SOLUTIONS TO PROBLEM SET 1

1. Let $f : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ be any bounded measurable function. In order to show (X, S) is Markovian, it suffices to show

$$E[f(X_u, S_u) \,|\, \mathcal{F}_t] = E[f(X_u, S_u) \,|\, X_t, S_t], \qquad \text{for any } u \geq t.$$

To this end, note that $X_u = X_t + (X_u - X_t)$ and $S_u = \max(S_t, \sup_{t \leq s \leq u} X_s)$. Moreover, $X_u - X_t$ and $\sup_{t \leq s \leq u} X_s$ are $\mathcal{F}'_t$ measurable. Then

$$\begin{aligned} E[f(X_u, S_u) \,|\, \mathcal{F}_t] &= E\left[f(X_t + (X_u - X_t), \max(S_t, \sup_{t \leq s \leq u} X_s) \,|\, \mathcal{F}_t \right] \\ &= E\left[f(X_t + (X_u - X_t), \max(S_t, \sup_{t \leq s \leq u} X_s) \,|\, X_t, S_t \right]. \end{aligned}$$

Here, given S_t, $f(X_t + (X_u - X_t), \max(S_t, \sup_{t \leq s \leq u} X_u))$ is $\mathcal{F}'_t$ measurable. Hence the second identity follows from Proposition 1.1.1 i) in the note and the Markov property of X. Therefore, (X, S) is Markovian.

As for the transition density, we need to find $P(X_u \leq a, S_u \geq b \,|\, X_t = x, S_t = y)$ with $b \geq y, a \leq b$, and $x \leq y$. Let us first consider $b > y$ case first. To this end,

$$\begin{aligned} P(X_u \leq a, S_u \geq b \,|\, X_t = x, S_t = y) &= P(X_u \leq a, \sup_{0 \leq s \leq u} X_s \geq b \,|\, X_t = x, S_t = y) \\ &= P(X_u \leq a, \sup_{t \leq s \leq u} X_s \geq b \,|\, X_t = x) \\ &= P^x(X_{u-t} \leq a, T_b \leq u - t) \\ &= P^0(X_{u-t} \leq a - x, T_{b-x} \leq u - t) \\ &= P^0(X_{u-t} \geq 2b - a - x) \\ &= \frac{1}{\sqrt{2\pi(u-t)}} \int_{2b-a-x}^{+\infty} e^{-\frac{z^2}{2(u-t)}} \, dz, \end{aligned}$$

where P^x is the probability measure generated by X with $X_0 = x$, $T_y = \inf\{t \geq 0; X_t = y\}$, the second equality is due to the Markov property of X since $\sup_{t \leq s \leq u} X_s$ is $\mathcal{F}'_t$ measurable, and the second last equality follows from the reflection principle. Then differentiation yields

$$p_{t,u}(x, y; da, db) = \frac{2(2b - a - x)}{\sqrt{2\pi(u-t)^3}} \exp\left(-\frac{(2b - a - x)^2}{2(u-t)} \right) da \, db.$$

When $b = y$, we have

$$\begin{aligned} P(X_u \le a, S_u = y \,|\, X_t = x, S_t = y) &= P(X_u \le a, \sup_{t \le s \le u} X_s \le y \,|\, X_t = x) \\ &= P^x(X_{u-t} \le a, T_y \ge u - t) \\ &= P^0(X_{u-t} \le a - x) - P^0(X_{u-t} \le a - x, T_{y-x} \le u - t) \\ &= P^0(X_{u-t} \le a - x) - P^0(X_{u-t} \ge 2y - a - x) \\ &= \frac{1}{\sqrt{2\pi(u-t)}} \int_{2y-a-x}^{a-x} e^{-\frac{z^2}{2(u-t)}} dz. \end{aligned}$$

Taking differentiation with respect to a gives the transition density

$$p_{t,u}(x, y; da, y) = \frac{1}{\sqrt{2\pi(u-t)}} \left(e^{-\frac{(a-x)^2}{2(u-t)}} - e^{-\frac{(2y-a-x)^2}{2(u-t)}} \right) da = \epsilon_{\{y\}}(b) q_t(y - a, y - x) da,$$

where ϵ is the point mass and q is the densit of Brownian motion killed at 0 (see Example 1.4 in the note). In this case, $y - X$ becomes a Brownian motion killed at 0.

2. The proof is similar to that in 1. noticing that $A_u = A_t + \int_t^u f(X_s) ds$ where $\int_t^u f(X_s) ds$ is $\mathcal{F}_t'$ measurable.
3. Note that $X_u = x + N_u = x + N_t + (N_u - N_t) = X_t + (N_u - N_t)$ where $(N_u - N_t)$ is independent of $\mathcal{F}_t$. Then for any bounded measurable function f,

$$E[f(X_u) \,|\, \mathcal{F}_t] = E[f(X_t + (N_u - N_t)) \,|\, \mathcal{F}_t] = g(X_t) = E[f(X_t + (N_u - N_t)) \,|\, X_t],$$

where $g(x) = E[f(x + N_u - N_t)]$ and the second identity follows from the independence between $N_u - N_t$ and $\mathcal{F}_t$. Hence $(X_t, \mathcal{F}_t)$ is Markovian.

Since N_t has Poisson distribution with parameter λt, the transition probabilities then read

$$p_t(x, y) = \begin{cases} e^{-\lambda t} \frac{(\lambda t)^{y-x}}{(y-x)!}, & \text{if } y \ge x, \\ 0, & \text{otherwise} \end{cases}.$$

4. The Chapman-Kolmogorov equation in this case reads

$$P_{s,u}(x, A) = \int_E P_{s,t}(x, dy) P_{t,u}(y, A),$$

where $P_{s,t}(x, dy) = p_{t-s}(x, y) m(dy)$. Then the right side of the previous equation reads

$$\begin{aligned} \int_E P_{s,t}(x, dy) P_{t,u}(y, A) &= \int_E \int_A p_{t-s}(x, y) p_{u-t}(y, z) m(dz) m(dy) \\ &= \int_A \int_E p_{t-s}(x, y) p_{u-t}(y, z) m(dy) m(dz) \\ &= \int_A p_{u-s}(x, z) m(dz) = P_{s,u}(x, A), \end{aligned}$$

where the second identity follows from the Fubini theorem, the third identity holds if and only if

$$\int_E p_s(x, y) p_t(y, z) m(dy) = p_{s+t}(x, z).$$

5. A process X is Gaussian if and only if for any finite set of time indices $t_1, \cdots, t_k$, $(X_{t_1}, \cdots, X_{t_k})$ is a multivariate Gaussian random variable. For $s > t$, from $E^x X_t = e^{-t/2} x$ and $Cov(X_s, X_t) = e^{-(s+t)/2}(e^{s \wedge t} - 1)$, we have

$$E^{X_t} X_s = e^{-(s-t)/2} X_t, \quad Cov(X_s, X_u \,|\, X_t) = 0, \, \forall s \leq t \leq u \quad \text{and} \quad Var(X_s \,|\, X_t) = 1 - e^{-(s-t)}.$$

Since the distribution of a Guassian random variable is determined by its mean and covariance matrix, then the previous calculations yield that the conditional law of X_s only depends on X_t. Hence X is Markovian.

The transition density then reads

$$p_t(x, y) = \frac{1}{\sqrt{2\pi(1 - e^{-t})}} \exp\left(-\frac{(y - e^{-t/2}x)^2}{2\pi(1 - e^{-t})} \right).$$