AN INTRODUCTION TO MARKOV PROCESSES AND THEIR APPLICATIONS IN MATHEMATICAL ECONOMICS

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In this section we will focus on one-dimensional continuous Markov processes on real line. Our aim is to better understand their extended generators, transition functions, and to construct diffusion process from a Brownian motion by a change of time and space.

We will deal with a Markov process, X, whose state space, $\mathbf{E}$, is an interval (l, r) in $\mathbb{R}$, which may be closed, open, semi-open, bounded or unbounded. The life-time is denoted with ζ as usual. The following is the standing assumptions on X throughout the section:

(1) X is continuous on $[0, \zeta[$;
(2) X has strong Markov property;
(3) X is regular; i.e. if $T_x := \inf\{t > 0 : X_t = x\}$, then $P^x(T_y < \infty) > 0$ for any $x \in \text{int}(E)$ and $y \in E$.

The last assumption on the regularity of X is very much like the concept of *irreducibility* for Markov chains. Indeed, when X is regular, its state space cannot be decomposed into smaller sets from which X cannot exit.

For any interval $I =]a, b[$ such that $[a, b] \subset E$, we denote by σ_I the exit time of I. Note that for $x \in I$, $\sigma_I = T_a \wedge T_b, P^x$-a.s., and for $x \notin I$, $\sigma_I = 0, P^x$-a.s.. We also put $m_I(x) := E^x[\sigma_I]$.

Proposition 6.1. *m_I is bounded on I. In particular σ_I is finite almost surely.*

Proof. Let y be a fixed point in I. Since X is regular, we can find $\alpha < 1$ and $t > 0$ such that

$$\max\{P^y(T_a > t), P^y(T_b > t)\} = \alpha.$$

If $y < x < b$, then

$$P^x(\sigma_I > t) \leq P^x(T_b > t) \leq P^y(T_b > t) \leq \alpha.$$

Applying the same reasoning to $a < x < y$, we thus get

$$\sup_{x \in I} P^x(\sigma_I > t) \leq \alpha < 1.$$

Now, since $\sigma_I = u + \sigma_I \circ \theta_u$ on $[\sigma_I > u]$, we have

$$P^x(\sigma_I > nt) = P^x\left([\sigma_I > (n-1)t] \bigcap [(n-1)t + \sigma_I \circ \theta_{(n-1)t} > nt]\right).$$

Thus, in view of the Markov property

$$P^x(\sigma_I > nt) = E^x\left[\mathbf{1}_{[\sigma_I>(n-1)t]} E^{X_{(n-1)t}}\left[\mathbf{1}_{[\sigma_I>t]}\right]\right].$$

However, on $[\sigma_I > (n-1)t]$, $X_{(n-1)t} \in I$, thus, $E^{X_{(n-1)t}}\left[\mathbf{1}_{[\sigma_I>t]}\right] \le \alpha$ so that

$$P^x(\sigma_I > nt) \le \alpha P^x(\sigma_I > (n-1)t),$$

and consequently $P^x(\sigma_I > nt) \le \alpha^n$ for every $x \in I$. Therefore,

$$\begin{aligned}
\sup_{x\in I} E^x[\sigma_I] &= \sup_{x\in I}\int_0^\infty P^x(\sigma_I > s)ds = \sup_{x\in I}\sum_{n=0}^\infty \int_{nt}^{(n+1)t} P^x(\sigma_I > s)ds \\
&\le \sum_{n=0}^\infty tP^x(\sigma_I > nt) \le t\frac{1}{1-\alpha}.
\end{aligned}$$

□

In view of the above proposition, for any $l \le a < x < b \le r$, we have

$$P^x[T_a < T_b] + P^x[T^b < T^a] = 1.$$

Theorem 6.1. *There exists a continuous and strictly increasing function s on $\mathbf{E}$ such that for any a, b, x in $\mathbf{E}$ with $l \le a < x < b \le r$*

$$P^x(T_b < T_a) = \frac{s(x) - s(a)}{s(b) - s(a)}.$$

Moroever, if $\tilde{s}$ is another function with the same properties, then $\tilde{s} = \alpha s + \beta$ for some $\alpha > 0$.

