SOLUTIONS TO PROBLEM SET 3

The following treatment follows from Chapter 23 of "Stochastic Processes" by Richard Bass.

1. Let $g_n = n(f - P_{1/n}f)$. Since f is excessive, i.e. $f \geq P_t f$ for any $t \geq 0$, then $g_n \geq 0$. We have
$$Ug_n = n\int_0^\infty P_s f\, ds - n\int_0^\infty P_{s+1/n} f\, ds = n\int_0^{1/n} P_s f\, ds.$$
Since f is excessive, the right side of the previous identity is less than f for each n. Recall that X is Feller, hence $\lim_{s\downarrow 0} P_s f = f$. Therefore $\lim_{n\to\infty} n\int_0^{1/n} P_s f\, ds = f$.
2. Let us first understand the set of time $T_n = \{k2^{-n} : 0 \leq k \leq n2^n\}$. For each n, the time interval $[0, n]$ is divided into $n2^n$ sub-intervals. Then T_n contains all time on grid points.

   Let us now prove g_n is increasing. Observe that $g_n(x) \geq P_0 g_{n-1}(x) = E^x[g_{n-1}(X_0)] = g_{n-1}(x)$. Then g is increasing.

   Now define $H(x) := \lim_{n\to\infty} g_n(x)$. Since the limit of an increasing sequence of continuous functions is lower semi-continuous[1], it suffices to show each g_n is continuous function. When $n = 1$, $g_1(x) = \max_{t\in\{0,1/2,1\}} P_t g(x)$. Since X is Feller, $P_t g(x)$ is then continuous, and the maximum of finite continuous functions is continuous. Therefore g_1 is continuous. The rest proof follows from induction in n.
3. Let us first fix m and take $t \in T_m$. Then for $n \geq m$,
$$H(x) \geq g_n(x) \geq P_t g_{n-1}(x) = E^x g_{n-1}(X_t).$$
Sending $n \uparrow \infty$ and using the monotone convergence theorem, we obtain
$$H(x) \geq E^x H(X_t) \quad \text{for} \quad t \in T_m.$$
Moreover the previous inequality also holds when $t \in \cup_m T_m$. Now for an arbitrary t, there exists a sequence $\cup_m T_m \ni t_k \to t$. Then Fatou's lemma yields
$$H(x) \geq \liminf_{k\to\infty} E^x H(X_{t_k}) \geq E^x[\liminf_{k\to\infty} H(X_{t_k})] \geq E^x H(X_t),$$
where the last inequality utilises $\liminf_{k\to\infty} H(X_{t_k}) \geq H(X_t)$ since H is lower semi-continuous from Problem 2.

   It then remains to show $\lim_{t\downarrow 0} P_t H(x) = H(x)$. For $a \in \mathbb{R}$, let $E_a = \{y : H(y) > a\}$, which is an open set. If $a \leq H(x)$, then
$$P_t H(x) = E^x H(X_t) \geq aP^x(X_t \in E_a) \to a \quad \text{as} \quad t \downarrow 0.$$
Therefore $\liminf_{t\downarrow 0} P_t H(x) \geq a$ for any $a < H(x)$. This implies that
$$\liminf_{t\downarrow 0} P_t H(x) \geq H(x).$$

[1] The function f is *lower semi-continuous* if $\liminf_{x\to x_0} f(x) \geq f(x_0)$ for any $x_0 \in \mathbb{R}$.

On the other hand, we have already show $H(x) \geq E^x H(X_t)$, hence $H(x) \geq \limsup_{t\downarrow 0} P_t H(x)$. Therefore we confirmed $\lim_{t\downarrow 0} P_t H(x) = H(x)$. Thus H is excessive.

We have shown that H is excessive and H dominates g by its construction. Therefore H is an excessive majorant of g. Let us show H is the least one. Take an arbitrary excessive function F such that $F \geq g$. If $F \geq g_{n-1}$, then $F(x) \geq P_t F(x) \geq P_t g_{n-1}(x)$ for every $t \in T_n$, hence $F(x) \geq g_n(x)$. By an induction argument, $F(x) \geq g_n(x)$ for all n, hence $F(x) \geq H(x)$. Therefore H is the least excessive majorant of g.

