SOLUTIONS TO PROBLEM SET 4

1. Without loss of generality suppose $f \in \mathbb{C}_0^2$. First, note that

$$\int_0^\infty e^{-\alpha t}\frac{1}{\sqrt{2\pi t}}e^{-\frac{(x-y)^2}{2t}}dt = \frac{1}{\sqrt{2\alpha}}e^{-\sqrt{2\alpha}|x-y|}.$$

Thus,

$$\begin{aligned} U^\alpha f(x) &= \int_{-\infty}^\infty f(y)\frac{1}{\sqrt{2\alpha}}e^{-\sqrt{2\alpha}|x-y|}dy \\ &= \int_{-\infty}^x f(y)\frac{1}{\sqrt{2\alpha}}e^{-\sqrt{2\alpha}(x-y)}dy \qquad (1) \\ &\quad + \int_x^\infty f(y)\frac{1}{\sqrt{2\alpha}}e^{\sqrt{2\alpha}(x-y)}dy. \qquad (2) \end{aligned}$$

In order to find the derivative of the first integral with respect to x, note that for $h > 0$:

$$\begin{aligned} &\int_{-\infty}^{x+h} f(y)\frac{1}{\sqrt{2\alpha}}e^{-\sqrt{2\alpha}(x+h-y)}dy - \int_{-\infty}^{x} f(y)\frac{1}{\sqrt{2\alpha}}e^{-\sqrt{2\alpha}(x-y)}dy \\ = &\int_{x}^{x+h} f(y)\frac{1}{\sqrt{2\alpha}}e^{-\sqrt{2\alpha}(x-y)} + \left\{e^{-\sqrt{2\alpha}h} - 1\right\}\int_{-\infty}^{x+h} f(y)\frac{e^{-\sqrt{2\alpha}(x-y)}}{\sqrt{2\alpha}}dy \end{aligned}$$

The first integral on the right hand side divided by h converges to

$$f(x)\frac{1}{\sqrt{2\alpha}}$$

by the fundamental theorem of calculus. The second term divided by h converges to

$$-\int_{-\infty}^x f(y)e^{-\sqrt{2\alpha}(x-y)}dy$$

by the continuity of the integral with respect to dy. Taking $h < 0$ and repeating similar steps ends up with the same limit. Therefore, the derivative of (1) equals

$$f(x)\frac{1}{\sqrt{2\alpha}} - \int_{-\infty}^x f(y)e^{-\sqrt{2\alpha}(x-y)}dy.$$

Similarly, the derivative of (2) equals

$$-f(x)\frac{1}{\sqrt{2\alpha}} + \int_x^\infty f(y)e^{\sqrt{2\alpha}(x-y)}dy.$$

Hence, $U^\alpha f$ is differentiable and its derivative is given by

$$-\int_{-\infty}^x f(y)e^{-\sqrt{2\alpha}(x-y)}dy + \int_x^\infty f(y)e^{\sqrt{2\alpha}(x-y)}dy.$$

After a change of variable the above can be rewritten as

$$\int_0^\infty \{f(x+u) - f(x-u)\}\, e^{-\sqrt{2\alpha}}du.$$

Since f is bounded and continuous, and vanishing at infinity, the Dominated Convergence Theorem implies the derivative of $U^\alpha f \in \mathbb{C}_0$.

Repeating the above arguments yields

$$\frac{d^2}{dx^2}U^\alpha f(x) = -2f(x)+\int_{-\infty}^{x}\sqrt{2\alpha}f(y)e^{-\sqrt{2\alpha}(x-y)}dy+\int_{x}^{\infty}f(y)\sqrt{2\alpha}e^{\sqrt{2\alpha}(x-y)}dy = \int_{-\infty}^{\infty}\sqrt{2\alpha}f(y)e^{-\sqrt{2\alpha}|x-y|}dy.$$

Hence,

$$\frac{1}{2}\frac{d^2}{dx^2}U^\alpha f(x) = \alpha U^\alpha f - f.$$

2. To show the Markov property note that

$$P(|B_t| \in dy|\mathcal{F}_s) = P(|B_t| \in dy|B_s).$$

Then, for all $x \in \mathbb{R}$

$$P(|B_t| \in dy|B_s = x) = p(t-s,x,y)dy + p(t-s,x,-y)dy = p(t-s,|x|,y)dy + p(t-s,|x|,-y)dy$$

where p is the transition density of standard Brownian motion. This shows that the conditional distribution only depends on B_s, hence the Markov property. If we denote the transition density of X by q, then we have

$$\begin{aligned} q(t,x,dy) &= p(t,x,dy) + p(t,x,-dy) \\ &= \frac{1}{\sqrt{2\pi t}}\left[e^{-\frac{(x-y)^2}{2t}} + e^{-\frac{(x+y)^2}{2t}}\right]dy. \end{aligned}$$

To show the Feller property take an arbitrary $f \in \mathbb{C}_0(\mathbb{R}_+)$, i.e. a continuous function on $\mathbb{R}_+$ which vanishes at ∞. Then, $g : \mathbb{R} \mapsto \mathbb{R}$ defined by $g(x) = f(|x|)$ belongs to $\mathbb{C}_0(\mathbb{R})$. Note that

$$Q_t f(x) = P_t g(x), \ x \geq 0.$$

Thus, the desired Feller property follows from the Feller property of Brownian motion. One can also have a direct proof of this fact by using the explicit form of q using the arguments leading to the solution of Q5 in Problem Set 2.

