SPEEDING UP THE EULER SCHEME FOR KILLED DIFFUSIONS

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Abstract. Let X be a linear diffusion taking values in (ℓ, r) and consider the standard Euler scheme to compute an approximation to $\mathbb{E}[g(X_T)\mathbf{1}_{[T<\zeta]}]$ for a given function g and a deterministic T, where $\zeta = \inf\{t \geq 0 : X_t \notin (\ell, r)\}$. It is well-known since Gobet [21] that the presence of killing introduces a loss of accuracy and reduces the weak convergence rate to $1/\sqrt{N}$ with N being the number of discretisatons. We introduce a drift-implicit Euler method to bring the convergence rate back to $1/N$, i.e. the optimal rate in the absence of killing, using the theory of recurrent transformations developed in [9]. Although the current setup assumes a one-dimensional setting, multidimensional extension is within reach as soon as a systematic treatment of recurrent transformations is available in higher dimensions.
Keywords: diffusions with killing, Euler-Maruyama scheme, drift-implicit scheme, weak convergence, recurrent transformations, strict local martingales, Kato classes, barrier options.

1. Introduction

Let X be a diffusion on some filtered probability space taking values in (ℓ, r) and solving

$$X_t = x + \int_0^t \sigma(X_s)dB_s + \int_0^t b(X_s)ds, \quad t < \zeta, \tag{1.1}$$

where B is a Brownian motion, and $\zeta := \inf\{t \geq 0 : X_t \notin (\ell, r)\}$ is the first exit time from the interval (ℓ, r). The process is *killed* at ζ and sent to a cemetery state.

Let's assume that at least one of the boundaries are accessible and ζ is finite a.s. and consider $\mathbb{E}[g(X_T)\mathbf{1}_{[T<\zeta]}]$ for a given function g and a deterministic T. Such computations appear very naturally in many applied problems of science, engineering, and finance. For instance, in Mathematical Finance theory, such an expectation corresponds to the price of a barrier option with payoff g and maturity T written on a stock whose price process is given by X. The barrier feature renders the option worthless if the stock price hits one of the accessible boundaries before the maturity of the option.

A closed form expression for $\mathbb{E}[g(X_T)\mathbf{1}_{[T<\zeta]}]$ is rarely available even in this one-dimensional setting. Thus, one needs to resort to an approximation scheme for an answer. Arguably the easiest approach is to run a standard Euler-Maruyama scheme on the SDE (1.1) by setting

$$\bar{X}_{t_{n+1}} = \bar{X}_{t_n} + \sigma(\bar{X}_{t_n})(B_{t_{n+1}} - B_{t_n}) + b(\bar{X}_{t_n})\frac{T}{N},$$

where $\bar{X}_0 = x$, $t_0 = 0$, $N > 0$ is an integer, $t_n = \frac{nT}{N}$ for $n = 1, \ldots N$, and compute $\mathbb{E}[g(\bar{X}_T)\mathbf{1}_{[T<\tau]}]$, where τ is the first time that the discrete-time process $(\bar{X}_{t_n})_{n=0}^N$ hits any of the barriers. Under standard regularity conditions on the diffusion process and g, such a scheme indeed converges as $N \to \infty$. However, it converges at a rate much slower than a standard Euler-Maruyama scheme applied to a diffusion process that is not killed at accessible boundaries.

Indeed it was shown by Gobet [21] that under standard hypothesis the above scheme for the killed diffusion converges weakly at rate $N^{-1/2}$ as opposed to N^{-1}, which is the rate of weak

Date: June 16, 2023.

convergence for the Euler-Maruyama scheme in the absence of killing (see, e.g., Talay and Tubaro [43] or Mikulevičius and Platen [33]). This rate is optimal since it is reached when X is a Brownian motion and g is an indicator function of a set strictly contained in (ℓ, r) (see Siegmund and Yuh [42]).

Çetin [9] conjectured that using a *recurrent transformation* would bring the convergence rate back to N^{-1}. A recurrent transformation at heart is a change of measure that keeps the Markovian structure intact while transforming the process into a recurrent one. In particular X never touches the boundaries of (ℓ, r) under the new measure $\mathbb{Q}$. [9] shows that $\mathbb{Q}$ is locally absolutely continuous with respect to the original measure $\mathbb{P}$, and X follows

$$dX_t = \sigma(X_t)dW_t + \left(b(X_t) + \sigma^2(X_t)\frac{h'}{h}(X_t)\right)dt, \tag{1.2}$$

for some function h and a $\mathbb{Q}$-Brownian motion W. That the above claim was a conjecture and not following immediately from the standard results on Euler-Maruyama schemes is that $\frac{h'}{h}$ is explosive near boundaries and is not Lipschitz, which is in fact needed for X not to touch the previously accessible boundaries after the measure change. This can create significant difficulties with approximation and may even lead to divergence (see, e.g., the potential issues that may arise with non-Lipschitz drivers and methods on how to resolve them in Hutzenthaler et al. [27] and [28]).

In this paper we prove this conjecture with a slight "twist". Note that if one applies the Euler-Maruyama scheme naively to (1.2), one obtains, as usual, a Brownian motion with drift whose parameters change at times of discretisation. This process will hit finite boundaries with positive probability, and therefore will exit the state space of X with positive probability. One way to overcome this is to impose an ad hoc reflection on the boundaries. However, this will introduce a local time term in computations requiring additional estimates on its convergence rate to 0. Moreover, it is far from obvious that reflection is the optimal resolution of problems arising from the discretised process exiting the domain.

We instead study a drift-implicit method that keeps the state space intact after discretisation. To see this, suppose that $b \equiv 0$, which can be obtained by changing the scale if necessary, and consider the backward Euler-Maruyama scheme

$$\widehat{X}_{t_{n+1}} = \widehat{X}_{t_n} + \sigma(\widehat{X}_{t_n})(B_{t_{n+1}} - B_{t_n}) + \frac{T}{N}\sigma^2(\widehat{X}_{t_n})\frac{h'}{h}(\widehat{X}_{t_{n+1}}), \tag{1.3}$$

where h becomes a concave function.

Note that different than what one would expect from a backward scheme (see, e.g. Mao and Szpruch [32], Alfonsi [2], Alfonsi [3], and Neuenkirch and Szpruch [35] to name a few) the σ^2-term in the drift of (1.2) is still evaluated at $\widehat{X}_{t_n}$. This stems from the fact that (1.2) with $b \equiv 0$ should be viewed as a time-changed version of

$$dY_t = dW_t + \frac{h'}{h}(Y_t)dt,$$

where the time change is given by $\int_0^t \sigma^2(Y_s)ds$. We make an extensive use of this correspondence in our proofs.

Our main result is Theorem 3.1 which proves that the rate of weak convergence of the above backward Euler-Maruyama scheme is N^{-1} under standard assumptions on the diffusion process. Moreover, there is no single h function that achieves this rate. We show that any non-negative concave h vanishing at accessible boundaries can be used to obtain this convergence rate as long as it satisfies some mild growth conditions. Such functions are easy to construct and we study in

Section 5 construction of some particular h-functions to compute approximate prices for barrier options in a Black-Scholes framework. In fact we observe fast convergence in our numerical studies even in the absence of the growth conditions imposed by our theoretical analysis. Our numerical results are very promising and error terms very rapidly converge to 0 even with a small number of iterations. Moreover, in the case of a particular local volatility model with double barriers, our method yields smaller error terms than the so-called *Brownian bridge method* when the number of discretisations is reasonably large.

We are not the first to consider implicit schemes for studying diffusions with infinite lifetime and taking values in a strict subset of $\mathbb{R}$. Alfonsi [2, 3] and Neunkirch and Szpruch [35] consider such scalar processes whose SDE representation is given by

$$dY_t = dW_t + f(Y_t)dt, \tag{1.4}$$

and f satisfying the conditions of a Feller test ensuring that Y takes values in (ℓ, r) (see also Dereich et al. [15] in the special case of Cox-Ingersoll-Ross (CIR) process). [3] and [35] show that a the drift implicit Euler scheme for Y converge strongly with rate N^{-1} if f satisfies certain integrability conditions including

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T (f'(Y_t))^2 dt\right] < \infty. \tag{1.5}$$

However, this condition cannot be satisfied by h that paves the way for a recurrent transformation rendering X recurrent and following (1.2) (See Proposition C.1 in the Appendix for a proof in case $b \equiv 0$, and $\sigma \equiv 1$.).

The estimates obtained by the authors in [3] and [35] rely on the Burkholder-Davis-Gundy (BDG) inequality which requires the corresponding local martingale be a true martingale. As $\frac{1}{h(X)}$ is a strict local submartingale under $\mathbb{Q}$, one needs to develop new techniques to arrive at the needed estimate for convergence theorem.

This brings to the fore another novelty of our paper. Given the impossibility of the use of BDG inequality we use potential theoretic methods that yield the boundedness of inverse moments of $h(X)$ under $\mathbb{Q}$, which is crucial for obtaining the weak convergence result in our paper (or a strong convergence type results considered by Alfonsi, Neunkirch and Szpruch). We use the theory of *Kato class* potentials to show the boundedness of required moments. Kato potentials are one of the fundamental objects in the study of Schrödinger operators (see, e.g. Aizenman and Simon [1], Cranston et al. [13], Chen [11], and Chen and Song in [12]). We show in Theorem 2.1 that the additive functional $dA_t = -\frac{1}{2}\frac{h''(X_t)}{h(X_t)}dt$ belongs to a particular Kato class defined in [11], which in turn yields the boundedness of the inverse moment of $\frac{1}{h}(\widehat{X}_{t_n})$ (uniformly in N) in conjunction with a comparison argument via Lemma 4.2. The potential theory also helps us to prove uniform bounds on the moments of integral functionals of $h^{-2-p}(\widehat{X}_t)$ (see Theorem 4.1 for an exact description).

Our methodology offers hope to study the convergence rates for CIR processes or diffusions that live in a bounded interval or half space) that do not satisfy (1.5). We show in this paper that if one considers the 3-dimensional Bessel process,

$$dX_t = dW_t + \frac{1}{X_t}dt,$$

the implicit scheme in (1.3) converges weakly at rate N^{-1}. Clearly, (1.5) is violated since the reciprocal of a 3-dimensional Bessel process is a prime example of a strict local martingale. This process satisfies the conditions of Theorem 3.1 and one obtains the optimal convergence rate for sufficiently smooth g.

To the best of our knowledge, the use of BDG inequality seems to be almost the only method to control the bounds of the moments in the literature concerning the numerical analysis of SDEs. The novel potential theoretic approach taken in the paper avoids the use of the BDG inequality in the computation of inverse moments and instead make use of the concept of Kato classes. As a result, the appearance of local martingale terms do not introduce an extra difficulty in our framework. This is an important contribution in its own right and presents a potential to be useful in other contexts as well. We leave the investigation of the convergence rate for conservative diffusions on $(0,\infty)$ satisfying (1.4) in the absence of condition (C.1) to a future study.

Although our analysis assumes a one-dimensional framework, a close look into our technical analysis reveals that our convergence result does not depend heavily on this assumption apart from the comparison argument used in Lemma 4.2. In particular it is relatively clear how to obtain a version of Theorem 2.2 in the multidimensional case using well-known potential theoretic arguments on Kato classes. However, our main obstacle in extending our results to a multidimensional setting is the absence of a systematic study of recurrent transformations in higher dimensions. Also note that Lemma 4.2 is only used to obtain estimates on $h(\widehat{X})$, which is always a one-dimensional object with $\widehat{X}$ referring to the continuous Euler scheme. Such a study and its applications to Euler methods for killed diffusions will be the subject of future research.

