SOLUTIONS TO PROBLEM SET 5

1. It follows from Ito's formula that

$$M_t^f = \sum_{i=1}^d \sum_{j=1}^r M_t^{(i,j)}, \qquad \text{with} \qquad M_t^{(i,j)} = \int_0^t \sigma_{ij}(s, X_s) \frac{\partial}{\partial x_i} f(s, X_s) dW_s^j.$$

Then the expression of the covariation follows. When σ_{ij} are bounded on the support of f, then the integrand in each $M^{(i,j)}$ is bounded. It then follows from Ito isometry that

$$E^x\left[(M_t^f)^2\right] = E^x\left[\langle M^f, M^f\rangle_t\right] = \sum_{i,j} \int_0^t a_{ij} \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}(s, X_s)\, ds < \infty.$$

Hence M^f is a square integrable martingale.

2. For any $f \in C^2((0,\infty)\times\mathbb{R}^d)$, define $\sigma_n = \inf\{t \geq 0 \,|\, \|\sigma\nabla f(t, X_t)\| \geq n\}$. The definition yields $\lim_{n\to\infty}\sigma_n = \infty$. Then the same argument as in Problem 1 implies that $\{M^f_{\sigma_n\wedge t}\}_{t\geq 0}$ is a $\mathbb{P}$-martingale. Hence M^f is a $\mathbb{P}$-local martingale. Since f is chosen arbitrarily, $\mathbb{P}$ is a solution to the local martingale problem.

Let us show the second assertion. For any $f \in C_0^2((0,\infty)\times\mathbb{R}^d)$, since σ_{ij} are locally bounded and f has compact support, $\|\sigma\nabla f\|$ is bounded. It then follows from Problem 1 that M^f is a $\mathbb{P}$-martingale. Hence $\mathbb{P}$ is a solution to the martingale problem.

3. For any $p > 0$ we have

$$|a_1|^p + \ldots + |a_n|^p \leq n(|a_1| + \ldots + |a_n|)^p \leq n^{p+1}\left(|a_1|^p + \ldots + |a_n|^p\right). \tag{1}$$

Applying above inequality to

$$\|X_t\|^{2m} = \|X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s\|$$

yields

$$\|X_t\|^{2m} \leq K\left(\|X_0\| + \|\int_0^t b(s, X_s)ds\|^{2m} + \|\int_0^t \sigma(s, X_s)dW_s\|^{2m}\right),$$

for some K that depends on m and d. Moreover,

$$\begin{aligned}
\|\int_0^t b(s, X_s)ds\|^{2m} &= \left(\sum_{i=1}^d \left(\int_0^t b_i(s, X_s)ds\right)^2\right)^m \\
&\leq t^m \left(\sum_{i=1}^d \int_0^t b_i^2(s, X_s)ds\right)^m = t^m \int_0^t \|b(s, X_s)\|^2 ds \\
&\leq t^{2m-1}\int_0^t \|b(s, X_s)\|^{2m} ds,
\end{aligned}$$

where the last inequality follows from Holder's inequality. Thus,

$$\mathbb{E}\left[\max_{0\leq s\leq t}\|X_s\|^{2m}\right] \leq K\left\{t^{2m-1}\int_0^t \mathbb{E}\|b(s, X_s)\|^{2m}ds + \mathbb{E}\|X_0\|^{2m} + \mathbb{E}\left[\max_{0\leq s\leq t}\|\int_0^s \sigma(u, X_u)dW_u\|^{2m}\right]\right\}$$

Using Burkholder-Davis-Gundy inequality along with Holder inequality we have

$$\mathbb{E}\left[\max_{0\le s\le t}\|\int_0^t \sigma(u,X_u)dW_u\|^{2m}\right]\le C\int_0^t \mathbb{E}\|\sigma(u,X_u)\|^{2m}du,$$

for some constant that depends on t and m only. The linear growth condition now gives

$$\mathbb{E}\left[\max_{0\le s\le t}\|X_s\|^{2m}\right]\le K\left\{1+\mathbb{E}\|X_0\|^{2m}+\int_0^t \mathbb{E}\|X_u\|^{2m}du\right\}\le K\left\{1+\mathbb{E}\|X_0\|^{2m}+\int_0^t \mathbb{E}\left(\sup_{s\le u}\|X_s\|^{2m}\right)du\right\},$$

where K is a constant depending only on t, m and d. Application of Gronwall's inequality yields the claim.

The second inequality can be proved in the same manner since

$$\|X_t-X_s\|^{2m}\le K\left(\|\int_s^t b(u,X_u)du\|^{2m}+\|\int_s^t \sigma(u,X_u)dW_u\|^{2m}\right).$$

4. We begin with a d-dimensional Brownian family $X=\{X_t,\mathcal{F}_t;0\le t\le T\},(\Omega,\mathcal{F}),\{P^x\}_{x\in\mathbb{R}^d}$. According to Corollary 3.5.16 in "Brownian Motion and Stochastics Calculus" by Karatzas and Shreve that

$$Z_t=\exp\left(\sum_{j=1}^d\int_0^t b_j(s,X_s)dX_s^j-\frac{1}{2}\int_0^t\|b(s,X_s)\|^2\,ds\right)$$

is a martingale under each measure P^x, so the Girsanov theorem implies that under Q^x given by $(dQ^x/dP^x)=Z_T$, the process

$$W_t=X_t-X_0-\int_0^t b(s,X_s)\,ds;\quad 0\le t\le T$$

is a Brownian motion with $Q^x(W_0=0)=1$. We can then rewrite the previous equation as

$$X_t=X_0+\int_0^t b(s,X_s)\,ds+W_t;\quad 0\le t\le T.$$

Therefore the triple $(X,W),(\Omega,\mathcal{F},Q),\{\mathcal{F}_t\}$ is a weak solution to the previous stochastic differential equation.

