SOLUTIONS TO PROBLEM SET 6

Let x be a 1-dimensional diffusion satisfying assumptions in the lecture note. For $l \le a < x < b \le r$, there exists a function s such that

$$P^x(T_b < T_a) = \frac{s(x) - s(a)}{s(b) - s(a)}. \tag{1}$$

1. For $x < y$, observe that $P^x(T_y < T_r) = 1$. Then

$$P^x(T_r < T_l) = P^x(T_r < T_l, T_y < T_l).$$

Note $T_r = T_y + T_r \circ \theta_{T_y}$, and on the event $[T_r < T_l]$, $T_l = T_y + T_l \circ \theta_{T_y}$. Thus,

$$P^x(T_r < T_l) = E^x[1_{[T_y<T_l]} 1_{[T_r<T_l]} \circ \theta_{T_y}] = P^x(T_y < T_l) P^y(T_r < T_l),$$

where the second identity follows from the strong Markov property and $X_{T_y} = y$ when $T_y < \infty$. Recalling $s(x) = P^x(T_r < T_l)$, the previous identity then reads

$$s(x) = s(y) P^x(T_y < T_l). \tag{2}$$

Identity (2) obviously yields $s(x) \le s(y)$. If $s(x) = s(y)$, for any $b > y$, (1) yields $P^y(T_b < T_x) = 0$ which contradicts with the regularity of the process X. Indeed, let $S_0 = 0$, $R_{n+1} = \inf\{t > S_n : X_t = y\}$, $S_{n+1} = \inf\{t \ge R_{n+1} : X_t \in \{x, b\}\}$. Note that

$$P^x(T_b < \infty) = P^x(\text{for some } n,\, S_n < \infty, X_{S_n} = b).$$

However,

$$P^x(S_n < \infty, X_{S_n} = b) = E^x\left[\mathbb{I}_{[R_n<\infty]} P^y(T_b < T_x)\right] = 0,$$

implying $P^x(T_b < \infty) = 0$, which is a contradiction to the regularity. Therefore, s is strictly increasing.

To show s is right-continuous, suppose $x < y$. In view of (2), we need to show $\lim_{y\downarrow x} P^x(T_y < T_l) = 1$. Moreover, this limit equals $P^x(\sup_{t\le T_l} X_t > x)$. If $P^x(\sup_{t\le T_l} X_t > x) < 1$, then $P^x(X_t \le x, \forall t \le T_l) > 0$, which contradicts the regularity of X. The left-continuity is shown in the same way.

2. It follows from Theorem 6.2 in the lecture notes that s is the scale function if and only if $s(X)^R$ is a local martingale. Using Ito's formula, one can check the drift of $s(X)$ is zero, hence $s(x)^R$ is a local martingale. Therefore the scale function must be

$$s(x) = \int_c^x \exp\left(-\int_c^y 2b(z)\sigma^{-2}(z)\,dz\right) dy.$$

3. Choosing $c = 1$ and calculating via formula in Problem 2, we obtain $s(x) = -1/x + 1$. Since scale function is defined up to affine transformations, the we can set $s(x) = 1/x$.

4. Via the homoeomorphism $\phi : E \to \tilde{E}$, $\tilde{X} = \phi(X)$. Let $\tilde{s}$ be the scale function of $\tilde{X}$, then $\tilde{s}(\tilde{X})$ is a local martingale. On the other hand, $\tilde{s}(\tilde{X}) = \tilde{s}(\phi(X))$, which implies $\tilde{s} \circ \phi$ is a

local martingale, hence $\tilde{s} \circ \phi = s$. Therefore we conclude $\tilde{s} = s \circ \phi^{-1}$. For the speed measure, let $\tilde{I} = \phi(I)$ and $\tilde{\sigma}_{\tilde{I}} = \inf\{t \geq 0; \tilde{X}_t \notin \tilde{I}\}$. Since ϕ is a one-to-one map, $\tilde{\sigma}_{\tilde{I}} = \sigma_I$. Therefore, $\tilde{m}(\tilde{I}) = m(I) = m(\phi^{-1}(\tilde{I}))$.