The outline of the paper is as follows. Section 2 fixes the setting, gives a brief summary of results for recurrent transformations needed for this paper together with novel inverse moment estimates, and introduces the backward Euler-Maruyama scheme that is tailored for our purposes. Section 4 obtains the moment estimates that will be needed for the weak convergence analysis performed in Section 3. Theoretical results are confirmed via numerical studies in Section 5 and Section 6 concludes the paper.

2. Preliminaries

Let X be a regular diffusion on (ℓ, r), where $-\infty \leq \ell < r \leq \infty$. We assume that infinite boundaries are inaccessible and if any of the boundaries are reached in finite time, the process is killed and sent to the cemetery state Δ. This is the only instance when the process can be 'killed', we do not allow killing inside (ℓ, r). The set of points that can be reached in finite time starting from the interior of (ℓ, r) and entrance boundaries will be denoted by I. That is, I is the union of (ℓ, r) with the regular, exit and entrance boundaries. The law induced on $C(\mathbb{R}_+, I)$, the space of I-valued continuous functions on $[0,\infty)$, by X with $X_0 = x$ will be denoted by P^x as usual, while ζ will correspond to its lifetime, i.e. $\zeta := \inf\{t > 0 : X_t \notin (\ell, r)\}$. We also introduce the set $I_\Delta := I \cup \{\Delta\}$ and extend any I-valued Borel measurable function f to I_Δ by setting $f(\Delta) = 0$ unless stated otherwise. The filtration $(\mathcal{F}^0_t)_{t\geq 0}$ will correspond to the natural filtration of X, $\tilde{F}_t$ will be the universal completion of $\mathcal{F}^0_t$, and $\mathcal{F}_t = \tilde{F}_{t+}$ so that $(\mathcal{F}_t)_{t\geq 0}$ is a right continuous filtration. We will also set $\mathcal{F} := \bigvee_{t\geq 0} \mathcal{F}_t$. We refer the reader to Borodin and Salminen [7] (see e.g Chapters I and II) for a summary of results and references on one-dimensional diffusions. The definitive treatment of such diffusions is, of course, contained in Itô and McKean [29] (see e.g Chapter 3).

Since we are only concerned with the diffusion process until it is killed, we can assume without any loss of generality that X is on natural scale. The extra regularity conditions imposed in the following assumption are standard in the theory of Euler discretisations for SDEs.

Assumption 2.1. *X is a regular one-dimensional diffusion on (ℓ, r) such that*

$$X_t = X_0 + \int_0^t \sigma(X_s)dB_s, \qquad t < \zeta,$$

where $\sigma : (\ell, r) \to (0, \infty)$ is continuously differentiable with a bounded derivative, B is a standard Brownian motion, and $\zeta = \inf\{t > 0 : X_t \in \{l, r\}\}$. Moreover, $\sigma(\ell+)$ (resp. $\sigma(r-)$) exists and is finite if ℓ (resp. r) is finite.

Note that the speed measure m associated with X is given by $m(dx) = 2\sigma^{-2}(x)dx$ on the Borel subsets of (ℓ, r).

Since we are interested in diffusions with killing, the following assumption is needed to ensure that we are not dealing with a vacuous problem.

Assumption 2.2. *$P^x[\zeta < \infty] > 0$ for each $x \in (\ell, r)$.*

Let I_0 be the set of points in I that can be reached from its interior in finite time. Note that under Assumptions 2.1 and 2.2 there are only two cases to consider:

Case 1: Both ℓ and r are accessible, which in turn implies ℓ and r are finite and $I_0 = [\ell, r]$.
Case 2: Only one of ℓ and r is accessible, which can be assumed to be ℓ without any loss of generality. In particular, $I^0 = [\ell, r)$.

As ℓ is always finite as a result of the above convention, the following will also be assumed for convenience:

Assumption 2.3. *$\ell = 0$.*

As a transient diffusion on $(0, r)$, X has a finite potential density, $u : (0, r)^2 \to \mathbb{R}_+$, with respect to its speed measure (see Paragraph 11 in Section II.1 of [7]). That is, for any non-negative and measurable f vanishing at accessible boundaries

$$Uf(x) := \int_0^\infty E^x[f(X_t)]dt = \int_0^r f(y)u(x, y)m(dy).$$

The potential density is symmetric and is explicitly known in terms of the scale function and the speed measure of X. This leads to the following specification of the potential density:

$$u(x, y) = \begin{cases} (x \wedge y)\left(1 - \frac{x \vee y}{r}\right), & \text{if } r < \infty, \\ x \wedge y, & \text{otherwise.} \end{cases}$$

Definition 2.1. *Suppose that Assumptions 2.1-2.3 are in force. Let $\mathcal{S}$ be the space of continuous functions $f : (0, r) \to (0, \infty)$ such that $\int_{(0,r)} f(y)m(dy) < \infty$ and $\int_{(0,r)} yf(y)m(dy) < \infty$. We define*

$$\mathcal{H}_0 := \{h : h = Uf,\ f \in \mathcal{S}\}.$$

Moreover, we denote by $\mathcal{H}$ the union of $\mathcal{H}_0$ and the identity function if $r = \infty$. If $r < \infty$, $\mathcal{H} = \mathcal{H}_0$.

Note that any $h \in \mathcal{H}_0$ is a concave function that is twice continuously differentiable and satisfies on (ℓ, r)

$$\frac{1}{2}\sigma^2 h'' = -f.$$

The following lemma, whose proof is delegated to the Appendix, lists some important properties shared by the functions that belong to $\mathcal{H}$.

Lemma 2.1. *Let $h \in \mathcal{H}$.*

(1) For any given $z > 0$ consider the function H defined by

$$H(x) = x - z\frac{h'(x)}{h(x)}, \qquad x \in (0, r).$$

H is strictly increasing and $H((0, r)) = \mathbb{R}$.

(2) h is increasing if $r=\infty$. However, h' is bounded. In particular, for $h \in \mathcal{H}_0$, we have

$$h'(0)=\begin{cases} \int_0^\infty f(y)m(dy), & \text{if } r=\infty,\\ \int_0^\infty \frac{r-y}{r} f(y)m(dy)>0, & \text{otherwise.}\end{cases}$$

$$h'(r)=\begin{cases} 0, & \text{if } r=\infty,\\ -\frac{1}{r}\int_0^r yf(y)m(dy)<0, & \text{otherwise.}\end{cases}$$

(3) For any $\alpha \geq 0$ and $h \in \mathcal{H}_0$

$$\int_{(0,r)} (u(y,y)\wedge 1)\frac{\alpha|h'(y)|-h''(y)}{h(y)}dy<\infty. \tag{2.1}$$

We are now ready to state the transformations that we shall use in the sequel.

Theorem 2.1. *Suppose that Assumptions 2.1-2.3 are in force, and consider $h \in \mathcal{H}$. Then, the following hold:*

1. *There exists a probability measure $Q^{h,x}$ on $\mathcal{F}$ that is locally absolutely continuous with respect to P^x such that*
$$dX_t=\sigma(X_t)dW_t+\sigma^2(X_t)\frac{h'(X_t)}{h(X_t)}dt \tag{2.2}$$
and W is an $Q^{h,x}$-Brownian motion.
2. *For any $x \in (\ell,r)$, $Q^{h,x}[\zeta<\infty]=0$.*
3. *Let $g: I_0 \to \mathbb{R}$ be a continuous function vanishing at accessible boundaries. Then, for any deterministic $T>0$, we have*
$$E^x[g(X_T)\mathbf{1}_{[T<\zeta]}]=h(x)E^{h,x}\left[\frac{g(X_T)}{h(X_T)}\exp\left(\frac{1}{2}\int_0^T \frac{\sigma^2(X_s)h''(X_s)}{h(X_s)}ds\right)\right], \tag{2.3}$$
where $E^{h,x}$ is the expectation operator associated with $Q^{h,x}$.

Proof. If $h \in \mathcal{H}_0$, the claims follow from Theorem 3.2 in [9].

When $r=\infty$ and h is the identity function, the stated transformation is the well-known Doob's h-transform, and the reader is referred to Theorem 6.2 in Evans and Hening [18] for a proof in a much more general setting. □

Remark 2.1. *When $h \in \mathcal{H}_0$, [9] shows that $h(X)\exp(-\frac{1}{2}\int_0^\cdot \frac{\sigma^2(X_s)h''(X_s)}{h(X_s)}ds)$ is a P^x-martingale, and X is a recurrent process under $Q^{h,x}$, where $Q^{h,x}$ is as in Theorem 2.1. This measure transformation via the* excessive *function h and the* multiplicative functional $\exp(-\frac{1}{2}\int_0^\cdot \frac{\sigma^2(X_s)h''(X_s)}{h(X_s)}ds)$ *is called a* recurrent transformation.

Although similar to a Doob's h-transform at heart, the concept of recurrent transformation is fundamentally different than an h-transform. Recall that an h-transform is a change of measure by an excessive function $h:(0,r)\to\mathbb{R}_+$. In particular, $h(X)$ is at most a local martingale and can be a supermartingale that is not a local martingale. The latter case leads to a loss of mass after the change of measure that appears in the killing measure of the resulting diffusion (see Section 6 of Evans and Hening [18] for details). First, a recurrent transformation always yields a recurrent diffusion while an h-transform yields a transient one. Indeed, when $r=\infty$ and one uses the identity function in above theorem, $Q^{Id,x}[\lim_{t\to\infty}X_t=\infty]=1$ since the corresponding scale function under $Q^{Id,x}$ is finite at ∞.

Second, the excessive function that is associated with an h-transformation is in general not harmonic. That is, $h(X)$ is a strict P^x-supermartingale. This leads to a killing, i.e. a probability *mass loss, under $Q^{h,x}$ at some particular last passage time. We refer the reader to Section 3.1 of* [9] *for more details on this point and the potential theoretic connection between the recurrent transformations and Doob's h-transformations.*

Finally, akin to what we are doing in this paper, if one is interested in an h-transformation that does not involve any killing as in the preceding paragraph and prevents the diffusion from hitting the boundaries, one needs to find an h-function such that h is harmonic, i.e. $h(X)$ is a P^x-local martingale, and h vanishes at accessible boundaries. However, the only harmonic functions of a diffusion under natural scale are affine functions. Combined with the requirement that h vanish at 0 *and r, this implies $h \equiv 0$. That is, there is no h-transform that yields a conservative diffusion that does not hit boundaries when $r < \infty$. On the other hand, any function $h \in \mathcal{H}_0$ yields a recurrent diffusion process that avoids hitting the boundaries via a recurrent transformation.*

In order to approximate the expectation on the right side of (2.3) we shall use a backward Euler-Maruyama (BEM) scheme:

Let $N > 1$ be an integer and define $t_n := \frac{n}{N}T$ for $n = 0, \dots, N$. Set $\bar{X}_0 = X_0$ and proceed inductively by setting

$$\widehat{X}_t = \widehat{X}_{t_n} + \sigma(\widehat{X}_{t_n})(W_t - W_{t_n}) + (t - t_n)\sigma^2(\widehat{X}_{t_n})\frac{h'(\widehat{X}_t)}{h(\widehat{X}_t)} \tag{2.4}$$

for $t \in (t_n, t_{n+1}]$ and $n = 0, \dots N - 1$.

Note that in view of Lemma 2.1 the mapping $x \mapsto x - z\frac{h'}{h}(x)$ is one-to-one and onto for any given $z > 0$. Thus, the above scheme is well-defined since $\sigma(x) > 0$ for all $x \in (0, r)$.