Recall that the domain of the infinitesimal generator are the functions in $\mathbb{C}_0(\mathbb{R}_+)$ for which the $\lim_{t\to 0}\frac{Q_t f(x)-f(x)}{t}$ exists and belongs to $\mathbb{C}_0(\mathbb{R}_+)$. Note that if f belongs to the infinitesimal generator of X, then the function g defined by $g(x) = f(|x|)$ should belong to the infinitesimal generator of B. On the other hand, g is $\mathbb{C}_0^2(\mathbb{R})$ if and only if f is $\mathbb{C}_0^2(\mathbb{R}_+)$ with $f'(0) = 0$. Thus, the domain of the generator are twice continuously differentiable functions on $[0,\infty)$ with zero derivative at 0. Moreover, for any such f

$$\lim_{t\to 0}\frac{Q_t f(x) - f(x)}{t} = \frac{1}{2}f''(x).$$

3. Let N be a Poisson process with parameter λ. Then, using the fact that $e^{-\lambda t} = 1 - \lambda t + o(t)$, where $o(t)$ is the collection of terms having a higher order than t,

$$\frac{E^x[f(N_t)] - f(x)}{t} = \frac{1}{t}\left(f(x)(1-\lambda t) + f(x+1)\lambda t - f(x) + o(t)\right) \to \lambda(f(x+1)-f(x)), \quad \text{as} \quad t \to 0.$$

Therefore the infinitesimal generator is

$$\mathcal{L}f(x) = \lambda(f(x+1) - f(x)).$$

4. To show the Chapman-Kolmogorov identity, it suffices to show that $Q_{t+s}f = Q_t Q_s f$ for all bounded and measurable f. Note that

$$Q_s f(x) = E^x \left[f(X_s) \exp\left(- \int_0^s c(X_r)\, dr \right) \right].$$

Then,

$$\begin{aligned} Q_t Q_s f(x) &= E^x \left[Q_s f(X_t) \exp\left(- \int_0^t c(X_r)\, dr \right) \right] \\ &= E^x \left[E^{X_t} \left[f(X_s) \exp\left(- \int_0^s c(X_r)\, dr \right) \right] \exp\left(- \int_0^t c(X_r)\, dr \right) \right] \\ &= E^x \left[E^x \left[f(X_{t+s}) \exp\left(- \int_t^{t+s} c(X_r)\, dr \right) \Big| \mathcal{F}_t \right] \exp\left(- \int_0^t c(X_r)\, dr \right) \right] \\ &= E^x \left[f(X_{t+s}) \exp\left(- \int_0^{t+s} c(X_r)\, dr \right) \right] = Q_{t+s} f(x). \end{aligned}$$

To calculate the infinitesimal generator of Q, observe

$$\frac{Q_t f - f}{t} = \frac{1}{t} E^x \left[f(X_t) \left(\exp\left(- \int_0^t c(X_s) ds \right) - 1 \right) \right] + \frac{1}{t} \left(E^x[f(X_t)] - f(x) \right).$$

Let C be an upper bound of c. Then,

$$\frac{1 - \exp\left(- \int_0^t c(X_s) ds \right)}{t} \leq \frac{1 - \exp\left(-tC \right)}{t}.$$

Moreover, the function $\frac{1-\exp(-tC)}{t}$ is continuous and bounded on $[0,1]$. Thus, sending $t \downarrow 0$, we obtain via the Dominated Convergence Theorem,

$$\lim_{t \downarrow 0} \frac{Q_t f - f}{t} = -c(x) f(x) + A f(x).$$

5. The statement follows from Hille-Yosida theorem. In what follows we work with the space without one-point compactification. Let us verify its three conditions. First, Since $\mathbb{C}^2 \subset \mathcal{D}(A)$, $\mathcal{D}(A)$ is dense in $\mathbb{C}^b$ (Banach space of bounded continuous functions). Second, A satisfies the positive maximum principle. Indeed, if $f(x_0) = \sup\{f(x); x \in \mathbb{R}\}$, then $f''(x_0) \leq 0$ and $f'(x_0) = 0$, which implies $\frac{1}{2} a(x_0) f''(x_0) + b(x_0) f'(x_0) \leq 0$. Third, given g belonging to some fixed dense subset of $\mathbb{C}^b$ and some fixed $\alpha > 0$, we want to find a function $f \in \mathbb{C}^b$ such that it solves the ordinary differential equation $(\alpha - A) f = g$. When $a(x) > 0$ on E, the previous equation is a second order ordinary differential equation

$$\frac{1}{2} a(x) f''(x) + b(x) f'(x) - \alpha f + g = 0.$$

Bounded solution to such equation exists for $\alpha > 0$ by Theorem 13.1 of Feller, W. *The Parabolic Differential Equations and the Associated Semi-Groups of Transformations*, Annals of Mathematics, Second Series, Vol. 55, No. 3 (May, 1952) , pp. 468-519.