Proof. We will only give the proof of the formula when $\mathbf{E}$ is a closed and bounded interval. For the rest of the proof and the extension to the general case see the proof of Proposition 3.2 in Chap. VII of Revuz & Yor (or better try it yourself!).

First, observe that

$$[T_r < T_l] = [T_r < T_l, T_a < T_b] \cap [T_r < T_l, T_b < T_a],$$

and that $T_l = T_a + T_l \circ \theta_{T_a}$ and $T_r = T_a + T_r \circ \theta_{T_a}$ on $[T_a < T_b]$. Thus,

$$P^x(T_r < T_l, T_a < T_b) = E^x\left[\mathbf{1}_{[T_a<T_b]}\mathbf{1}_{[T_r<T_l]} \circ \theta_{T_a}\right].$$

Using the strong Markov property at T_a and that $X_{T_a} = a$ when $T_a < \infty$, we obtain

$$P^x(T_r < T_l, T_a < T_b) = P_x(T_a < T_b)P_a(T_r < T_l).$$

Similarly,

$$P^x(T_r < T_l, T_b < T_a) = P_x(T_b < T_a)P_b(T_r < T_l),$$

so that

$$P^x(T_r < T_l) = P_x(T_a < T_b)P_a(T_r < T_l) + P_x(T_b < T_a)P_b(T_r < T_l).$$

Setting

$$s(x) = P^x(T_r < T_l)$$

and solving for $P_x(T_b < T_a)$, we get the formula in the statement. □

Definition 6.1. *The function s in the above result is called the* scale function *of X.*

Observe that s is unique only upto an affine transformation. We will say that X is on *natural scale* when $s(x) = x$. Moreover, if we apply the scale function to X, we get the following

Proposition 6.2. *Let $\tilde{X} := s(X)$. Then, $\tilde{X}$ satisfies the standing assumptions of this section and it is on natural scale.*

Let's put $R = T_r \wedge T_l$.

Theorem 6.2. *Let f be a locally bounded and increasing Borel function. Then, f is a scale function if and only if $f(X)^R$ is a local martingale.*

Proof. We refer to the proof of Proposition 3.5 in Chap. VII of Revuz & Yor for the proof of that $s(X)^R$ is a local martingale. We will now give the proof of the converse.

Suppose that $f(X)^R$ is a local martingale and let $[a, b]$ be a closed interval in the interior of $\mathbf{E}$. Then, $f(X)^{T_a \wedge T_b}$ is a bounded martingale. Thus, an application of the Optional Stopping Theorem yields

$$f(x) = f(a)P^x(T_a < T_b) + f(b)P^x(T_b < T_a).$$

Since $P^x(T_a < T_b) + P^x(T_b < T_a) = 1$, solving for $P^x(T_b < T_a)$ shows that f is a scale function. □

Exercise 6.1. *Suppose that X is a diffusion with the infinitesimal generator*

$$L = \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx},$$

where σ and b are locally bounded functions on $\mathbf{E}$ and $\sigma > 0$. Prove that the scale function is given by

$$s(x) = \int_c^x \exp\left(-\int_c^y 2b(z)\sigma^{-2}(z)dz\right) dy$$

where c is an arbitrary point in the interior of $\mathbf{E}$.

Exercise 6.2. *A 3-dimensional Bessel process is a Markov process on $(0, \infty)$ satisfying*

$$X_t = x + \int_0^t \frac{1}{X_s} ds + B_t, \qquad x > 0.$$

Compute its scale function.

