(a)(s(b) - s(x)) + f(b)(s(x) - s(a)) - f(x)(s(b) - s(a))
= (s(b) - s(a)) \int_I G_I(x, y) Af(y) m(dy).
\]
(6.1)

It can be directly verified that
\[
\frac{f(b) - f(x)}{s(b) - s(x)} + \frac{f(x) - f(a)}{s(x) - s(a)} = \int_I H_I(x, y) Af(y) m(dy),
\]
where

\[ H_t(x, y) = \begin{cases} \frac{s(y)-s(a)}{s(x)-s(a)}, & \text{if } a < y \leq x, \\ \frac{s(b)-s(y)}{s(b)-s(x)}, & \text{if } x \leq y < b, \\ 0, & \text{otherwise.} \end{cases} \]

Note that if \( b \downarrow x \), \( \frac{s(b)-s(y)}{s(b)-s(x)} \to 1_{\{x\}} \), which implies that the right \( s \)-derivative of \( f \) exists. Similarly, \( \frac{s(y)-s(a)}{s(x)-s(a)} \to 1_{\{x\}} \) as \( a \uparrow x \) and thus the left derivative exists as well. Letting \( a \) and \( b \) tend to \( x \) together, we obtain

\[ \frac{df^+}{ds}(x) - \frac{df^-}{ds}(x) = 2m(\{x\})Af(x), \]

which yields the first part of the statement.

In order to show the second claim, let \( h \) be such that \( a < x + h < b \) and apply (6.1) to \( x \) and \( x + h \) to get

\[ f(b) - f(a) - (s(b) - s(a)) f(x + h) - f(x) \]

\[ = (s(b) - s(a)) \int_I G_I(x + h, y) - G_I(x, y) \frac{1}{s(x + h) - s(x)} Af(y)m(dy). \]

Now, suppose that \( f \) is \( s \)-differentiable at \( x \) and let \( h \to 0 \). It follows from the dominated convergence theorem that

\[ f(b) - f(a) - (s(b) - s(a)) \frac{df}{ds}(x) \]

\[ = - \int_a^b (s(y) - s(a)) Af(y)m(dy) + \int_x^b (s(b) - s(y)) Af(y)m(dy) \]

\[ = - \int_a^b s(y) Af(y)m(dy) + s(a) \int_a^x Af(y)m(dy) + s(b) \int_x^b Af(y)m(dy). \]

If \( x_1 \) and \( x_2 \) are two such points, we obtain the claim by subtracting. \( \square \)

**Exercise 6.3.** If \( X \) is a Markov processes satisfying the hypothesis of this section and \( \phi \) is a homeomorphism from \( E \) to \( \tilde{E} \), then \( \tilde{X} = \phi(X) \) also satisfies these hypothesis on \( \tilde{E} \). Prove that \( \tilde{s} = s \circ \phi^{-1} \) and that \( \tilde{m} \) is the image of \( m \) under \( \phi \), i.e \( \tilde{m}(\tilde{I}) = m(\phi^{-1}(I)) \) for any \( \tilde{I} \subset \text{int}(\tilde{E}) \).

**Exercise 6.4.** Let \( X \) be as in Exercise 6.1. Show that \( m(dx) = \frac{2}{s'(x)\sigma^2(x)}dx \). (Hint: First find the speed measure for the process \( \tilde{X} = s(X) \) using the relationship between the speed measure and the infinitesimal generator by observing that \( \tilde{s}(x) = x \). Then, show that that \( m(dx) = \tilde{m}(s(x))s'(x) \) using the previous exercise.

As mentioned earlier \( \tilde{X} = s(X) \) is continuous local martingale, therefore, it is a time-changed Brownian motion, i.e.

\[ \tilde{X}_t = \beta_{\eta_t} \]

where \( \beta \) is a Brownian motion in some enlargement of the probability space. In our setting it is possible to find explicitly the process \( \gamma \). Theorem V.47.1 in volume 2 of Rogers &
Williams\textsuperscript{1} shows that
\[ \gamma_t = \inf \left\{ u : \frac{1}{2} \int_0^t \ell_z \hat{m}(dz) > t \right\}, \]
where \((\ell_z)_{t \geq 0}\) is the local time process of \(\beta\) at level \(z\), and \(\hat{m}\) is the speed measure of \(\tilde{X}\).

The speed measure can also be linked to the resolvent of \(X\) and its transition function.