Remark 2.2. *One may also consider an exact simulation method in Beskos and Roberts* [5] *to compute the right or left side of* (2.3)*. However, the singularity of the drift term in* (2.2) *is not consistent with the assumptions therein or its future extensions. Thus, one needs to consider the right side and use the original diffusion after assuming enough regularity on σ. Provided we can perform an exact simulation, one can get an exact simulation of the minimum or maximum of a path together with its terminal value. In* [5]*, for $T =$ 1 year, the time taking to run 50,000 simulations to compute the maximum of X on* $[0, 1]$ *is about 3.1 seconds in their C-language program. To get a good pricing accuracy (a few bps in errors), one may need to run 0.5 or 1 million simulations paths, which may amount to 30seconds to 1 minute. In our numerical calculations for the BEM method, we have used Octave software, which is much slower than the C-language. On the other hand, the numerical experiments have shown that we needed very few discretisation time steps (20 or 30 steps) and very few number of paths to get comparable accuracy (see Section 5). As a result our simulations took about 20 seconds to complete.*

Nevertheless, development of an exact simulation method for the recurrent transform to compute the right side of (2.3) *have the potential to significantly reduce the computation time by avoiding calculations using an inverse function. This interesting direction is left to future research.*

As we shall see in Section 4 the following type of diffusion processes on $(0, r)$ will play a crucial role:

$$dY_t = dW_t + \left\{\frac{h'(Y_t)}{h(Y_t)} + c\right\}dt, \quad t < \zeta(Y) \tag{2.5}$$

where $\zeta(Y)$ denotes the first hitting time of 0 or r. Note that $c = 0$ corresponds to the transformations defined in Theorem 2.1.

Theorem 2.2. *Suppose that Assumptions 2.1-2.3 are in force, $h \in \mathcal{H}$, and Y is a process defined by ((2.5)) with $Y_0 = X_0$. Assume further that $c \leq 0$ if $r = \infty$, and $c = 0$ if $h(x) = x$ for all x. Then the following statements are valid:*

(1) $Q^{h,X_0}[\zeta(Y)=\infty]=1$.
(2) For any stopping time S that is bounded Q^{h,X_0}-a.s. there exists a constant K that does not depend on X_0 such that
$$E^{h,X_0}\Big[\frac{1}{h(Y_S)}\Big]<\frac{K}{h(X_0)}.$$
(3) For any $t>0$ and $p\in[0,1)$
$$E^{h,X_0}\left[\int_0^t \frac{1}{h^{2+p}(Y_s)}ds\right]<\infty.$$

Proof. (1) First observe that a scale function and speed measure for Y can be chosen as
$$s_y(x)=\int_d^x \frac{e^{-2cy}}{h^2(y)}dy,\quad m_y(dx)=2h^2(x)\exp{(2cx)}dx,$$
where $d\in(0,r)$. Since $s_y(0)=-\infty$, 0 is an inaccessible boundary for Y. By the same token, r is also an inaccessible boundary when $s_y(r)=\infty$, which will be valid when $r<\infty$ or $c\leq 0$.
(2) Define Z by
$$Z_t:=\frac{1}{h(Y_t)}\exp\Big(\frac{1}{2}\int_0^t\frac{2ch'(Y_s)+h''(Y_s)}{h(Y_s)}ds\Big)$$
and note that Z is a non-negative Q^{h,X_0}-local martingale by a straightforward application of Ito's formula. By Theorem 62.19 in Sharpe [41] there exists a probability measure $\tilde{P}$ such that
$$dY_t=d\beta_t+cdt,\quad t<\zeta(Y),$$
where β is a $\tilde{P}$-Brownian motion, and whenever S is a stopping time that is finite Q^{h,X_0}-a.s., one has
$$\begin{aligned}E^{h,X_0}\Big[\frac{1}{h(Y_S)}\Big]&=\frac{1}{h(X_0)}\tilde{E}\left[\mathbf{1}_{[S<\zeta(Y)]}\exp\left(-\frac{1}{2}\int_0^S\frac{2ch'(Y_s)+h''(Y_s^N)}{h(Y_s)}ds\right)\right]\\
&\leq\frac{1}{h(X_0)}\tilde{E}\left[\mathbf{1}_{[S<\zeta]}\exp\left(\frac{1}{2}\int_0^S\frac{2(ch'(Y_s))^- - h''(Y_s)}{h(Y_s)}ds\right)\right],\end{aligned}$$
where x^- denotes the negative part of x and we drop the dependency on Y for ζ to ease the exposition.

Suppose that $S<R, Q^{h,X_0}$-a.s. where R is a deterministic constant, and observe that $\tilde{P}[S\geq R,S<\zeta]=0$. Thus,
$$\tilde{E}\left[\mathbf{1}_{[S<\zeta]}\exp\left(\frac{1}{2}\int_0^S\frac{2(ch'(Y_s))^- - h''(Y_s)}{h(Y_s)}ds\right)\right]\leq\tilde{E}\left[\exp\left(\frac{1}{2}\int_0^{R\wedge\zeta}\frac{2(ch'(Y_s))^- - h''(Y_s)}{h(Y_s)}ds\right)\right]$$

Let $\mathcal{W}^{c,y}$ denote the law of the process $\tilde{Y}$ starting at y, where $d\tilde{Y}_t=d\beta_t+cdt$ and gets killed at hitting 0 or r. Thus,
$$\tilde{E}\left[\exp\left(\frac{1}{2}\int_0^{R\wedge\zeta}\frac{2(ch'(Y_s))^- - h''(Y_s)}{h(Y_s)}ds\right)\right]=\mathcal{W}^{c,X_0}[\exp(C_R)],$$
where C is the positive continuous additive functional of $\tilde{Y}$ with $dC_t=\frac{1}{2}\frac{2(ch'(\tilde{Y}_t))^- - h''(\tilde{Y}_t)}{h(\tilde{Y}_t)}\mathbf{1}_{[t<\tilde{\zeta}]}dt$.

Note that the potential function u_C of C is given by
$$u_C(x)=\mathcal{W}^x[C_\infty]=\int_0^r v(x,y)\mu_C(y)\frac{d\tilde{m}}{dy},$$

where v is the potential density of $\tilde{Y}$, $\mu_C(y) = \frac{1}{2}\frac{(2ch'(y))^- - h''(y)}{h(y)}$, and $d\tilde{m}$ is the associated speed measure of Y. Since a scale function and a speed measure of $\tilde{Y}$ can be chosen as

$$\tilde{s}(x) = \frac{1-e^{-2cx}}{2c} \text{ and } \tilde{m}(dx) = 2e^{2cx}dx,$$

where $\tilde{s}(x) = x$ if $c = 0$, we obtain for $x \le y$

$$v(x,y) = \frac{\tilde{s}(x)(\tilde{s}(r) - \tilde{s}(y))}{\tilde{s}(r)},$$

with $\frac{\tilde{s}(r)-\tilde{s}(y)}{\tilde{s}(r)}$ being interpreted as 1 if $\tilde{s}(r) = \infty$.

First observe that $v(x,y) = u(x,y)$ if $c = 0$. On the other hand, if $r = \infty$ and $c < 0$

$$v(y,y)e^{2cy} = \frac{e^{2cy}-1}{2c} \le y \wedge \frac{1}{2|c|}. \tag{2.7}$$

Similarly, for $r < \infty$

$$v(y,y)e^{2cy} = \frac{e^{-2cr}}{2c(1-e^{-2cr})}(e^{2cy}-1)(e^{2c(r-y)}-1) \le K(c,r)y(1-\frac{y}{r}). \tag{2.8}$$

Thus, for some $K < \infty$,

$$\int_0^r v(y,y)\mu_C(y)2e^{2cy}dy \le K\int_0^r (u(y,y)\wedge 1)\frac{(2ch'(y))^- - h''(y)}{h(y)}dy < \infty, \tag{2.9}$$

by an application of (2.1) due to the bounds obtained via (2.7) and (2.8), and the assumption on the choice of c when $r = \infty$.

As

$$u_C(x) \le \int_0^r v(y,y)\mu_C(y)2e^{2cy}dy,$$

we deduce that u_c is bounded.

Now consider a decreasing sequence (D_n) of subsets of $(0,r)$ such that $D_n \to \emptyset$. Since

$$\int_0^r v(x,y)\mathbf{1}_{D_n}(y)\mu_C(y)2e^{2cy} \le \int_0^r v(y,y)\mathbf{1}_{D_n}(y)\mu_C(y)2e^{2cy}dy,$$

and the right side converges to 0 by the dominated convergence theorem due to (2.9), we establish that $\mu_C \in \mathbf{K}_1(\tilde{Y})$ (see Definition 2.2 in Chen [11]) by Proposition 2.4 in [11]. Therefore, by Proposition 2.3 in Chen and Song [12] we arrive at $\sup_{y\in(0,r)} \mathcal{W}^{c,y}\exp(C_t) \le d_1e^{d_2t}$ for some constants d_1 and d_2. This proves the claim.

(3) Since the semigroup is self-dual with respect to the speed measure, for any non-negative measurable f we have

$$\int_0^r dy2h^2(y)e^{2cy}f(y)E^{h,y}\int_0^t \frac{e^{-s}\mathbf{1}_{[Y_s\in D]}}{h^{2+p}(Y_s)}ds = \int_D dy2h^2(y)e^{2cy}\frac{1}{h^{2+p}(y)}E^{h,y}\int_0^t e^{-s}f(Y_s)ds$$
$$\le \int_D dy\frac{2e^{2cy}}{h^p(y)}E^{h,y}\int_0^t f(Y_s)ds,$$

where $D := \{y : h(y) < 1 \wedge \frac{1}{2}\|h\|_\infty\}$.

In particular, when $f(y)=q(\varepsilon,y,y^*)$ for some $\varepsilon>0$, where q is the transition density of Y with respect to its speed measure, we obtain

$$\int_0^r dy 2h^2(y)e^{2cy}q(\varepsilon,y,y^*)E^{h,y}\int_0^t \frac{e^{-s}\mathbf{1}_{[Y_s\in D]}}{h^{2+p}(Y_s)}ds \leq \int_D dy 2e^{2cy}h^{-p}(y)E^{h,y}(L^{y*}_{t+\varepsilon})$$
$$\leq E^{h,y^*}(L^{y*}_{t+\varepsilon})\int_D dy2e^{2cy}h^{-p}(y),$$

where L^{y*} is the diffusion local time with respect to the speed measure. Letting $\varepsilon\to 0$ we arrive at

$$E^{h,y^*}\int_0^t \frac{e^{-s}\mathbf{1}_{[Y_s\in D]}}{h^{2+p}(Y_s)}ds \leq E^{h,y^*}(L^{y*}_t)\int_D dy2e^{2cy}h^{-p}(y)<\infty,$$

provided $y\mapsto E^{h,y}\int_0^t \frac{e^{-s}\mathbf{1}_{[Y_s\in D]}}{h^{2+p}(Y_s)}ds$ is lower semi-continuous. Note that the finiteness of the integral on the right hand side follows from the fact that $|h'(y)|\geq\alpha$ for some $\alpha>0$ on D.

Observe that

$$E^{h,y}\int_0^t \frac{e^{-s}\mathbf{1}_{[Y_s\in D]}}{h^{2+p}(Y_s)}ds=\phi(y)-e^{-t}E^{h,y}(\phi(Y_t)),$$

where

$$\phi(y):=E^{h,y}\int_0^\infty ds\frac{e^{-s}\mathbf{1}_{[Y_s\in D]}}{h^{2+p}(Y_s)}ds=\int_D \frac{2e^{2cz}v_1(y,z)}{h^p(z)}dz,$$

where v_1 is the 1-potential density of Y. Since v_1 is jointly continuous (see Paragraphs 10-11 in Chapter II of [7]), the claimed semi-continuity follows.

Since

$$E^{h,y^*}\int_0^t \frac{1}{h^{2+p}(Y_s)}ds \leq e^t E^{h,y^*}\int_0^t \frac{e^{-s}\mathbf{1}_{[Y_s\in D]}}{h^{2+p}(Y_s)}ds+K,$$

for some K, the claim follows from the arbitrariness of y^*.