Since X has continuous paths, $s(X)$ is a continuous local martingale. Thus, it can be viewed as a time-changed Brownian motion. This time change can be explicitly written in terms of a so-called *speed measure.*

Recall that $m_I(x) = E^x[\sigma_I]$. Let now that $J =]c, d[$ be a subinterval of I. By the strong Markov property, for any $a < c < x < d < b$ one has

$$\begin{aligned} m_I(x) &= E^x[\sigma_J + \sigma_I \circ \theta_{\sigma_J}] \\ &= m_J(x) + \frac{s(d) - s(x)}{s(d) - s(c)} m_I(c) + \frac{s(x) - s(c)}{s(d) - s(c)} m_I(d) \\ &> \frac{s(d) - s(x)}{s(d) - s(c)} m_I(c) + \frac{s(x) - s(c)}{s(d) - s(c)} m_I(d), \end{aligned}$$

which says that m_I is an s-concave function. Thus, one can define a Green's function G_I on $\mathbf{E} \times \mathbf{E}$ by

$$G_I(x, y) = \begin{cases} \frac{(s(x)-s(a))(s(b)-s(y))}{s(b)-s(a)}, & \text{if } a \le x \le y \le b, \\ \frac{(s(y)-s(a))(s(b)-s(x))}{s(b)-s(a)}, & \text{if } a \le y \le x \le b, \\ 0, & \text{otherwise;} \end{cases}$$

so that there exists a measure μ_I on the interior of $\mathbf{E}$ such that

$$m_I(x) = \int G_I(x,y)\mu_I(dy).$$

In fact, we have more. The measure μ_I does not depend on I (see Theorem 3.6 in Chap. VII of Revuz & Yor) and one can write

$$m_I(x) = \int G_I(x,y)m(dy).$$

The measure in the above representation is called the *speed measure* of X. The next two results state some of the basic properties of the speed measure.

Proposition 6.3. *For any positive Borel function f*

$$E^x\left[\int_0^{T_a\wedge T_b} f(X_s)ds\right] = \int G_I(x,y)f(y)m(dy).$$

Proof. See the proof of Corollary 3.8 n Chap. VII of Revuz & Yor. □

Theorem 6.3. *For a bounded $f \in \mathbb{D}_A$ and $x \in int(\mathbf{E})$*

$$Af(x) = \frac{d}{dm}\frac{d}{ds}f(x)$$

in the sense that

i) *the s-derivative $\frac{df}{ds}$ exists except possibly on the set $\{x : m(\{x\}) > 0\}$,*
ii) *if x_1 and x_2 are two points for which this s-derivative exists, then*

$$\frac{df}{ds}(x_2) - \frac{df}{ds}(x_1) = \int_{x_1}^{x_2} Af(y)m(dy).$$

Proof. If $f \in$, then M^f is a martingale where

$$M_t^f = f(X_t) - f(X_0) - \int_0^t Af(X_s)\,ds.$$

Moreover, for any stopping time T with $E^x[T] < \infty$, $(M^f_{t\wedge T})$ is a uniformly integrable martingale since $|M^f_{t\wedge T}| \leq 2 \parallel f \parallel + T \parallel Af \parallel$. Thus, if we choose $T = T_a \wedge T_b$, by the Optional Stopping Theorem

$$E^x[f(X_T)] = f(x) + E^x\left[\int_0^T Af(X_s)\,ds\right].$$

Thus, it follows from the preceding result that

$$\begin{aligned} &f(a)(s(b)-s(x)) + f(b)(s(x)-s(a)) - f(x)(s(b)-s(a)) \\ = \;& (s(b)-s(a))\int_I G_I(x,y)Af(y)m(dy). \end{aligned} \tag{6.1}$$

It can be directly verified that

$$\frac{f(b)-f(x)}{s(b)-s(x)} + \frac{f(x)-f(a)}{s(x)-s(a)} = \int_I H_I(x,y)Af(y)m(dy),$$

where

$$H_I(x,y) = \begin{cases} \frac{s(y)-s(a)}{s(x)-s(a)}, & \text{if } a < y \le x, \\ \frac{s(b)-s(y)}{s(b)-s(x)} & \text{if } x \le y < b, \\ 0, & \text{otherwise.} \end{cases}$$

Note that if $b \downarrow x$, $\frac{s(b)-s(y)}{s(b)-s(x)} \to \mathbf{1}_{\{x\}}$, which implies that the right s-derivative of f exists. Similarly, $\frac{s(y)-s(a)}{s(x)-s(a)} \to \mathbf{1}_{\{x\}}$ as $a \uparrow x$ and thus the left derivative exists as well. Letting a and b tend to x together, we obtain

$$\frac{df^+}{ds}(x) - \frac{df^-}{ds}(x) = 2m(\{x\})Af(x),$$

which yields the first part of the statement.