To see this we first define
\[
\varphi^+(\alpha, x) := \begin{cases} 
E^x[\exp(-\alpha T_c)]: & x \leq c, x \in \text{int}(E) \\
1/E^x[\exp(-\alpha T_x)]: & x \geq c, x \in \text{int}(E) 
\end{cases}
\]
\[
\varphi^-(\alpha, x) := \begin{cases} 
E^x[\exp(-\alpha T_c)]: & x \geq c, x \in \text{int}(E) \\
1/E^x[\exp(-\alpha T_x)]: & x \leq c, x \in \text{int}(E) 
\end{cases}
\]
where \(c\) is an arbitrary point in \(\text{int}(E)\). Consequently, for \(a < b\)
\[ E^a[\exp(-\alpha T_b)] = \frac{\varphi^+(\alpha, a)}{\varphi^+(\alpha, b)}. \]

**Proposition 6.4.** \(\varphi^+(\alpha, \cdot)\) (resp. \(\varphi^-(\alpha, \cdot)\)) is continuous and increasing (resp. decreasing).
Moreover, \(\varphi^+\) and \(\varphi^-\), as functions of \(x\), solve
\[ \frac{d}{dt} \frac{d}{ds} f = \alpha f. \]

**Proposition 6.5.** Suppose that \(X\) is on natural scale. Then for any \(x \in \text{int}(E)\) and a bounded Borel function \(f\) with a support in \(\text{int}(E)\)
\[ U^\alpha f(x) = \int f(y) u_\alpha(x, y) m(dy), \]
where
\[
u_\alpha(x, y) = \begin{cases} 
c_\alpha \varphi^+(\alpha, x) \varphi^-(\alpha, y), & x \leq y \\
c_\alpha \varphi^+(\alpha, y) \varphi^-(\alpha, x), & x \geq y, 
\end{cases}
\]
\[
c_\alpha^{-1} = \varphi^-(\alpha, x) D\varphi^+(\alpha, x-) - \varphi^+(\alpha, x) D\varphi^-(\alpha, x+),
\]
and \(D\) denotes differentiation.

**Proposition 6.6.** There exists a jointly continuous map \(p(t, x, y)\), which is symmetric in \(x\) and \(y\) such that for all bounded Borel functions \(f\) supported in \(\text{int}(E)\)
\[ E^x[f(X_t)] = \int f(y) p(t, x, y) m(dy), \]
whenever \(x \in \text{int}(E)\).

**Exercise 6.5.** Let \(X\) be as in Exercise 6.1, \(I = ]l, r[\) and suppose \(X_0 = x \in \text{int}(I)\). We distinguish four cases:
1. \(s(l+) = -\infty, s(r-) = \infty.\) Then,
\[
P^x(\sigma_I = \infty) = P^x \left( \sup_{0 \leq t < \infty} X_t = r \right) = P^x \left( \inf_{0 \leq t < \infty} X_t = l \right) = 1.
\]

\textsuperscript{1}Observe that their definition of the speed measure results in a measure which is only half of the one according to our definition.
In particular, the process $X$ is recurrent, i.e. for any $y \in I$, we have

$$P^x(X_t = y; \text{ for some } t \in [0, \infty)) = 1.$$  

(2) $s(l+) > -\infty, s(r-) = \infty$. Then,

$$P^x\left(\lim_{t \to \sigma_I} X_t = l\right) = P^x\left(\sup_{t < \sigma_I} X_t < r\right) = 1.$$  

(3) $s(l+) = -\infty, s(r-) < \infty$. Then,

$$P^x\left(\lim_{t \to \sigma_I} X_t = r\right) = P^x\left(\inf_{t < \sigma_I} X_t > l\right) = 1.$$  

(4) $s(l+) > -\infty, s(r-) < \infty$. Then,

$$P^x\left(\lim_{t \to \sigma_I} X_t = l\right) = 1 - P^x\left(\lim_{t \to \sigma_I} X_t = r\right) = \frac{s(r-) - s(x)}{s(r-) - s(l+)}.$$  

(In parts 2, 3, and 4 there is no claim that $\sigma_I$ is finite. Note that in this exercise we do not necessarily assume that $I$ is bounded.)

**Remark 1.** The scale function $s$ plays a crucial role in determining the explosion properties of one-dimensional diffusions. See Lemma 5.5.26 and Theorem 5.5.29 (Feller’s test of explosions) in Karatzas & Shreve for more details.

**Remark 2.** In Mathematical Finance Theory one of the crucial questions is whether the stochastic exponential of a local martingale is a true martingale. When the underlying local martingale is strong Markov and continuous, the speed measure tells us whether its stochastic exponential is a true martingale or not. For details you can consult the following recent works:


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