□

3. Weak convergence of the BEM scheme

Consider the following stochastic differential equation on a filtered probability space $(\Omega,\mathcal{G},(\mathcal{G}_t)_{t\in[0,T]},\mathbb{P})$ satisfying the usual conditions:

$$X_t=X_0+\int_0^t\sigma(X_s)dW_s+\int_0^t\mu(X_s)ds$$

where $X_0\in(0,r)$, σ and μ are bounded and Lipschitz on $(0,r)$, and $\sigma(x)>\varepsilon$ for all $x\in(0,r)$. Let $\tau:=\inf\{t\geq 0: X_t\notin(0,r)\}$ for some $\varepsilon>0$. We are interested in a numerical approximation for $\mathbb{E}[\tilde{g}(X_T)\mathbf{1}_{[T<\tau]}]$, for a sufficiently regular $\tilde{g}$.

Observe that by a Girsanov transformation we can rewrite the above expression in terms of a diffusion process satisfying the conditions in earlier sections. Indeed, defining $\mathbb{Q}$ on $\mathcal{G}$ via

$$\frac{d\mathbb{Q}}{d\mathbb{P}}=\exp\Big(-\int_0^T\frac{\mu(X_s)}{\sigma(X_s)}dW_s-\frac{1}{2}\int_0^T\frac{\mu^2(X_s)}{\sigma^2(X_s)}ds\Big)$$

renders X solve $dX_t=\sigma(X_s)dB_s$, for a $\mathbb{Q}$-Brownian motion B. Therefore,

$$\mathbb{E}[\tilde{g}(X_T)\mathbf{1}_{[T<\tau]}]=\exp(-F(X_0))\mathbb{E}^{\mathbb{Q}}\Big[g(X_T)\exp\Big(\int_0^T\sigma^2(X_t)b(X_t)dt\Big)\mathbf{1}_{[T<\tau]}\Big],\text{ where}$$
$$g(x)=\tilde{g}(x)\exp(F(x)),F(x)=\int_c^x\frac{\mu(y)}{\sigma^2(y)}dy, b=-\frac{1}{2}\Big\{\big(\frac{\mu}{\sigma^2}\big)'+\frac{\mu^2}{\sigma^4}\Big\},$$

and $c \in (0, r)$.

Thus, we may assume $\mu \equiv 0$ and consider

$$E^{X_0}\Big[g(X_T)\exp\Big(\int_0^T \sigma^2(X_t)b(X_t)dt\Big)\mathbf{1}_{[T<\zeta]}\Big]$$
$$= E^{h,X_0}\Big[\frac{h(X_0)g(X_T)}{h(X_T)}\exp\Big(\int_0^T \sigma^2(X_t)\Big\{b(X_t)+\frac{h''(X_t)}{2h(X_t)}\Big\}dt\Big)\Big],$$

where X is a process satisfying Assumption 2.1, b is bounded, $\varepsilon < \sigma < K_\sigma$ and g is sufficiently regular.

We shall next introduce a set of conditions on the function h as well as the payoff g that will be needed for our analysis. The following proposition motivates some of the conditions stated in Assumption 3.1. Its proof is delegated to the end of this section.

Proposition 3.1. *Suppose $b \in C_b^4((0,r),\mathbb{R})$, $\sigma \in C_b^4((0,r)$, $h \in \mathcal{H}$ with*

$$\frac{|h^{(k)}|}{h} < \frac{K_h}{h^{k-2+p}}, \qquad k \in \{2,3,4\},$$

for some K_h and $p \in (0,1)$, $g \in C_b^6((0,r),\mathbb{R})$ is a bounded function with $g^{(k)}(0) = 0$ (and $g^{(k)}(r) = 0$ if $r < \infty$) for $k \in \{0,1,2,3,4\}$, and define for $t \le T$

$$v(T-t,x) := E^{h,x}\Big[\frac{g(X_t)}{h(X_t)}\exp\Big(\int_0^t \sigma^2(X_s)\Big\{b(X_s)+\frac{h''(X_s)}{2h(X_s)}\Big\}ds\Big)\Big]. \tag{3.1}$$

Then

$$v_t + \frac{\sigma^2}{2}v_{xx} + \sigma^2\frac{h'}{h}v_x = -\sigma^2 v\Big(b+\frac{h''}{2h}\Big). \tag{3.2}$$

Moreover, v and v_t are uniformly bounded and there exists a constant K such that

$$\sup_{t\le T}\Big|\frac{\partial^k}{\partial x^k}v_t(t,x)\Big| + \sup_{t\le T}\Big|\frac{\partial^k}{\partial x^k}v(t,x)\Big| \le Kh^{2-p-k}(x), \qquad k\in\{1,2\}.$$

In view of the above proposition, and for the convenience of the reader, we collect all the assumptions needed in Assumption 3.1 below to prove our convergence result.

Assumption 3.1. *The functions σ, b, h and g satisfy the following regularity conditions.*

(1) $h \in \mathcal{H} \cap C^4((0,r),(0,\infty))$ such that

$$\frac{|h^{(k)}|}{h} < \frac{K_h}{h^{p+k-2}}, \qquad k \in \{2,3,4\},$$

for some K_h and $p \in [0,\frac{1}{2}]$.

(2) $\sigma \in C_b^2((0,r),(0,\infty))$ is bounded away from 0, i.e. $\exists\varepsilon > 0$ s.t $\sigma(x) > \varepsilon$ for all $x \in (0,r)$.

(3) $b \in C_b^2((0,r),\mathbb{R})$.

(4) $g \in C((0,r),\mathbb{R})$ is of polynomial growth with $g(0) = 0$ (and $g(r) = 0$ if $r < \infty$).

(5) The function v defined by (3.1) belongs to $C^{1,4}((0,r),\mathbb{R})$, satisfies (3.2) as well as the growth conditions

$$\sup_{t\le T}\Big|\frac{\partial^k}{\partial x^k}v_t(t,x)\Big| + \sup_{t\le T}\Big|\frac{\partial^k}{\partial x^k}v(t,x)\Big| \le K(1+x^m)h^{2-p-k}(x), \qquad k\in\{1,2\},$$

for some constant K and integer $m \ge 0$.

Remark 3.1. *The first condition on the derivatives of h is not restrictive for practical purposes. Indeed, if a given $h \in \mathcal{H} \cap C^4((0,r),(0,\infty))$ does not satisfy this condition, one can always linearise this concave function near the boundaries at which h vanishes to obtain a new concave function satisfying the stated condition.*

Theorem 3.1. *Consider the BEM scheme defined by (2.4) as well as the associated error*

$$e(N) := \frac{g(\widehat{X}_T)}{h(\widehat{X}_T)} \exp\left(\sum_{n=0}^{N-1} \frac{T}{N}\sigma^2(\widehat{X}_{t_n})\Big\{b(\widehat{X}_{t_n})+\frac{h''(\widehat{X}_{t_n})}{2h(\widehat{X}_{t_n})}\Big\}\right) - \frac{g(X_T)}{h(X_T)} \exp\left(\int_0^T \sigma^2(X_t)\Big\{b(X_t)+\frac{h''(X_t)}{2h(X_t)}\Big\}dt\right).$$

Then $|E^{h,X_0}[e(N)| \leq \frac{KT}{N}$, for some constant K independent of N under Assumption 3.1.

Proof. Let $\pi_0(s) = 1$,

$$\pi_k(s) := \exp\left(\sum_{n=0}^{k-1} s\sigma^2(\widehat{X}_{t_n})\Big\{b(\widehat{X}_{t_n}) + \frac{h''(\widehat{X}_{t_n})}{2h(\widehat{X}_{t_n})}\Big\}\right), k = 1, \ldots, N,$$

with the convention that $\pi_k = \pi_k(TN^{-1})$, and observe that

$$\begin{aligned} E^{h,X_0}[e(N)] &= E^{h,X_0}[v(T,\widehat{X}_T)\pi_N] - v(0,X_0) = \sum_{n=0}^{N-1} E^{h,X_0}\left[v(t_{n+1},\widehat{X}_{t_{n+1}})\pi_{n+1} - v(t_n,\widehat{X}_{t_n})\pi_n\right] \\ &= \sum_{n=0}^{N-1} E^{h,X_0}\left[\pi_n\Big(v(t_{n+1},\widehat{X}_{t_{n+1}})\exp\Big(TN^{-1}\sigma^2(\widehat{X}_{t_n})\Big\{b(\widehat{X}_{t_n}) + \frac{h''(\widehat{X}_{t_n})}{2h(\widehat{X}_{t_n})}\Big\}\Big) - v(t_n,\widehat{X}_{t_n})\Big)\right] \end{aligned}$$

Next observe that

$$\begin{aligned} &E^{h,X_0}\left[\pi_n\Big(v(t_{n+1},\widehat{X}_{t_{n+1}})\exp\Big(TN^{-1}\sigma^2(\widehat{X}_{t_n})\Big\{b(\widehat{X}_{t_n}) + \frac{h''(\widehat{X}_{t_n})}{2h(\widehat{X}_{t_n})}\Big\}\Big) - v(t_n,\widehat{X}_{t_n})\Big)\Big|\mathcal{F}_n\right] \\ &= \pi_n E^{h,X_0}\left[\Big(v(t_{n+1},\widehat{X}_{t_{n+1}})\exp\Big(TN^{-1}\sigma^2(\widehat{X}_{t_n})\Big\{b(\widehat{X}_{t_n}) + \frac{h''(\widehat{X}_{t_n})}{2h(\widehat{X}_{t_n})}\Big\}\Big) - v(t_n,\widehat{X}_{t_n})\Big)\Big|\mathcal{F}_n\right]. \end{aligned}$$

Moreover, in view of (3.2) (in fact dividing both sides of the equality by σ^2) we have

$$v(t_{n+1},\widehat{X}_{t_{n+1}})\exp\Big(TN^{-1}\sigma^2(\widehat{X}_{t_n})\Big\{b(\widehat{X}_{t_n}) + \frac{h''(\widehat{X}_{t_n})}{2h(\widehat{X}_{t_n})}\Big\}\Big) - v(t_n,\widehat{X}_{t_n}) = M_{t_{n+1}} - M_{t_n} + I_1 + I_2 + I_3,$$

where M is a local martingale and

$$\begin{aligned} I_1 &= \int_{t_n}^{t_{n+1}} \frac{\pi_{n+1}(t-t_n)}{\pi_n(t-t_n)} \frac{\sigma^2(\widehat{X}_{t_n})v_x(t,\widehat{X}_t)\mu(t_n,\widehat{X}_{t_n};t,\widehat{X}_t)}{H_x^2(t_n,\widehat{X}_{t_n};t,\widehat{X}_t)}dt \\ I_2 &= \int_{t_n}^{t_{n+1}} \frac{\pi_{n+1}(t-t_n)}{\pi_n(t-t_n)}\sigma^2(\widehat{X}_{t_n})v_t(t,\widehat{X}_t)\Big(\frac{1}{\sigma^2(\widehat{X}_{t_n})} - \frac{1}{\sigma^2(\widehat{X}_t)H_x^2(t_n,\widehat{X}_{t_n};t,\widehat{X}_t)}\Big)dt \\ I_3 &= \int_{t_n}^{t_{n+1}} \frac{\pi_{n+1}(t-t_n)}{\pi_n(t-t_n)}\sigma^2(\widehat{X}_{t_n})v(t,\widehat{X}_t)\Big(b(\widehat{X}_{t_n}) + \frac{h''(\widehat{X}_{t_n})}{2h(\widehat{X}_{t_n})} - \Big(b(\widehat{X}_t) + \frac{h''(\widehat{X}_t)}{2h(\widehat{X}_t)}\Big)\frac{1}{H_x^2(t_n,\widehat{X}_{t_n};t,\widehat{X}_t)}\Big)dt. \end{aligned}$$