In order to show the second claim, let h be such that $a < x+h < b$ and apply (6.1) to x and $x+h$ to get

$$\begin{aligned} & f(b) - f(a) - (s(b)-s(a))\frac{f(x+h)-f(x)}{s(x+h)-s(x)} \\ = \quad & (s(b)-s(a)) \int_I \frac{G_I(x+h,y) - G_I(x,y)}{s(x+h)-s(x)} Af(y)m(dy). \end{aligned}$$

Now, suppose that f is s-differentiable at x and let $h \to 0$. It follows from the dominated convergence theorem that

$$\begin{aligned} & f(b) - f(a) - (s(b)-s(a))\frac{df}{ds}(x) \\ = \quad & -\int_a^x (s(y)-s(a))Af(y)m(dy) + \int_x^b (s(b)-s(y))Af(y)m(dy) \\ = \quad & -\int_a^b s(y)Af(y)m(dy) + s(a)\int_a^x Af(y)m(dy) + s(b)\int_x^b Af(y)m(dy). \end{aligned}$$

If x_1 and x_2 are two such points, we obtain the claim by subtracting. $\square$

Exercise 6.3. *If X is a Markov processes satisfying the hypothesis of this section and ϕ is a homeomorphism from $\mathbf{E}$ to $\tilde{\mathbf{E}}$, then $\tilde{X} = \phi(X)$ also satisfies these hypothesis on $\tilde{\mathbf{E}}$. Prove that $\tilde{s} = s \circ \phi^{-1}$ and that $\tilde{m}$ is the image of m under ϕ, i.e $\tilde{m}(\tilde{I}) = m(\phi^{-1}(\tilde{I}))$ for any $\tilde{I} \subset int(\tilde{\mathbf{E}})$.*

Exercise 6.4. *Let X be as in Exercise 6.1. Show that $m(dx) = \frac{2}{s'(x)\sigma^2(x)}dx$. (Hint: First find the speed measure for the process $\tilde{X} = s(X)$ using the relationship between the speed measure and the infinitesimal generator by observing that $\tilde{s}(x) = x$. Then, show that that $m(dx) = \tilde{m}(s(x))s'(x)$ using the previous exercise.*

As mentioned earlier $\tilde{X} = s(X)$ is continuous local martingale, therefore, it is a time-changed Brownian motion, i.e.

$$\tilde{X}_t = \beta_{\gamma_t}$$

where β is a Brownian motion in some enlargement of the probability space. In our setting it is possible to find explicitly the process γ. Theorem V.47.1 in volume 2 of Rogers &

Williams[1] shows that

$$\gamma_t = \inf\left\{u : \int \frac{1}{2}\ell_u^z \tilde{m}(dz) > t\right\},$$

where $(\ell_t^z)_{t\geq 0}$ is the local time process of β at level z, and $\tilde{m}$ is the speed measure of $\tilde{X}$.

The speed measure can also be linked to the resolvent of X and its transition function. To see this we first define

$$\begin{aligned}
\varphi^+(\alpha, x) &:= \begin{cases} E^x[\exp(-\alpha T_c)]; & x \leq c, x \in \mathrm{int}(\mathbf{E}) \\ 1/E^c[\exp(-\alpha T_x)]; & x \geq c, x \in \mathrm{int}(\mathbf{E}) \end{cases} \\
\varphi^-(\alpha, x) &:= \begin{cases} E^x[\exp(-\alpha T_c)]; & x \geq c, x \in \mathrm{int}(\mathbf{E}) \\ 1/E^c[\exp(-\alpha T_x)]; & x \leq c, x \in \mathrm{int}(\mathbf{E}) \end{cases}
\end{aligned}$$

where c is an arbitrary point in $\mathrm{int}(\mathbf{E})$. Consequently, for $a < b$