5. Without loss of generality suppose b_1 is Lipschitz and denote a common Lipschitz constant for b_1 and σ by K. Let $\{a_n\}_{n\ge0}$ be a sequence such that $\lim_{n\to\infty}a_n=0$, $a_0=1$ and $\int_{a_n}^{a_{n-1}}K^{-2}x^{-2}dx=n$. Then, there exists a continuous function ρ_n on $\mathbb{R}$ with support in (a_n,a_{n-1}) so that $0\le\rho_n(x)\le(2/nK^2x^2)$ holds for any $x>0$, and $\int_{a_n}^{a_{n-1}}\rho_n(x)dx=1$. Then the function

$$\psi_n(x)=\int_0^{|x|}\int_0^y\rho_n(u)dudy;\quad x\in\mathbb{R},$$

is even and twice continuously differentiable, with $|\psi_n'(x)|\le1$ and $\lim_{n\to\infty}\psi_n(x)=|x|$ for $x\in\mathbb{R}$. Also note that ψ is increasing on the positive real line. Moreover. $\psi''(x)=\rho_n(|x|)$.

Without loss of generality, we can assume that

$$\mathbb{E}\int_0^t|\sigma(s,X_s^{(i)})|^2ds<\infty;\quad 0\le t<\infty, i=1,2.$$

Otherwise, we may use a localization argument to reduce to the previous case. Define

$$\Delta_t=X_t^{(1)}-X_t^{(2)}=\int_0^t\left(b_1(s,X_s^{(1)})-b_2(s,X_s^{(2)})\right)ds+\int_0^t\left(\sigma(s,X_s^{(1)})-\sigma(s,X_s^{(2)})\right)dW_s,$$

and a new sequence of functions $\phi_n(x) = \psi_n(x) \cdot 1_{(0,\infty)}(x)$. By the Ito's formula, we have

$$\begin{aligned}\phi_n(\Delta_t) &= \int_0^t \phi_n'(\Delta_s) \left[b_1(s, X_s^{(1)}) - b_2(s, X_s^{(2)})\right] ds + \frac{1}{2}\int_0^t \phi_n''(\Delta_s) \left[\sigma(s, X_s^{(1)}) - \sigma(s, X_s^{(2)})\right]^2 ds \\ &\quad + \int_0^t \phi_n'(\Delta_s) \left[\sigma(s, X_s^{(1)}) - \sigma(s, X_s^{(2)})\right] dW_s.\end{aligned}$$

The expectation of the stochastic integral is zero, whereas the expectation of the second integral is bounded from above by $\mathbb{E}\int_0^t \phi_n''(\Delta_s) K^2 |\Delta_s|^2 ds \le 2t/n$ due to the bound on ρ_n. We than conclude

$$\mathbb{E}\phi_n(\Delta_t) - \frac{t}{n} \le \mathbb{E}\int_0^t \phi_n' \left[b(s, X_s^{(1)}) - b(s, X_s^{(2)})\right] ds + \frac{t}{n}.$$

The expectation on the right-hand side is can be bounded from above by considering

$$\mathbb{E}\int_0^t \phi_n'(\Delta_s) \left[b_1(s, X_s^{(1)}) - b_1(s, X_s^{(2)})\right] ds + \mathbb{E}\int_0^t \phi_n'(\Delta_s) \left[b_1(s, X_s^{(2)}) - b_2(s, X_s^{(2)})\right] ds \le K\int_0^t \mathbb{E}[\Delta_s^+] ds.$$

Send $n \to \infty$ to obtain $\mathbb{E}[\Delta_t^+] \le K\int_0^t \mathbb{E}[\Delta_s^+] ds$ and by Gronwall inequality, we conclude $\mathbb{E}[\Delta_t^+] = 0$; ie., $X_t^{(1)} \le X_t^{(2)}$ a.s..

6.

$$N_t = u(X_{t\wedge\tau_D}) E^x \left(-\int_0^{t\wedge\tau_D} k(X_s) ds\right) + \int_0^{t\wedge\tau_D} g(X_s) \exp\left(-\int_0^s k(X_r) dr\right) ds.$$

Since k is positive, u and g are continuous in bounded domain D, then for any stopping time T, $|N_T|$ is bounded from above by

$$\|u\|_\infty + \tau_D \|g\|_\infty,$$

which has finite expectation because $E^x \tau_D < \infty$. On the other hand, applying Ito's formula and utilizing the equation $Lu - ku = -g$ that u satisfies, we can verify N_t is a local martingale. Thus, N is a uniformly integrable martingale.

It then follows from the optional sampling theorem that

$$\begin{aligned}u(x) = N_0 = E^x N_{\tau_D} &= E^x \left[u(X_{\tau_D}) \exp\left(-\int_0^{\tau_D} k(X_s) ds\right) + \int_0^{\tau_D} g(X_s) \exp\left(-\int_0^s k(X_r) dr\right) ds\right] \\ &= E^x \left[f(X_{\tau_D}) \exp\left(-\int_0^{\tau_D} k(X_s) ds\right) + \int_0^{\tau_D} g(X_s) \exp\left(-\int_0^s k(X_r) dr\right) ds\right].\end{aligned}$$