First note that M is martingale due to (4.5) by the hypothesis on v and that h is bounded. Moreover, Lemma 4.3 shows (for a generic constant K that may change from line to line albeit remaining bounded uniformly in N) such that

$$
\begin{aligned}
&\Big|E^{h,X_0}[I_1+I_2+I_3|\mathcal{F}_n]\Big| \leq K\frac{T}{N}E^{h,X_0}\Bigg[\int_{t_n}^{t_{n+1}}\frac{\sigma^2(\widehat{X}_{t_n})(h^{-2-p}(\widehat{X}_s)+\widehat{X}_s^m)}{H_x^2(t_n,\widehat{X}_{t_n};s,\widehat{X}_s)}ds\Bigg|\mathcal{F}_n\Bigg]\\
&+KE^{h,X_0}\Bigg[\int_{t_n}^{t_{n+1}}dt\frac{\pi_{n+1}(t-t_n)}{\pi_n(t-t_n)}\sigma^2(\widehat{X}_{t_n})\int_{t_n}^{t}\frac{\sigma(\widehat{X}_{t_n})^2(h^{-2-p}(\widehat{X}_s)+\widehat{X}_s^m)}{H_x^2(t_n,\widehat{X}_{t_n};s,\widehat{X}_s)}ds\Bigg|\mathcal{F}_n\Bigg]\\
&-KE^{h,X_0}\Bigg[\int_{t_n}^{t_{n+1}}dt\frac{\pi_{n+1}(t-t_n)}{\pi_n(t-t_n)}\frac{h''}{h}(\widehat{X}_{t_n})\sigma^2(\widehat{X}_{t_n})\int_{t_n}^{t}\frac{\sigma(\widehat{X}_{t_n})^2(h^{-p}(\widehat{X}_s)+\widehat{X}_s^m)}{H_x^2(t_n,\widehat{X}_{t_n};s,\widehat{X}_s)}ds\Bigg|\mathcal{F}_n\Bigg]\\
&-KE^{h,X_0}\Bigg[\int_{t_n}^{t_{n+1}}dt\frac{\pi_{n+1}(t-t_n)}{\pi_n(t-t_n)}\frac{h''}{h}(\widehat{X}_{t_n})\sigma^2(\widehat{X}_{t_n})\int_{t_n}^{t}\frac{\sigma(\widehat{X}_{t_n})^2(h^{-2}(\widehat{X}_s)+\widehat{X}_s^m)(s-t_n)}{H_x^2(t_n,\widehat{X}_{t_n};s,\widehat{X}_s)}ds\Bigg|\mathcal{F}_n\Bigg]\\
&\leq K\frac{T}{N}E^{h,X_0}\Bigg[\int_{t_n}^{t_{n+1}}\frac{\sigma^2(\widehat{X}_{t_n})(h^{-2-p}(\widehat{X}_s)+\widehat{X}_s^m)}{H_x^2(t_n,\widehat{X}_{t_n};s,\widehat{X}_s)}ds\Bigg|\mathcal{F}_n\Bigg]\\
&+E^{h,X_0}\Bigg[\int_{t_n}^{t_{n+1}}\Big(1-\exp\big((s-t_n)\sigma^2(\widehat{X}_{t_n})\frac{h''}{2h}(\widehat{X}_{t_n})\big)\Big)\frac{\sigma(\widehat{X}_{t_n})^2(h^{-p}(\widehat{X}_s)+\widehat{X}_s^m)}{H_x^2(t_n,\widehat{X}_{t_n};s,\widehat{X}_s)}ds\Bigg|\mathcal{F}_n\Bigg]\\
&+K\frac{T}{N}E^{h,X_0}\Bigg[\int_{t_n}^{t_{n+1}}\frac{\sigma(\widehat{X}_{t_n})^2(h^{-2}(\widehat{X}_s)+\widehat{X}_s^m)}{H_x^2(t_n,\widehat{X}_{t_n};s,\widehat{X}_s)}ds\Bigg|\mathcal{F}_n\Bigg],
\end{aligned}
$$

where we have used the boundedness of π_{n+1}/π_n several times and the last two lines follow from the interchange of the order of integration on the third and the fourth lines.

This proves the assertion in view of Theorem 4.1 and, in particular (4.5) and (4.7), since (π_n)s are non-negative and uniformly bounded, and $H_x \geq 1$. □

We end this section with

Proof of Proposition 3.1. Note that $v(T-t,x)=\frac{u(T-t,x)}{h(x)}$, where

$$u(T-t,x):=E^x\Big[g(X_t)\exp\Big(\int_0^t\sigma^2(X_s)b(X_s)ds\Big)\mathbf{1}_{[t<\zeta]}\Big].$$

Note that $u(t,0)=0$ for $t\leq T$. Moreover, it follows from Theorem 5.2 in Ladyženskaja et al. [31] that u is the unique solution of

$$u_t+\frac{1}{2}\sigma^2u_{xx}+\sigma^2ub=0, \tag{3.3}$$

and that

$$\sup_{t\leq T,x\in(0,r)}\Big|\frac{\partial^l}{\partial x^l}\frac{\partial^k}{\partial t^k}u\Big|<\infty,\quad 0\leq 2k+l\leq 5. \tag{3.4}$$

Also note that since

$$
\begin{aligned}
&E^x\Big[g(X_t)\exp\Big(\int_0^t\sigma^2(X_s)b(X_s)ds\Big)\mathbf{1}_{[t<\zeta]}\Big]=g(x)\\
&+\frac{1}{2}E^x\Big[\int_0^t\sigma^2(X_u)\exp\Big(\int_0^u\sigma^2(X_s)b(X_s)ds\Big)(g''(X_u)+2g(X_u)b(X_u))\mathbf{1}_{[u<\zeta]}du\Big],
\end{aligned}
$$

we have

$$u_t(t,x) = -\frac{1}{2}E^x\left[\sigma^2(X_{T-t})\exp\Big(\int_0^{T-t}\sigma^2(X_s)b(X_s)ds\Big)(g''(X_{T-t})+2g(X_{T-t})b(X_{T-t}))\mathbf{1}_{[u<\zeta]}\right]. \tag{3.5}$$

In particular, $u_t(\cdot,0)=0$, which in turn implies $u_{xx}(\cdot,0)=0$. Analogous boundary conditions also holds at r if r is finite.

Let $w:=u_t$ and note that w solves (3.3) with the boundary condition $w(t,0)=0$ and $w(T,\cdot)=-\frac{1}{2}\sigma^2g''-\sigma^2gb$. Using the stochastic representation in (3.5) and analogous arguments we again arrive at w_t vanishing at finite boundaries .

Using the PDE for u it is straightforward to establish that v solves (3.2) and is bounded. Moreover, as $v_x=\frac{hu_x-uh'}{h^2}$, using integration by parts we arrive at

$$v_x(t,x)=\frac{\int_0^x\{h(y)u_{xx}(t,y)-u(t,y)h''(y)\}dy}{h^2(x)}$$

Since $h'(0)<\infty$ and u and u_{xx} vanish at 0 (and are jointly continuous near $t=T$), there exists a neighbourhood of 0 in which $|h''|(y)\leq Kh^{1-p}(y)\leq K^2y$, $|u(\cdot,y)|+|u_{xx}(\cdot,y)|<Ky$ (due to Lipschitz continuity), and $h(y)>cy$. Thus, whenever x belongs to this neighbourhood, we have

$$\frac{v_x(t,x)}{h^{1-p}(x)}\leq\frac{K\int_0^x\{y(Ky+K^2y^{1-p})\}dy}{c^{3-p}x^{3-p}}=\frac{K^2/3x^3+K^3/(3-p)x^{3-p}}{c^{3-p}x^{3-p}}.$$

Thus, v_x/h^{1-p} is bounded near 0. Analogous considerations when $r<\infty$ shows that the ratio is bounded over $(0,r)$.

Next observe that v_t is bounded since u_t vanishes at finite boundaries and u_{tx} is bounded. In particular, v_th^p remain bounded near finite boundaries (uniformly in t). Multiplying (3.2) by h^p and using the fact that v_x/h^{1-p} is bounded demonstrate that

$$\sup_{t\leq T,x\in(0,r)}|v_{xx}(t,x)h^p(x)|<\infty.$$

Finally, since $v_t=\frac{w}{h}$, repeating the above arguments and using the fact that w_{xx} vanish at finite boundaries and is Lipschitz continuous in view of (3.4), we deduce v_{tx}/h^{1-p} is bounded. Similar arguments (due to the boundedness of $w_{tx}=u_{ttx}$ in view of (3.4) also lead to

$$\sup_{t\leq T,x\in(0,r)}|v_{txx}(t,x)h^p(x)|<\infty.$$

□

4. Moment estimates for the continuous BEM scheme

In this section we obtain some moment estimates, including inverse ones, that are necessary to establish the weak rate of convergence (see Section 3). We start with the following consequence of Ito's formula.

Lemma 4.1. *Suppose that $h\in C_b^2((0,r),(0,\infty))$, $h^{(3)}$ exists and satisfies $|h^{(3)}|\leq K(1+h^{-p})$ for some constant K and $p\in[0,1)$. Consider the BEM scheme defined by (2.4) for $h\in\mathcal{H}$. Then*

$$d\widehat{X}_t=\frac{\sigma(\widehat{X}_{t_n})}{H_x(t_n,\widehat{X}_{t_n};t,\widehat{X}_t)}dW_t+\frac{\sigma^2(\widehat{X}_{t_n})}{H_x^2(t_n,\widehat{X}_{t_n};t,\widehat{X}_t)}\Big\{\frac{h'}{h}(\widehat{X}_t)+\mu(t_n,\widehat{X}_{t_n};t,\widehat{X}_t)\Big\}dt,\quad t\in(t_n,t_{n+1}], \tag{4.1}$$

where

$$H(t_n,z;t,x):=x-\sigma^2(z)(t-t_n)\frac{h'}{h}(x)$$
$$\mu(t_n,z;t,x):=(H_x(t_n,z;t,x)-1)\frac{h'}{h}(x)+\frac{1}{2}\frac{\sigma^2(z)(t-t_n)}{H_x(t_n,z;t,x)}\left(\frac{h'}{h}\right)''(x).$$

Consider the sets $O_1:=\{x:h'(x)>0\}$ *and* $O_2:=\{x:h'(x)<0\}$. *Then*

$$\inf_{x\in O_1}\mu(t_n,z;t,x)\geq c_1 \text{ and } \sup_{x\in O_2}\mu(t_n,z;t,x)\leq c_2$$

for some constants $c_1\leq 0\leq c_2$ *that do not depend on* t_n,t *or* z. *In particular,* $c_1=0$ *when* $h(x)=x$.

Proof. The decomposition (4.1) follows from Ito's formula and straightforward calculations regarding the derivatives of the inverse function.

To prove the second assertion first observe that $H_x(t,x)-1=-\sigma^2(z)(t-t_n)\left(\frac{h'}{h}\right)'(x)\geq 0$, where we drop the dependency on t_n and z to ease the exposition.

Observe that

$$\mu=-\frac{\sigma^2(z)(t-t_n)\left(\frac{h'}{h}\right)'}{H_x}\left(H_x\frac{h'}{h}-\frac{1}{2}\frac{\left(\frac{h'}{h}\right)''}{\left(\frac{h'}{h}\right)'}\right), \tag{4.2}$$

and that the claim follows immediately if $h(x)=x$ since the term in the parenthesis in (4.2) becomes non-negative . Thus, it remains to show the assertion when $h\in\mathcal{H}_0$.