$$E^a[\exp(-\alpha T_b)] = \frac{\varphi^+(\alpha, a)}{\varphi^+(\alpha, b)}.$$

Proposition 6.4. *$\varphi^+(\alpha, \cdot)$ (resp. $\varphi^-(\alpha, \cdot)$) is continuous and increasing (resp. decreasing). Moreover, φ^+ and φ^-, as functions of x, solve*

$$\frac{d}{dm}\frac{d}{ds} f = \alpha f.$$

Proposition 6.5. *Suppose that X is on natural scale. Then for any x in int$(\mathbf{E})$ and a bounded Borel function f with a support in int$(\mathbf{E})$*

$$U^\alpha f(x) = \int f(y) u_\alpha(x, y) m(dy),$$

where

$$\begin{aligned}
u_\alpha(x, y) &= c_\alpha \varphi^+(\alpha, x)\varphi^-(\alpha, y), \qquad x \leq y \\
&= c_\alpha \varphi^+(\alpha, y)\varphi^-(\alpha, x), \qquad x \geq y, \\
c_\alpha^{-1} &= \varphi^-(\alpha, x) D\varphi^+(\alpha, x-) - \varphi^+(\alpha, x) D\varphi^-(\alpha, x+)
\end{aligned}$$

and D denotes differentiation.

Proposition 6.6. *There exists a jointly continuous map $p(t, x, y)$, which is symmetric in x and y such that for all bounded Borel functions f supported in int$(\mathbf{E})$*

$$E^x[f(X_t)] = \int f(y) p(t, x, y) m(dy),$$

whenever $x \in$ int$(\mathbf{E})$.

Exercise 6.5. *Let X be as in Exercise 6.1, $I =]l, r[$ and suppose $X_0 = x \in$ int(I). We distinguish four cases:*

(1) $s(l+) = -\infty, s(r-) = \infty$. *Then,*

$$P^x(\sigma_I = \infty) = P^x\left(\sup_{0\leq t<\infty} X_t = r\right) = P^x\left(\inf_{0\leq t<\infty} X_t = l\right) = 1.$$

[1] Observe that their definition of the speed measure results in a measure which is only half of the one according to our definition.

In particular, the process X is recurrent, i.e. for any $y \in I$, we have

$$P^x(X_t = y;\ \textit{for some} t \in [0,\infty)) = 1.$$

(2) $s(l+) > -\infty, s(r-) = \infty$. *Then,*

$$P^x\left(\lim_{t\to\sigma_I} X_t = l\right) = P^x\left(\sup_{t<\sigma_I} X_t < r\right) = 1.$$

(3) $s(l+) = -\infty, s(r-) < \infty$. *Then,*

$$P^x\left(\lim_{t\to\sigma_I} X_t = r\right) = P^x\left(\inf_{t<\sigma_I} X_t > l\right) = 1.$$

(4) $s(l+) > -\infty, s(r-) < \infty$. *Then,*

$$P^x\left(\lim_{t\to\sigma_I} X_t = l\right) = 1 - P^x\left(\lim_{t\to\sigma_I} X_t = r\right) = \frac{s(r-) - s(x)}{s(r-) - s(l+)}.$$

*(*In parts 2,3, and 4 there is no claim that σ_I is finite. Note that in this exercise we do not necessarily assume that I is bounded.*)*

Remark 1. *The scale function s plays a crucial role in determining the explosion properties of one-dimensional diffusions. See Lemma 5.5.26 and Theorem 5.5.29 (Feller's test of explosions) in Karatzas & Shreve for more details.*

Remark 2. *In Mathematical Finance Theory one of the crucial questions is whether the stochastic exponential of a local martingale is a true martingale. When the underlying local martingale is strong Markov and continuous, the speed measure tells us whether its stochastic exponential is a true martingale or not. For details you can consult the following recent works:*

(1) *S. Blei and H-J. Engelbert.* On exponential local martingales associated with strong Markov continuous local martingales*. Stochastic Proc. Appl., 119 (2009), pp. 2859–2880.*
(2) *A. Mijatovic and M. Urusov.* On the martingale property of certain local martingales*. To appear in Prob. Th. Rel. Fields.*

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