SOLUTIONS TO PROBLEM SET 2

1. Since $\mathcal{G}_t \subset \mathcal{F}_t$, it follows from the tower property that, for any bounded measurable function f,

$$E[f(X_s)\,|\,\mathcal{G}_t] = E\left[E[f(X_s)\,|\,\mathcal{F}_t]\,|\,\mathcal{G}_t\right] = E\left[E[f(X_s)\,|\,X_t]\,|\,\mathcal{G}_t\right] = E[f(X_s)\,|\,X_t],$$

where the second identity follows from the Markov property of $(X_t, \mathcal{F}_t)$, the third identity holds since X_t is $\mathcal{G}_t$-measurable.

2. It follows from the Markov property of $(X_t, \mathcal{F}_t)$ that

$$M_t = P_{t_0-t}f(X_t) = E^{X_t}[f(X_{t_0})] = E^x[f(X_{t_0})\,|\,\mathcal{F}_t].$$

Hence M is a P^x-martingale.

3. For any $s < t$,

$$\begin{aligned} P^x(\sigma_x > t) &= P^x(\sigma_x > s)P(\sigma_x > t\,|\,\sigma_x > s) = P^x(\sigma_x > s)P(\sigma_x > t\,|\,X_s = x) \\ &= P^x(\sigma_x > s)P^x(\sigma_x > t-s), \end{aligned}$$

where $X_s = x$ when $\sigma_x > s$ and the Markov property of $(X_t, \mathcal{F}_t)$ are used to obtain the second identity. The previous identity yields that $f(t) := P^x(\sigma_x > t)$ satisfies the following equation

$$f(t) = f(s)f(t-s), \qquad \text{for any } s < t,$$

impying f is an exponentail function. Thus,

$$P^x(\sigma_x > t) = e^{-\alpha t}, \qquad \text{for some } \alpha \in [0, \infty].$$

When $a = 0$, X stays at x forever. Such point x is an *absorbing* point. When $a = \infty$, X leaves x immediately. Such point X is then *regular*.

4. From the definition of resolvent

$$pU^p f(x) = \int_0^\infty pe^{-pt}P_t f(x)dt = E^x\left[\int_0^\infty pe^{-pt} f(X_t)\,dt\right] = E^x[f(X_{e_p})], \tag{1}$$

where the second identity follows from Fubini's theorem. On the other hand,

$$\begin{aligned} pqU^pU^q f(x) &= pU^p E^x[f(X_{e_q})] = E^x\left[E^{X_{e_p}}[f(X_{e_q})]\right] \\ &= E^x\left[E^x\left[f(X_{e_q}\circ\theta_{e_p})\,|\,\mathcal{F}_{e_p}\right]\right] = E^x[f(X_{e_q+e_p})], \end{aligned}$$

where both second and third identities utilise (1).

Let us now prove the resolvent equation

$$U^p f(x) - U^q f(x) = (q-p)U^pU^q f(x). \tag{2}$$

First,

$$U^pU^qf(x) = \int_0^\infty e^{-pt}P_t\left(\int_0^\infty e^{-ps}P_sf\,ds\right)(x)\,dt = \int_0^\infty e^{-pt}\int_0^\infty P_tP_sf(x)\,dsdt$$
$$= \int_0^\infty e^{-pt}\int_0^\infty e^{-ps}P_{s+t}f(x)\,dsdt.$$

On the other hand,

$$U^pf(x) - U^qf(x) = \int_0^\infty (e^{-pt} - e^{-qt})P_tf(x)\,dt = \int_0^\infty e^{-pt}(1 - e^{-(q-p)t})P_tf(x)\,dt$$
$$= (q-p)\int_0^\infty e^{-pt}\frac{1-e^{-(q-p)t}}{q-p}P_tf(x)\,dt = (q-p)\int_0^\infty e^{-pt}\int_0^t e^{-(q-p)s}dsP_tf(s)dt$$
$$= (q-p)\int_0^\infty ds\,e^{-(q-p)s}\int_s^\infty dt\,e^{-pt}P_tf(x) = (q-p)\int_0^\infty ds\,e^{-qs}\int_s^\infty dt\,e^{-p(t-s)}P_tf(x)$$
$$= (q-p)\int_0^\infty ds\,e^{-qs}\int_0^\infty dv\,e^{-pv}P_{v+s}f(x),$$

where we change the order of integrals to obtain the fifth identity and introduce new variable $v = t - s$ to get the last identity. Therefore, the resolvent equation (2) is confirmed.