First consider the case $r=\infty$, and let $u:=\frac{h'}{h}$ and note that $\lim_{x\to\infty}u'(x)=0$ by Lemma 2.1. Moreover, $|u'(x)|\leq Kx^{-2}$ for some $K<\infty$, which in turn implies

$$\lim_{x\to\infty}\frac{\log(-u'(x))}{x}=0=\lim_{x\to\infty}\frac{u''(x)}{u'(x)}, \tag{4.3}$$

where the second equality is an application if L'Hospital's rule. Thus,

$$-\frac{1}{2}\frac{\left(\frac{h'}{h}\right)''}{\left(\frac{h'}{h}\right)'}>c \text{ on } (\frac{x^*}{2},\infty)$$

for some $c<0$ where $x^*:=\inf\{x:h'(x)=0\}>0$ by Lemma 2.1.

An alternative representation for μ is given by

$$\mu=\sigma^2(z)(t-t_n)\left(-\frac{h'}{h}\left(\frac{h'}{h}\right)'\frac{1+H_x}{H_x}+\frac{1}{2H_x}\frac{h'''h-h''h'}{h^2}\right). \tag{4.4}$$

Thus, we will be done if

$$r(t,x):=\frac{\sigma^2(z)(t-t_n)}{2H_x}\frac{h'''h-h''h'}{h^2}$$

is bounded from below on $(0,\frac{x^*}{2})$. Indeed, as h' is bounded away from 0 on this interval, the hypothesis on h''' implies

$$r(t,x)\geq -K\frac{\sigma^2(z)(t-t_n)\left(\frac{h'}{h}\right)^2}{1+\sigma^2(z)(t-t_n)(\frac{h'}{h})^2}$$

leading to the desired lower bound.

When $r < \infty$, we have in particular that σ is bounded. Moreover, $|r(t,x)| \le K\frac{\sigma^2(z)(t-t_n)\frac{1}{h^2}}{2H_x}$, for some constant K, which renders r bounded. Observing that the remaining terms in (4.4) has the correct sign completes the proof. □

The next result is a key comparison result that relates the inverse moments of the BEM scheme to those of the process (2.5) and thereby provide estimates that are valid uniformly in N.

Lemma 4.2. *Suppose that h satisfies the conditions of Lemma 4.1, σ is bounded, $r = \infty$, and consider the BEM scheme defined by* (2.4) *for $h \in \mathcal{H}$. Then for any non-decreasing and measurable function ϕ that does not change sign, we have*

$$E^{h,X_0}[\phi(\widehat{X}_{A_t^{-1}})] \ge E^{h,X_0}[\phi(Y_t)],$$

where Y is the process defined by (2.5) *with $c = c_1$, c_1 is as in Lemma 4.1, and A is a continuous time-change defined by $A_0 = 0$ and*

$$dA_t = \frac{\sigma^2(\widehat{X}_{t_n})}{H_x^2(t_n, \widehat{X}_{t_n}; t, \widehat{X}_t)}dt, \qquad t \in (t_n, t_{n+1}].$$

Moreover, $Q^{h,X_0}[A_t \le t\|\sigma\|_\infty^2] = 1$.

Proof. Consider the process $\widehat{Y}$ defined by $\widehat{Y}_t = \widehat{X}_{A_t^{-1}}$.

Dambis, Dubins-Schwarz Theorem (cf. Theorem V.1.6 in Revuz and Yor [38]) yields

$$d\widehat{Y}_t = d\beta_t + \Big(\frac{h'}{h}(\widehat{Y}_t) + \mu_t\Big)dt, \quad t \in (t_n, t_{n+1}],$$

where $\mu_t \ge c_1$ and β is a standard Brownian motion adapted to the filtration $(\mathcal{F}_{A_t^{-1}})_{t\ge 0}$.

Then the comparison theorem for stochastic differential equations (cf. Theorem 2.10 in [10]) show that

$$P^{h,X_0}[\widehat{Y}_t \ge Y_t, t \le T] = 1,$$

where

$$Y_t = X_0 + \beta_t + \int_0^t \Big(\frac{h'}{h}(Y_s) + c_1\Big)ds.$$

Since $H_x \ge 1$, it follows that $A_t \le \|\sigma\|_\infty^2 t$. This completes the proof. □

The main moment estimates are collected in the following theorem.

Theorem 4.1. *Suppose that h satisfies the conditions of Lemma 4.1, σ is bounded, and consider the BEM scheme defined by* (2.4)*. Then for any $T > 0$ and $p \in [0,1)$, the following statements are valid:*

(1) For each $m \in \mathbb{N}$

$$\sup_{t\le T,N} E^{h,X_0}\left[\frac{1}{h}(\widehat{X}_t) + \sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}} \frac{\sigma^2(\widehat{X}_{t_n})h^{-2-p}(\widehat{X}_t)}{H_x^2(t_n, \widehat{X}_{t_n}; t, \widehat{X}_t)}dt + |\widehat{X}_t|^m\right] < \infty. \tag{4.5}$$

(2) For each n

$$\text{ess sup}_{\tau\in\mathcal{T}_n} E^{h,X_0}\left[\frac{1}{h}(\widehat{X}_\tau) + X_\tau^m \Big| \mathcal{F}_{t_n}\right] < \infty, \tag{4.6}$$

where $m \ge 0$ is an integer and $\mathcal{T}_n := \{\tau : \tau$ is a stopping time such that $\tau \in [t_n, t_{n+1}], Q^{h,X_0}$-a.s..$\}$.

(3) Suppose further that $p \le \frac{1}{2}$ and that $\frac{h''}{h^{1-p}}$ is bounded. Then for each $n \in \mathbb{N}$ and $m \ge 0$

$$E^{h,X_0}\Big[\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}}\Big(1-\exp\big((s-t_n)\sigma^2(\widehat{X}_{t_n})\frac{h''}{2h}(\widehat{X}_{t_n})\big)\Big)\frac{\sigma(\widehat{X}_{t_n})^2(h^{-p}(\widehat{X}_s)+\widehat{X}_s^m)}{H_x^2(t_n,\widehat{X}_{t_n};s,\widehat{X}_s)}ds\Big] < \frac{KT}{N}, \quad (4.7)$$

where K is independent of N.

Proof of the above theorem is lengthy and is delegated to the Appendix. We end this section with the following lemma that will be useful in our PDE approach to weak convergence rate in the following section.

Lemma 4.3. *Suppose that h satisfies the conditions of Lemma 4.1, σ is bounded, and consider the BEM scheme defined by (2.4). Then for any $T > 0$ the following statements are valid:*

(1) Let $p \in [0,1)$ and $m \ge 0$ be an integer. For each n

$$E^{h,X_0}\Big[\int_{t_n}^{t_{n+1}}\Big|\frac{h^{1-p}(\widehat{X}_t)(1+\widehat{X}_t^m)\mu(t_n,\widehat{X}_{t_n};t,\widehat{X}_t)}{H_x^2(t_n,\widehat{X}_{t_n};t,\widehat{X}_t)}\Big|dt\Big|\mathcal{F}_n\Big]$$
$$\le \frac{KT}{N}E^{h,X_0}\Big[\int_{t_n}^{t_{n+1}}\frac{\sigma^2(\widehat{X}_{t_n})(h^{-2-p}(\widehat{X}_t)+\widehat{X}_t^m)}{H_x^2(t_n,\widehat{X}_{t_n};t,\widehat{X}_t)}dt\Big|\mathcal{F}_n\Big],$$

with K being a constant independent of n.

(2) Assume further that $h \in C^4((0,r),(0,\infty))$. Consider $p \in [0,1)$ and suppose

$$\frac{|h^{(k)}|}{h} < \frac{K}{h^{k-2+p}}, \qquad k \in \{2,3,4\},$$

for some K. Let $f \in C^2((0,r),\mathbb{R})$ be a bounded function such that

$$|f^{(k)}(x)| \le K(1+x^m)h^{2-p-k}(x), \qquad k \in \{1,2\},$$

for some $m \ge 0$. Then for each n and $t \in [t_n, t_{n+1}]$

$$\Big|E^{h,X_0}\Big[f(\widehat{X}_t)\Big\{\frac{h''}{h}(\widehat{X}_{t_n})-\frac{h''(\widehat{X}_t)}{H_x^2(t_n,\widehat{X}_{t_n};t,\widehat{X}_t)h(\widehat{X}_t)}\Big\}\Big|\mathcal{F}_n\Big]\Big|$$
$$\le KE^{h,X_0}\Big[\int_{t_n}^{t}\frac{\sigma(\widehat{X}_{t_n})^2(h^{-(2+p)}(\widehat{X}_s)+\widehat{X}_s^m)}{H_x^2(t_n,\widehat{X}_{t_n};s,\widehat{X}_s)}ds\Big|\mathcal{F}_n\Big]$$
$$-K\frac{h''}{h}(\widehat{X}_{t_n})E^{h,X_0}\Big[\int_{t_n}^{t}\frac{\sigma(\widehat{X}_{t_n})^2\big(h^{-p}(\widehat{X}_s)+\widehat{X}_s^m+(s-t_n)\big(h^{-2}(\widehat{X}_s)+\widehat{X}_s^m\big)\big)}{H_x^2(t_n,\widehat{X}_{t_n};s,\widehat{X}_s)}ds\Big|\mathcal{F}_n\Big]$$

for some constant K independent of n.

(3) Suppose f and h satisfy the conditions of the previous part and $b \in C_b^2((0,r),\mathbb{R})$. Then for each n and $t \in [t_n, t_{n+1}]$,

$$\Big|E^{h,X_0}\Big[f(\widehat{X}_t)\Big\{b(\widehat{X}_{t_n})-\frac{b(\widehat{X}_t)}{H_x^2(t_n,\widehat{X}_{t_n};t,\widehat{X}_t)}\Big\}dt\Big|\mathcal{F}_n\Big]\Big|$$
$$\le KE^{h,X_0}\Big[\int_{t_n}^{t}\frac{\sigma(\widehat{X}_{t_n})^2(h^{-2-p}(\widehat{X}_s)+\widehat{X}_s^m)}{H_x^2(t_n,\widehat{X}_{t_n};s,\widehat{X}_s)}ds\Big|\mathcal{F}_n\Big],$$

for some constant K independent of n.