4. Let G be the least excessive majorant of g. We are going to show that G is the value of the optimal stopping problem, and one optimal stopping time is the first time that process X hits the set $\{x : g(x) = G(x)\}$.

   In order to prove this result, let us prepare the following lemma:

Lemma 1. (a) *If f is excessive, T is a finite stopping time, and $h(x) = E^x f(X_T)$, then h is excessive.*

(b) *If f is excessive and T is a finite stopping time, then $f(x) \geq E^x f(X_T)$.*

(c) *If f is excessive, then $f(X_t)$ is a supermartingale.*

Proof. (a) First suppose $f = Ug$ for some nonnegative g. Then

(1)
$$\begin{aligned} h(x) = E^x Ug(X_T) &= E^x E^{X_T} \int_0^\infty g(X_s)\, ds \\ &= E^x \int_0^\infty g(X_{s+T})\, ds = E^x \int_T^\infty g(X_s)\, ds \end{aligned}$$

by the strong Markov property and a change of variables. The same argument shows that

$$P_t h(x) = E^x h(X_t) = E^x E^{X_t} \int_T^\infty g(X_s)\, ds = E^x \int_{T+t}^\infty g(X_s)\, ds.$$

This is less than $E^x \int_T^\infty g(X_s)\, ds = h(x)$ and increases up to $h(x)$ as $t \downarrow 0$.

Now let f be excessive but not necessarily of the form Ug. In the paragraph above, replace g by g_n in Problem 1 to conclude

$$P_t h(x) = \lim_{n\to\infty} E^x \int_{T+t}^\infty g_n(X_s)\, ds \leq \lim_{n\to\infty} E^x \int_T^\infty g_n(X_s)\, ds = h(x).$$

That $P_t h$ increases up to h is proved similarly.

(b) When T is a real number, this has been proved in Proposition 2.1.2 in the lecture note. Let us also prove it here. As in the proof of (1), it suffices to consider the case where $f = Ug$ and then take limits. It follows from (1) that

$$E^x Ug(X_T) = E^x \int_T^\infty g(X_s)\, ds \leq E^x \int_0^\infty g(X_s)\, ds = Ug(x).$$

(c) This can be proved using Proposition 2.1.3 in the lecture note. We also give a direct argument here. By the Markov property,

$$E^x[f(X_t) \mid \mathcal{F}_t] = E^x f(X_{t-s}) = P_{t-s} f(X_s) \leq f(X_s).$$

□

Let us now come back to our optimal stopping problem. The proof of Problem 4 is split into the following three steps.

Step 1. Let $D_\epsilon = \{x : g(x) < G(x) - \epsilon\}$ and $\tau_\epsilon = \inf\{t : X_t \notin D_\epsilon\}$. Define

$$H_\epsilon = E^x[G(X_{\tau_\epsilon})].$$

It then follows from Lemma 1 (a) that H_ϵ is excessive.

In this step, we show

(2) $$g(x) \le H_\epsilon(x) + \epsilon, \quad x \in D.$$

To prove this, we suppose not, that is, we let

$$b = \sup_{x \in D}(g(x) - H_\epsilon(x))$$

and suppose $b > \epsilon$. Choose $\eta < \epsilon$, ad then choose x_0 such that

$$g(x_0) - H_\epsilon(x_0) \ge b - \eta.$$

Since $H_\epsilon + b$ is an excessive majorant of g be the definition of b, and G is the least excessive majorant, then

$$G(x_0) \le H_\epsilon(x_0) + b.$$

From the previous two inequalities, we conclude

(3) $$G(x_0) \le g(x_0) + \eta.$$

By the Blumenthal 0-1 Law (HW2, Problem 9), $P^{x_0}(\tau_\epsilon = 0)$ is either 0 or 1. In the first case, for any $t > 0$,

$$g(x_0) + \eta \ge G(x_0) \ge E^{x_0}[G(X_{t \wedge \tau_\epsilon})] \ge E^{x_0}[g(X_{t \wedge \tau_\epsilon} + \epsilon); \tau_\epsilon > t],$$

where the first inequality is (3), the second is due to G being excessive, and the third because $G > g + \epsilon$ until τ_ϵ. Sending $t \downarrow 0$ and use the fact that g is continuous, we get $g(x_0) + \eta \ge g(x_0) + \epsilon$, which contradicts with the choice of η.