5. We prove this for Brownian motion, the same technique works for Poisson process as well. Recall that the transition density of Brownian motion is homogeneous in space and time. Let $p(t, x-y)$ be this density. Thus, for $f \in \mathbb{C}$ and $x \neq \Delta$

$$P_tf(x) = \int_{\mathbb{R}} f(y)p(t, y-x)dy = \int_{\mathbb{R}} f(y+x)p(t,y)dy,$$

and $P_tf(\Delta) = f(\Delta)$. Since f is bounded, the desired continuity property of P_tf is a direct consequence of bounded convergence theorem. $\|P_tf - f\| \to 0$ follows from the fact that the density $p(t, y)$ converging to the delta function at 0 weakly as $t \to 0$.

6. Let T be an exponential random variable with parameter 1. Consider

$$X_t = (t - T)_+, \quad t \in \mathbb{R}_+.$$

This X is a Markov process. Its semi-group P_t can be computed as follows:

$$P_tf(x) = E[f(X_{s+t}) \,|\, X_s = x].$$

If $x > 0$, then $X_{s+t} = x + t$. If $x = 0$, X has not left 0 yet, hence $T > s$. Because of the memoryless property of T, we get

$$P_tf(x) = \begin{cases} f(x+t) & \text{if } x > 0, \\ e^{-t}f(0) + \int_0^t du\,e^{-u}f(t-u) & \text{if } x = 0. \end{cases}$$

Therefore X is a Markov process. But X is not strong Markov. Then $X_T = X_0 = 0$. If X were strong Markov, the future after T would have the same law as the future at $t = 0$. But this is not the case, future at $t = 0$ starts with an exponential delay, whereas the future at T is that X starts immediate motion after T.

Another example:

Let $(B_t, t \geq 0)$ be a Brownian motion not necessarily starting from 0. Consider

$$X_t = B_t \, \mathbb{I}_{\{B_0 \neq 0\}} = \begin{cases} B_t, & \text{if } B_0 \neq 0 \\ 0, & \text{if } B_0 = 0 \end{cases}$$

Its transition density is

$$p_t(x, dy) = \begin{cases} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}} dy, & \text{if } x \neq 0 \\ \delta_0 dy, & \text{if } x = 0 \end{cases}$$

The process X is a Markov process because for any Borel set B,

$$\begin{aligned}
&E[1_B(X_{t+s}) \,|\, \mathcal{F}_s] \\
&= E[1_B(X_{t+s}) \cdot 1_{B_0 \neq 0} \,|\, \mathcal{F}_s] + E[1_B(X_{t+s}) \cdot 1_{B_0 = 0} \,|\, \mathcal{F}_s] \\
&= 1_{B_0 \neq 0} \cdot \int_B \frac{1}{\sqrt{2\pi t}} e^{-\frac{X_s - y^2}{\sqrt{2t}}} dy + 1_{B_0 = 0} \cdot 1_B(0) \\
&= 1_{X_0 \neq 0} \cdot \int_B \frac{1}{\sqrt{2\pi t}} e^{-\frac{(X_s - y)^2}{2t}} dy + 1_{B_0 \neq 0, X_s = 0} \cdot \int_B \frac{1}{\sqrt{2\pi t}} e^{-\frac{(X_s - y)^2}{2t}} dy + (1_{X_s = 0} - 1_{B_0 \neq 0, X_s = 0}) \cdot 1_B(0) \\
&= 1_{X_s \neq 0} \cdot \int_B \frac{1}{\sqrt{2\pi t}} e^{-\frac{(X_s - y)^2}{2t}} dy + 1_{X_s = 0} \cdot 1_B(X_s) + 1_{B_0 \neq 0, X_s = 0} \cdot \left(\int_B \frac{1}{\sqrt{2\pi t}} e^{-\frac{(X_s - y)^2}{2t}} dy - 1_B(0) \right) \\
&= E[1_B(X_s)].
\end{aligned}$$

Here the third identity uses $\{B_0 \neq 0\} = \{B_0 \neq 0, X_0 \neq 0\} \cup \{X_0 = 0, B_0 \neq 0\} = \{X_0 \neq 0\} \cup \{X_0 = 0, B_0 \neq 0\}$, $\{X_s = 0\} = \{B_0 \neq 0, X_s = 0\} \cup \{B_0 = 0, X_s = 0\} = \{B_0 \neq 0, X_s = 0\} \cup \{B_0 = 0\}$; the fifth identity follows from $1_{B_0 \neq 0, X_s = 0} = 0$. Therefore X is a Markov process.