Proof. (1) It follows directly from the definition of μ and the hypothesis on h''' that

$$h^{1-p}(\widehat{X}_t)|\mu(t_n,\widehat{X}_{t_n};t,\widehat{X}_t)| \leq K\sigma^2(\widehat{X}_{t_n})(t-t_n)h^{-2-p}(\widehat{X}_t), \quad t\in[t_n,t_{n+1}],$$

for some K. Also note that if $m \geq 1$ and $r=\infty$, there exists a K such that we have $x^m h^{-(2+p)} \leq K h^{m-(2+p)}$ for $x\in[0,1]$. An analogous bound can be obtained near r when r is finite. Thus,

$$x^m h^{-(2+p)} \leq K(x^m + h^{-(2+p)}). \tag{4.8}$$

(2) Let $\mu_s := \mu(t_n,\widehat{X}_{t_n};s,\widehat{X}_s)$, $u:=\frac{h'}{h}$, and $\eta_s := H_x(t_n,\widehat{X}_{t_n};s,\widehat{X}_s)$. Then Ito's formula yields

$$f(\widehat{X}_t)\left(\frac{h''}{h}(\widehat{X}_{t_n}) - \frac{h''(\widehat{X}_t)}{h(\widehat{X}_t)\eta_t^2}\right) = M_t + A_t,$$

where M is a local martingale with $M_{t_n}=0$ since $\eta_{t_n}=1$, and

$$\begin{aligned}
A_t = &\int_{t_n}^t \frac{\sigma^2(\widehat{X}_{t_n})f(\widehat{X}_s)}{2\eta_s^4}\left\{\frac{2h''h'}{h^2}(\widehat{X}_s)\mu_s + \frac{(h'')^2-hh^{(4)}}{h^2}(\widehat{X}_s)\right\}ds\\
&-\int_{t_n}^t \frac{\sigma(\widehat{X}_{t_n})^2 f(\widehat{X}_s)}{\eta_s^4}\frac{h^{(3)}}{h}(\widehat{X}_s)\left\{\mu_s + 2\sigma^2(\widehat{X}_{t_n})(s-t_n)\frac{u''(\widehat{X}_s)}{\eta_s}\right\}ds\\
&-\int_{t_n}^t \frac{\sigma^4(\widehat{X}_{t_n})(s-t_n)f(\widehat{X}_s)h''(\widehat{X}_s)}{h(\widehat{X}_s)\eta_s^5}\left\{2\mu_s u''(\widehat{X}_s) + \frac{3\sigma^2(\widehat{X}_{t_n})(s-t_n)(u'')^2(\widehat{X}_s)}{\eta_s} + u^{(3)}(\widehat{X}_s)\right\}ds\\
&+\int_{t_n}^t \left(\frac{h''}{h}(\widehat{X}_{t_n}) - \frac{h''(\widehat{X}_s)}{h(\widehat{X}_s)\eta_s^2}\right)\frac{\sigma^2(\widehat{X}_{t_n})}{\eta_s^2}\left\{f'(\widehat{X}_s)(u(\widehat{X}_s)+\mu_s) + \frac{1}{2}f''(\widehat{X}_s)\right\}ds\\
&+\int_{t_n}^t \frac{\sigma^2(\widehat{X}_{t_n})f'(\widehat{X}_s)}{\eta_s^4}\left\{\frac{h''h'-hh^{(3)}}{h^2}(\widehat{X}_s) - 2\frac{\sigma^2(\widehat{X}_{t_n})(s-t_n)u''(\widehat{X}_s)}{\eta_s}\frac{h''}{h}(\widehat{X}_s)\right\}ds.
\end{aligned}$$

Observe that the hypothesis on h implies that

$$|u^{(k)}| \leq K h^{-1-k}, \qquad k\in\{0,1,2,3\},$$

for some constant K. Moreover, $|\mu_s| \leq K\sigma^2(\widehat{X}_{t_n})(s-t_n)h^{-3}$ and $\sigma^2(\widehat{X}_{t_n})(s-t_n)h^{-2}\eta_s^{-1} \leq K$, for some other constant K that does not depend on s.

Thus, combined with the assumption on f we arrive at

$$\begin{aligned}
|A_t| \leq -K\frac{h''}{h}(\widehat{X}_{t_n})\int_{t_n}^t \frac{\sigma(\widehat{X}_{t_n})^2(1+\widehat{X}_s^m)}{H_x^2(t_n,\widehat{X}_{t_n};s,\widehat{X}_s)}\Big(h^{-p}(\widehat{X}_s)(1+(s-t_n)h^{-2}(\widehat{X}_s))\Big)ds\\
+K\int_{t_n}^t \frac{\sigma^2(\widehat{X}_{t_n})(1+\widehat{X}_s^m)}{H_x^2(t_n,\widehat{X}_{t_n};s,\widehat{X}_s)h^{2+p}(\widehat{X}_s)}ds
\end{aligned}$$

for some constant K. This in particular implies M is a martingale since we can deduce from the estimates (4.5) and (4.6) that the set $\{f(\widehat{X}_\tau)\frac{h''(\widehat{X}_\tau)}{h(\widehat{X}_\tau)\eta_\tau^2} : \tau \in (t_n,t_{n+1}] \text{ is a stopping time}\}$ is uniformly integrable as soon as we once again recall that $|h''/h| < Kh^{-p}$ for some $p<1$. Hence, the claim holds in view of (4.8).

(3) Applying Ito's formula and repeating the similar estimates yields the claim.

□

5. Numerical analysis

This section is dedicated to the numerical experiments illustrating the above technical analysis. As we shall see, one does not really need to satisfy all the conditions assumed in Theorem 3.1 in order to achieve the advertised convergence rate in practice. The experiments below will compare our methodology developed in this paper to standard numerical approaches for pricing of barrier options.

We shall consider the classical Black-Scholes model in the first part. As barrier option values are quite sensitive to the market skew/smile of volatility, the time-homogeneous *hyperbolic local volatility* model will also be studied in the second part.

Remark 5.1. *In this one dimensional setting, the value function of a "plain" knock-out option, i.e.* $E^x[g(X_T)\mathbf{1}_{[T<\zeta]}]$*, can be rather easily found by applying a finite difference scheme to the associated PDE with vanishing boundary conditions at accessible boundaries.*

The Monte Carlo BEM scheme introduced in this paper will have a clear advantage over the PDE method in higher dimensions. However, already in the one-dimensional case, it is quite flexible with respect to additional complications compared to the PDE method. In practice, a barrier- type payoff can be combined with various features like Asianing and forward start. As an example, one can consider

$$1_{\zeta>T}\left[\frac{1}{m}\sum_{i=1}^{N}X(T_i)-X(T_0)\right]_+$$

with $0<T_0<T_1<...<T_m=T$ *where the strike is not fixed today but at future date* T_0*.*

Computation of the price of the above derivate can be made without much extra effort using the BEM method provided the discretisation includes the time points T_i *for* $i=0,\dots,m$*.*

While the pricing of such a derivative can still be done via a PDE approach using a finite difference scheme, its implementation is relatively complex involving essentially a 3-d PDE solver: one dimension for the spot price X*, one to capture the possible values of the Asianing* $\frac{1}{N}\sum_{i=1}^{N}X(t_i)$*, and another to incorporate the possible values of the strike value at* t_0 *(see e.g Chapter 25 in Wilmott* [44]*, as well as Sections 6.2 and 7.1.10 in Musiela and Rutkowski* [34] *or Section 11 in De Weert* [14]*). Besides, this can be intensive in computation time.*

5.1. **Black-Scholes model for barrier options.** For expository purposes, let's assume the log price $X_t=\ln(S_t)$, under risk neutral probability $\mathbb{P}$, given by

$$dX_t=-\frac{1}{2}\sigma^2dt+\sigma dW_t,\quad X_0=x=0 \tag{5.1}$$

with volatility $\sigma>0$, bounded away from 0 (cf. point (2) in Assumption 3.1). We remark here that deterministic interest rates, dividend yield or borrowing costs can be incorporated without difficulty. The value of the barrier option with payoff $\tilde{g}$ is

$$price=\mathbb{E}^{\mathbb{P}}[\tilde{g}(X_T)\mathbb{1}_{\zeta>T}] \tag{5.2}$$

with $\zeta=\inf\{t>0:X_t\notin(\ell,r)\}$.

To remove the drift in (5.1), we follow the Girsanov transformation, described at beginning of Section 3, and obtain

$$dX_t=\sigma dW_t,\quad X_0=x \tag{5.3}$$

under $\mathbb{Q}$, where $\frac{d\mathbb{Q}}{d\mathbb{P}}=e^{-\frac{1}{8}\sigma^2T+\frac{1}{2}\sigma W_T}$. Consequently,

$$price=e^{\frac{1}{2}x-\frac{1}{8}\sigma^2T}\mathbb{E}^{\mathbb{Q}}[g(X_T)\mathbb{1}_{\zeta>T}],$$

with $g(x) = \tilde{g}(x)e^{-\frac{1}{2}x}$.

We shall perform a path transformation method described in earlier section that either produces a recurrent process or generates a transient process with infinite lifetime (see Theorem 2.1).

5.1.1. *Specification of the recurrent transformation.* For single barrier with l finite and $r = +\infty$, we shall choose $h(x) = e^{-l} - e^{-x}$ with $h'(x) = e^{-x}$ and $h''(x) = -e^{-x}$. Note that with this choice of h, Condition (1) of Assumption 3.1 is not satisfied for $k = 2$ and any $p \in [0, \frac{1}{2}]$.

Nevertheless, we will apply the implicit scheme (2.4) so that the price (5.2) is approximated by

$$price \approx e^{\frac{1}{2}x - \frac{1}{8}\sigma^2 T} h(x)\mathbb{E}^{h,x}\left[\frac{g}{h}(\widehat{X}_{t_N})e^{\frac{\sigma^2}{2}\frac{T}{N}\sum_{n=0}^{N-1}\frac{h''}{h}(\widehat{X}_{t_n})}\right]$$

and still obtain the optimal convergence rate in the numerical experiments.

Remark 5.2. *In the Black-Scholes model with $X_t = \ln(S_t)$, the H function is identical at each time step and needs to be computed once. In the implementation, we introduce a dense grid covering the interval (l, r), calculate the values of H on these points and H^{-1} is computed by piecewise constant approximation.*

5.1.2. *The transient transformation.* In the single barrier case of a down-and-out option, we can also consider transformation via $h(x) = x - l$ when l is finite and $r = +\infty$, as in Theorem 2.1. Under $Q^{h,x}$, the the process X defined in ((5.3)) follows

$$dX_t = \sigma dW_t + \frac{\sigma^2}{X_t - l}dt, \quad X_0 = x.$$

One advantage of this transformation is that the inverse of the function H appearing in the implicit scheme (2.4) can be computed analytically and is given by

$$H^{-1}(x) = \frac{1}{2}\left(\sqrt{4\sigma^2\frac{T}{N} + (x-l)^2} + x + l\right).$$

5.2. **Down and out put option.** For a down-and-out put barrier option, the payoff is given by $\max(K - S_T, 0)\mathbb{1}_{\zeta > T}$ where $\zeta := \inf\{t > 0 : S_t \notin (b, +\infty)\}$, $0 < b = e^l$, K is the option strike and T the maturity. As mentioned at the beginning of this section, to put our methodology in perspective we have also implemented two other approaches to the numerical pricing of the barrier option:

- Standard Euler without hitting probability: It consists of discretizing the SDE (5.1) according to the Euler scheme

$$\begin{aligned}\widehat{X}_0 &= \ln(S_0)\\ \widehat{X}_{t_{i+1}} &= \widehat{X}_{t_i} - \frac{1}{2}\sigma^2\frac{T}{N} + \sigma(W_{t_{i+1}} - W_{t_i}).\end{aligned} \tag{5.4}$$

  and evaluating $\tilde{g}(X_T)\mathbb{1}_{\zeta > T}$ by $\tilde{g}(\widehat{X}_{t_N})\mathbb{1}_{\zeta^N > T}$ where $\zeta^N = \inf(t_i > 0 : \widehat{X}_{t_i} \notin (\ell = \log(b), \infty)))$.

  This numerical scheme for barrier option pricing had been studied in [21], where it was shown to have a convergence rate of $\mathcal{O}(\frac{1}{\sqrt{N}})$. This loss of accuracy is mainly due to the fact that it is possible for X to cross the barriers l or r at some time t between grid points t_i and t_{i+1} and never be below the barrier at any of the dates t_i for $i = 1, .., N$.
- Standard Euler with hitting probability: Although this is still based on the Euler scheme simulations (5.4), it applies a further correction to remove the barrier crossing biases via the

conditional no-hitting probability $\hat{p}_i$ using the Brownian bridge technique (see e.g, section 1.1 of [21]). More precisely, the $\hat{p}_i$ are defined and can be computed analytically as

$$\hat{p}_i := \mathbb{P}[\forall t \in [t_i, t_{i+1}], \widehat{X}_t > l | \widehat{X}_{t_i} = x_i, \widehat{X}_{t_{i+1}} = x_{i+1}] = 1 - e^{-2\frac{(x_i - l)(x_{i+1} - l)}{\sigma^2 (t_{i+1} - t_i)}}$$

where the process $(\widehat{X}_t)_{0 \le t \le T}$ is the continuous Euler scheme which interpolates $(\widehat{X}_{t_i})_{0 \le i \le N}$ in the following way:

$$\forall t \in [t_i, t_{i+1}[: \qquad \widehat{X}_t = \widehat{X}_{t_i} - \frac{1}{2}\sigma^2 (t - t_i) + \sigma (W_t - W_{t_i}).$$

It then *corrects* the payoff $\tilde{g}(X_T)\mathbb{1}_{\zeta > T}$ by considering instead $\tilde{g}(\widehat{X}_{t_N}) \prod_{i=0}^{N-1} \hat{p}_i$. As shown in [22], this bias correction brings the convergence rate back to of order N^{-1}, which is the rate of weak convergence for the Euler-Maruyama scheme in the absence of killing. Moreover, in this specific Black-Scholes implementation, the simulation is exact, i.e no discretisation error occurs due to constant σ.