5. Let $\phi(x) = s(x)$. As we have seen in problem 1, s is strictly increasing and continuous. Note that the scale function of $\tilde{x}$ is $\tilde{s}(x) = x$. It then follows from Theorem 3.5.3 in the lecture note that

$$\frac{df}{dx}(x_2) - \frac{df}{dx}(x_1) = \frac{df}{d\tilde{s}}(x_2) - \frac{df}{d\tilde{s}}(x_1) = \int_{x_1}^{x_2} \tilde{A}f(y)\tilde{m}(dy) = \int_{x_1}^{x_2} \frac{1}{2}(s'\sigma(s^{-1}(y)))^2 \partial_x^2 f(y)\tilde{m}(dy).$$

The last identity yields $\tilde{m}(ds(x)) = 2(s'(x)\sigma(x))^{-2}s'(x)dx$. On the other hand, the previous problem gives $m(dx) = \tilde{m}(ds(x))$ since $m(I) = \tilde{m}(s(I))$ for any I in $\mathrm{int}(\mathbf{E})$. Therefore, $m(dx) = \frac{2}{s'(x)\sigma^2(x)}dx$.

6. For case (a), we have from (1) for $l \leq a < x < b \leq r$:

$$P^x[\inf_{0\leq t<\sigma_I} X_t \leq a] \geq P^x[X_{T_{a,b}} = a] = \frac{1 - s(x)/s(b)}{1 - s(a)/s(b)}. \tag{3}$$

Letting $b \uparrow r$, we obtain $P^x[\inf_{0\leq t<\sigma_I} X_t \leq a] = 1$ for every $a \in I$. Now we let $a \downarrow l$ to get $P^x[\inf_{0\leq t<\sigma_I} X_t = l] = 1$. A similar argument shows that $P^x[\sup_{0\leq t<\sigma_I} X_t = r] = 1$. Suppose that $P^x[\sigma_I < \infty] > 0$; then the event $[\sup_{0\leq t<\sigma_I} X_t = r]$ and $[\inf_{0\leq t<\sigma_I} X_t = l]$ cannot both have probability one. This is contradiction shows that $P^x[\sigma_I < \infty] = 1$.

For case (b), we first observe that (3) still implies $P^x[\inf_{0\leq t<\sigma_I} X_t = l] = 1$. On the other hand, we recall $P^x[X_{T_{a,b}} = b] = (s(x) - s(a))/(s(b) - s(a))$ from (1) and let $a \downarrow l$ to see that

$$P^x[X_t = b;\ \text{some } 0 \leq t < \sigma_I] = \frac{s(x) - s(l+)}{s(b) - s(l+)}.$$

(Observe that for $b > x$, $[\sup_s X_s \geq b] = [X_t = b \text{ for some } t]$ in above) Letting now $b \uparrow r$, we conclude that $P^x[\sup_{0\leq t<\sigma_I} X_t = r] = 0$. We have then shown

$$P^x[\inf_{0\leq t<\sigma_I} X_t = l] = P^x[\sup_{0\leq t<\sigma_I} X_t < r] = 1.$$

It remains to show that $\lim_{t\to\sigma_I} X_t = \inf_{0\leq t<\sigma_I} X_t$, and for this it suffices to establish that the limit exists, almost surely. With $\sigma_n = \inf\{t \geq 0; X_t \notin (l_n, r_n)\}$; $n \geq 1$, for sequence $(l_n) \downarrow l$ and $(r_n) \uparrow r$, the process $Y^{(n)} = s(X_{t\wedge\sigma_n}) - s(l+)$; $0 \leq t < \infty$ is for each $n \geq 1$ a nonnegative martingale. Letting $n \to \infty$ and using Fatou's lemma, we see that $Y_t = s(X_{t\wedge\sigma}) - s(l+)$; $0 \leq t < \infty$, is a nonnegative supermartingale. Therefore it converges almost surely as $t \to \infty$. Because $s : [l, r] \to \infty$ has a continuous inverse, $\lim_{t\to\infty} X_{t\wedge\sigma_I}$ must exists.

Case (c) is similar to (b), and case (d) is obtained by taking limits in (1).