In the second case, where $\tau^\epsilon = 0$ with probability 1, we have

$$H_\epsilon(x_0) = E^{x_0}G(X_{\tau_\epsilon}) = G(x_0) \ge g(x_0) \ge H_\epsilon(x_0) + b - \eta,$$

a contradiction since we choose $\eta < b$.

In either cases, we must have (2) hold.

Step 2. A conclusion we reach from (2) is that $H_\epsilon + \epsilon$ an excessive majorant of g. Therefore,

(4) $$G(x) \le H_\epsilon(x) + \epsilon = E^x[G(X_{X_{\tau_\epsilon}})] + \epsilon \le E^x[g(X_{\tau_\epsilon}) + \epsilon] + \epsilon \le g^*(x) + 2\epsilon.$$

The first inequality holds since G is the least excessive majorant, the second inequality follows from $g(X_{\tau_\epsilon}) + \epsilon \ge G(X_{\tau_\epsilon})$ by the definition of τ_ϵ, and the third b the definition of g^*. Since ϵ is arbitrarily chosen, we see that $G(x) \le g^*(x)$.

Step 3. For any stopping time T, because G is excessive and majorizes g,

$$G(x) \ge E^x[G(X_T)] \ge E^x g(X_T).$$

Taking the supremum over all stopping times T, $G(x) \geq g^*(x)$ and therefore $G(x) = g^*(x)$.

Step 4. Because τ_D is finite almost surely, the continuity of g tells us that $E^x g(X_{\tau_\epsilon}) \to E^x[g(X_{\tau_D})]$ as $\epsilon \downarrow 0$. This is due to the fact that a strong Markov process does not jump at stopping times which can be approximated from below by an increasing sequence of stopping times. By the definition of g^*, we know that $E^x g(X_{\tau_D}) \leq g^*(x)$.

On the other hand, by the definitions of τ_ϵ and H_ϵ,

$$E^x g(X_{\tau_\epsilon}) \geq E^x G(X_{\tau_\epsilon}) - \epsilon = H_\epsilon(x) - \epsilon.$$

By the first inequality in (4), the right-hand side is greater than or equal to $G(x) - 2\epsilon = g^*(x) - 2\epsilon$. Sending $\epsilon \downarrow 0$, we obtain

$$E^x g(X_{\tau_D} \geq g^*(x))$$

as desired.

5. Let G be the least excessive majorant of g. Then $h(x) \geq G$. However,

$$h(x) = E^x g(X_{\tau_A}) \leq \sup_T E^x[g(X_T)] = g^*(x) = G(x),$$

by results in Problem 4.

6. Let F be the smallest concave function which dominates g. By Jensen's inequality

$$P_t F(x) = E^x F(X_t) \leq F(E^x[X_t]) = F(x).$$

Because F is concave, it is continuous, and so $P_t F(x) = E^x F(X_t) \to F(x)$ as $t \to 0$. Therefore F is excessive. If $\tilde{F}$ is another excessive function larger than g and $a \leq c < x < d \leq h$, we have $\tilde{F}(x) \geq E^x \tilde{F}(X_S)$, where S is the first time that X leaves $[c, d]$ by Lemma 1. Since X is a Brownian motion, we have

$$\tilde{F}(x) \geq E^x \tilde{F}(X_S) = \frac{d-x}{d-c}\tilde{F}(c) + \frac{x-c}{d-c}\tilde{F}(d).$$

Rearranging this inequality shows that $\tilde{F}$ is concave. Recall that the minimum of two concave functions is concave, so $F \wedge \tilde{F}$ is a concave function larger than g. But F is the smallest concave function dominating g, hence $F = F \wedge \tilde{F}$, or $F \leq \tilde{F}$. Therefore F is the least excessive majorant of g. It then follows from results in Problem 4 that $F = g^*$.