However, X is not strong Markov. Indeed, consider $\tau = \inf\{t > 0, X_t = 0\}$. Then for any $x > 0$, since $P_x(X_1 = 0) = 0$,

$$0 \leq P_x(\tau \leq 1) = P_x(X_1 \neq 0, \tau \leq 1).$$

However, if X were a strong Markov process,

$$P_x(X_1 \neq 0, \tau \leq 1) = P_{X_\tau}(X_1 \neq 0) = 0,$$

which is a contradiction.

7. We only need to verify Chapman-Kolmogorov equation for the first part, which follows from Fubini's theorem.

Now suppose X is a Feller process with such transition function. For any $f \in \mathbb{C}$ and t,

$$E^x[f(X_t - X_0)] = E^x[f(X_t - x)] = \int_{\mathbb{R}^d} f(x + y - x) \mu_t(dy) = \int_{\mathbb{R}^d} f(y) \mu_t(dy) = \mu_t(f).$$

Hence the distribution of $X_t - X_0$ does not depend on X_0, which implies X has independent increments. Therefore, by the Markov property, for $s < t$,

$$E^v \left[f(X_t - X_s) \,|\, \mathcal{F}_s \right] = \mathbb{E}^{X_s} \left[f(X_t - X_s) \right] = \mathbb{E}^{X_s} \left[f(X_{t-s} - X_0) \right] = \mu_{t-s}(f),$$

where the second identity uses the identity four lines above. Hence X has stationary increments.

On the other hand, if X has independent stationary increments, for any $A \in \mathcal{F}$,

$$\mu_{t+s}(A) = P(X_{t+s} - X_0 \in A) = P(X_{t+s} - X_t + X_t - X_0) \in A) = \mu_s * \mu_t(A).$$

To find the semigroup, note that

$$P_t f(x) = E^x[f(X_t - X_0 + x)] = \int_{\mathbb{R}^d} f(x+y)\mu_t(dx),$$

using the distribution of $X_t - X_0$ defined above.

8. Since the Brownian will reach any level a in a finite time, $X_{T_a} = a$. For $b > a$, the strong Markov property yields that

$$P(T_b - T_a \in dt \,|\, \mathcal{F}_{T_a}) = P^{X_{T_a}}(T_b - T_a \in dt) = P^a(T_b - T_a \in dt) = P^0(T_{b-a} \in dt).$$

Therefore, the distribution of $T_b - T_a$ is independent of a and only depends on $b - a$. Hence $(T_a)_{a>0}$ has independent and stationary increments.

9. This follows from *Blumental 0-1 Law* since

$$[T_A = 0] = \cap_{n=1}^{\infty}[T_A < \frac{1}{n}] \in \tilde{\mathcal{F}}_0.$$

10. When $a \in (0, \infty)$, we have $P^x(\sigma_x < \infty) = 1$. Observe that on $[X_{\sigma_x} = x]$, $\sigma_x \circ \theta_{\sigma_x} = 0$. It then follows from the strong Markov property that

$$\begin{aligned} P^x[\sigma_x < \infty, X_{\sigma_x} = x] &= P^x[\sigma < \infty, X_{\sigma_x} = x, \sigma_x \circ \theta_{\sigma_x} = 0] \\ &= \mathbb{E}^x\left[1_{[\sigma_x < \infty, X_{\sigma_x} = x]} P_{X_{\sigma_x}}[\sigma_x = 0]\right] \\ &= P^x[\sigma_x = 0] P^x[\sigma_x < \infty, X_{\sigma_x} = x]. \end{aligned}$$

Therefore, if $P^x[X_{\sigma_x} = x] > 0$, we must have $P^x[\sigma_x = 0] = 1$, which contradicts with $P^x[\sigma_x > 0] = 1$.