SOLUTIONS TO PROBLEM SET 7

1. It follows from Theorem 3.5.3 and Proposition 3.5.4 in the lecture note that we need to find increasing and decreasing solutions to the equation

$$Af = \lambda f, \qquad \text{where } A = 2x\frac{d^2}{dx^2} + \delta\frac{d}{dx}. \tag{1}$$

Let $\nu = \delta/2 - 1$ and function $g(x)$ via

$$f(x) = x^{-\nu/2}g(x).$$

Calculation shows that g satisfies

$$g''(x) + \frac{g'(x)}{x} - \left(\frac{\lambda}{2x} + \frac{\nu^2}{4x^2}\right)g = 0.$$

Let us also define $h(y) = g(x)$, where $y = \sqrt{2\lambda x}$, then calculation shows that g solves

$$y^2h''(y) + yh'(y) - (y^2 + \nu^2)h(y) = 0.$$

The independent solutions to this ODE are so called *modified Bessel functions* $I_\nu(y)$ and $K_\nu(y)$, where I_ν is a increasing function, K_ν is a decreasing function, and both of them are positive. Therefore,

$$\varphi^+(\lambda, x) = x^{-\nu/2}I_\nu(\sqrt{2\lambda x}) \quad \text{and} \quad \varphi^-(\lambda, x) = \quad x^{-\nu/2}K_\nu(\sqrt{2\lambda x})$$

are two linearly independent solutions to (1). To check that $\varphi^+(\lambda, x)$ is increasing, we use the relation between I_ν and $I_{\nu+1}$: $I'_\nu = I_{\nu+1} + (\nu/x)I_\nu$ to obtain

$$\varphi'(x) = \frac{d}{dx}\left(x^{-\nu/2}I(\sqrt{2\lambda x})\right) = \frac{\sqrt{2\lambda}}{2}x^{-\frac{\nu-1}{2}}I_{\nu+1}(\sqrt{2\lambda x}) > 0.$$

Similarly, we can use the relation: $K'_\nu = K_{\nu-1} - (\nu/x)K_\nu$ to check that $\varphi^-(x)$ is decreasing.

It then follows that

$$\mathbb{E}^x[e^{-\lambda T_y}] = \begin{cases} \frac{\varphi^+(\lambda,x)}{\varphi^+(\lambda,y)} = \left(\frac{x}{y}\right)^{-\nu/2}\frac{I_\nu(\sqrt{2\lambda x})}{I_\nu(\sqrt{2\lambda y})} & \text{when } x < y \\ \frac{\varphi^-(\lambda,x)}{\varphi^-(\lambda,y)} = \left(\frac{x}{y}\right)^{-\nu/2}\frac{K_\nu(\sqrt{2\lambda x})}{K_\nu(\sqrt{2\lambda y})} & \text{when } x > y. \end{cases}$$

Here when $x > y$, I_ν should be replaced by K_ν so that the Laplace transform on the left hand side is less than 1.

2 and 3. Note that Problem 2 will be a special case of Problem 3 once we solve the 3rd question with correlation.

Let X be a diffusion with generator L:

$$L = \frac{1}{2}\sum_{i,j}^{n} a_{ij}(x)\partial^2_{ij} + \sum_{i}^{n} b_i(x)\partial_i,$$

where $a = \sigma\sigma'$. The observation process is of the form

$$Y_t = \int_0^t h_s\, ds + W_t,$$

where W is a Brownian motion in $\mathbb{R}^n$ and $\mathbb{E}[\int_0^T |h_s|^2\, ds] < \infty$. We suppose that $f \in C_K^2(\mathbb{R}^n)$ satisfying $\sup_{t\le T} E[f(X_t)^2] < \infty$ and $E\int_0^T |Lf(X_t)|^2\, dt < \infty$. Then

$$M_t = f(X_t) - f(X_0) - \int_0^t Lf(X_s)\, ds \quad \text{is a } L^2 \text{ martingale,}$$

and there exists an $\mathcal{F}_t$-optional process $\alpha = (\alpha^1, \cdots, \alpha^n)$ such that

$$[M, W^i]_t = \int_0^t \alpha_s^i\, ds.$$

Define $\pi_t f = E[f(X_t)\,|\,\mathcal{F}_t^Y]$, then the Kushner-Stratonovic equation reads

$$\pi_t f = \pi_0 f + \int_0^t \pi_s Lf ds + \int_0^t (\pi_s hf - \pi_s h \pi_s f + \pi_s \alpha_s)\, dN_s, \tag{2}$$

where N is a $\mathcal{F}_t^Y$-Brownian motion in $\mathbb{R}^n$.

The proof is almost the same with that in Theorem 4.1.2 in the lecture note, except the second formula from the bottom of page 64 is replaced by

$$\begin{aligned} f(X_t)Y_t &= \int_0^t f(X_s)dY_s + \int_0^t Y_s df(X_s) + \int_0^t \alpha_s ds \\ &= \int_0^t f(X_s)(dW_s + h_s ds) + \int_0^t Y_s(dM_s + Lf(X_s)ds) + \int_0^t \alpha_s ds \\ &= \int_0^t \left(f(X_s)h_s + Y_s Lf(X_s) + \alpha_s\right)\, ds + \mathcal{F}_t - \text{martingale.} \end{aligned}$$

Specialize to the situation of Problem 2, where M has the following form:

$$M_t = f(X_t) - f(X_0) - \int_0^t Lf(X_s)\, ds - \int_0^t f'(X_s)\sigma(X_s) dB_s,$$

where $d[W, B]_t = \rho(X_t, Y_t)$. Then α in the previous general setting is

$$\alpha_t = -f'(X_t)\sigma(X_t)\rho(X_t, Y_t).$$

Therefore Kushner-Stratonovic equation is in (2) where α is given as above.

4.

a) Let $\widehat{X}$ denote the optional projection. Then

$$\begin{aligned} \widehat{X}_t &= \int_0^t \frac{1}{f(s)} \left((\widehat{X_s^2} - \widehat{X_s Y_s}) - \widehat{X}_s \widehat{X_s - Y_s} \right) dB_s^Y \\ &= \int_0^t \frac{1}{f(s)} (\widehat{X_s^2} - \widehat{X}_s^{\,2}) dB_s^Y, \end{aligned}$$

where B^Y is an $\mathcal{F}^Y$-Brownian motion given by

$$B_t^Y = Y_t - \int_0^t \frac{1}{f(s)} (\widehat{X}_s - Y_s) ds.$$

b) Using Ito's formula,

$$dX_t^2 = 2X_t dX_t + \sigma^2(t)dt.$$

Thus,

$$d\widehat{X_t^2} = \sigma^2(t)dt + \frac{\widehat{X_t^3} - \widehat{X_t^2}\widehat{X}_t}{f(t)} dB_t^Y.$$

The hint yields $\widehat{X_t^3} - \widehat{X_t^2}\widehat{X}_t = 2v(t)\widehat{X}_t$. Applying Ito's formula to $\widehat{X^2} - \widehat{X}^2$ gives

$$\begin{aligned} dv(t) &= \sigma^2(t)dt + \frac{2v(t)\widehat{X}_t}{f(t} dB_t^Y - 2\frac{\widehat{X}_t v(t)}{f(t)} dB_t^Y - \frac{v^2(t)}{f^2(t)} dt \\ &= \left(\sigma^2(t) - \frac{v^2(t)}{f^2(t)}\right) dt. \end{aligned}$$

c) $X_t \sim N(\widehat{X}_t, v(t))$ since X and Y are jointly Gaussian. When $f(t) = s(t) - t$, it is easily checked that $s(t) - t$ satisfies the differential equation for v. Next note that

$$\begin{aligned} \widehat{X}_t &= \int_0^t \frac{1}{f(s)} (\widehat{X_s^2} - \widehat{X_s}^2) dB_s^Y \\ &= \int_0^t \frac{1}{f(s)} v(s) dB_s^Y = B_t^Y. \end{aligned}$$