We shall next summarise the experiments details and comparison results.

5.2.1. *Set of parameters.* The numerical experiments are conducted using the following values for the parameters: $S_0 = 1$, $T = 1$ year, $l = \log(b = 0.8)$, $r = +\infty$ and $\sigma = 20\%$. For thoroughness, we have considered in-the-money ($K = 1.2$), at-the-money ($K = 1$) and out-the-money ($K = 0.9$) options. To reduce statistical noise, the simulations are run with 1 million Monte Carlo paths. The benchmark price is calculated analytically (see e.g Section 4.17.1 in Haug [23]).

As our final results do not show any significant dependency on the moneyness of the option, we shall only report the results for at-the-money (ATM) options. In particular the discrepancy between benchmark prices and the numerical value for ATM down-and-out put options is shown in Figure 1. We have not observed any stability issues with any of our h-transformation schemes. As discussed earlier, the standard Euler with hitting probability method has no discretisation error. The discrepancy is therefore essentially the statistical noise.

Our numerical results show the rapid convergence of the numerical approximation of prices given by the recurrent and transient transforms via the implicit scheme and demonstrate clearly its effectiveness over the standard Euler scheme without hitting probability correction. This confirms the findings of our theoretical analysis even without satisfying all the conditions of Theorem 3.1.

Moreover, the prices given by the recurrent and transient transforms are quite comparable as predicted by the theoretical analysis. Figure 2 show the log-log plot of the discrepancy associated to the recurrent and transient transforms respectively for ATM down-and-out put option, respectively. The respective numerical rates of convergence observed are 0.95 and 0.9.

5.3. **Time-homogeneous hyperbolic local volatility model.** Empirical asset returns distributions tend to exhibit fat tails (kurtosis) and skewness (asymmetric distribution). The *skew* or *smile* in implied volatility surfaces observed across various asset classes are market reality (see e.g Chapter 1 in Gatheral [19], Section 1.2 in Overhaus et al. [36] or Section 22.4 in Willmott [45] for a manifestation of these stylized facts). We need more convenient models than *Gaussian* models for the asset S able to produce more closely the implied volatility surfaces. Local volatility models, either parametric or non-parametric, (see e.g Dupire [17], Derman and Kani [16] or Rubinstein [40]), arguably capture the surface of implied volatilities more precisely than other approaches such as stochastic volatility models (see e.g Ren et al. in [37] or Romo [39]). Needless to say, the volatility surface has a significant impact on barrier option valuation. Indeed, the barrier hitting probability depends strongly on the dynamics of the volatility of the spot prices (see e.g Bossens [8]).

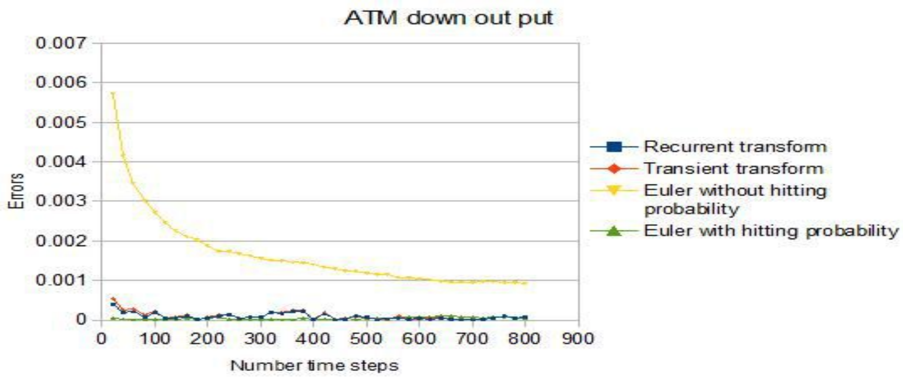

FIGURE 1. Absolute discrepancy between the benchmark price for ATM down-and-out put and those calculated with different numerical schemes when $S_0 = 1$, $K = 1$, $T = 1$ year, $l = \log(b = 0.8)$, $r = +\infty$ and $\sigma = 20\%$.

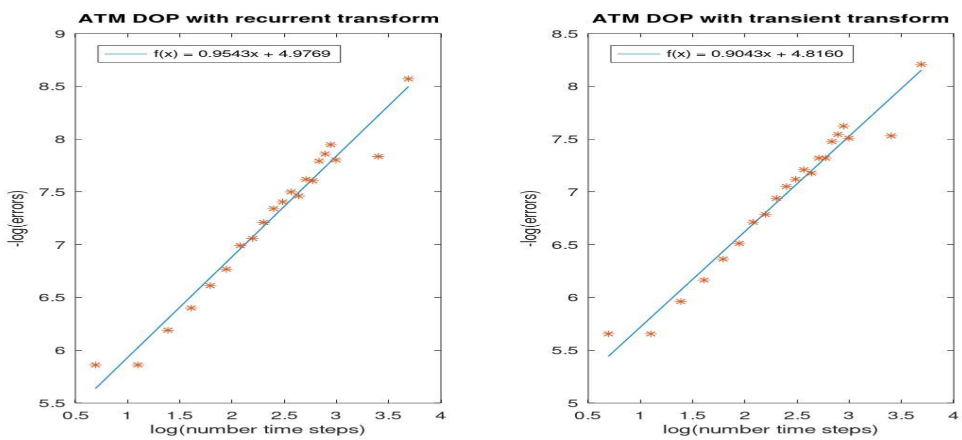

FIGURE 2. Log-log plot of the absolute discrepancy for ATM down-and-out put price with recurrent and transient transforms numerical scheme when $S_0 = 1$, $K = 1$, $T = 1$ year, $l = \log(b = 0.8)$, $r = +\infty$ and $\sigma = 20\%$.

For our analysis, we consider the time homogeneous hyperbolic local volatility model (HLV) where the dynamics of spot price under the risk neutral measure is given by

$$dX_t = \sigma(X_t)dW_t, \quad X_0 = 1$$

with

$$\sigma(x) = \nu\Big\{\frac{(1-\beta+\beta^2)}{\beta}x + \frac{(\beta-1)}{\beta}\big(\sqrt{x^2+\beta^2(1-x)^2}-\beta\big)\Big\}. \tag{5.5}$$

Above $\nu > 0$ is the level of volatility, $\beta \in (0, 1]$ is the skew parameter. First introduced in Jäckel [30] it behaves similarly to the Constant Elasticity of Variance (CEV) model and has been widely used in quantitative finance for numerical experiments in Hok et al. [25, 26, 24]. A practical advantage of this model is that zero is not an attainable boundary, which in turn avoids some numerical instabilities present in the CEV model when the underlying asset price is close to zero (see e.g. Andersen and Andreasen [4]). It corresponds to the Black-Scholes model for $\beta = 1$ and exhibits a skew for the implied volatility surface when $\beta \neq 1$. Figure 3 illustrates the impact of the parameter β on the skew of the volatility surface. We observe that the skew increases significantly with decreasing value of β. For example with $\nu = 0.3, \beta = 0.2$, the difference in volatility between strikes at 50% and at 100% is about 15%.

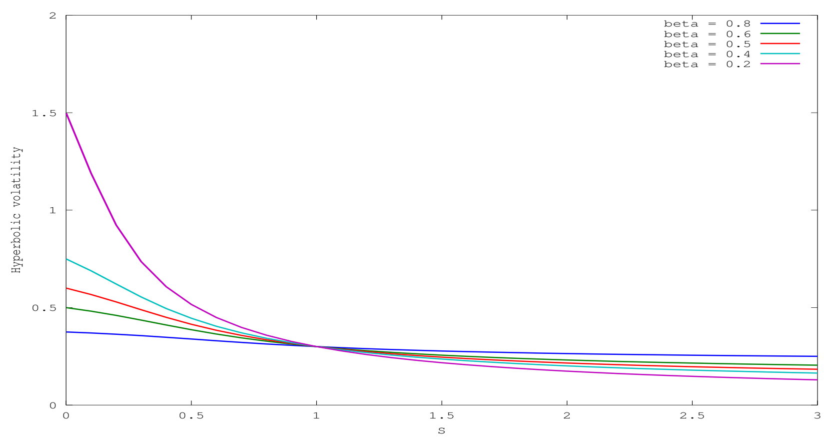

FIGURE 3. Impact of the value β on the hyperbolic local volatility for fixed volatility level $\nu = 0.3$.

5.3.1. *Down and up out double barrier call option.* In this implementation we shall set $h(x) = \frac{(x-l)(r-x)}{2(r-l)^2}$ with $h^{(1)}(x) = \frac{l+r-2x}{2(r-l)^2}$, $h^{(2)}(x) = -\frac{1}{(r-l)^2}$ and $h^{(3)}(x) = h^{(4)}(x) = 0$. Note that with this choice of h, Condition (1) in Assumption 3.1 is not satisfied for $k = 2$ and any $p \in [0, \frac{1}{2}]$. The associated BEM scheme will be then solved using bisection method with Octave vectorization for faster code execution. Consequently the price is approximated by

$$price \approx h(x)\mathbb{E}^{h,x}\Big[\frac{(\widehat{X}_{t_N} - K)_+}{h(\widehat{X}_{t_N})} e^{\frac{1}{2}\frac{T}{N}\sum_{n=0}^{N-1}\sigma^2(\widehat{X}_{t_n})\frac{h''}{h}(\widehat{X}_{t_n})}\Big]$$

For comparison, we compute also the numerical price given by the standard Euler scheme ((5.4)) with hitting probability where the expression is given as an infinite series in [21] [1]

$$\begin{aligned}\hat{p}_i &:= \mathbb{P}[\forall t \in [t_i, t_{i+1}], \widehat{X}_t \in (\ell, r) | \widehat{X}_{t_i} = x_i, \widehat{X}_{t_{i+1}} = x_{i+1}] \\ &= \mathbb{1}_{l<x_i,x_{i+1}<r} \sum_{n=-\infty}^{n=+\infty} \Big(e^{\frac{-2n(r-l)(n(r-l)+x_{i+1}-x_i)}{\sigma^2(t_{i+1}-t_i)}} - e^{\frac{-2(n(r-l)+x_i-r)(n(r-l)+x_{i+1}-r)}{\sigma^2(t_{i+1}-t_i)}} \Big) \qquad (5.6)\end{aligned}$$

Here σ is computed using the parametric local volatility function (5.5). The numerical studies of formula (5.6) suggests it suffices to calculate the leading two or three terms for most cases. To be conservative, in our tests, the $\hat{p}_i$ are estimated using n from -5 to 5. Experiment details and comparison results are described below.

[1] with a typography correction

5.3.2. *Set of parameters.* The numerical experiments are conducted using the following values for the parameters: $S_0 = 1$, $\nu = 20\%$, $\beta = 0.5$, $T = 1$ year, $b = 0.85$, $B = 1.25$. For thoroughness, we consider in-the-money ($K = 0.9$), at-the-money ($K = 1$) and out-the-money ($K = 1.05$) options. The benchmark prices for each numerical method are computed by the method itself with very dense time grid and high number of Monte Carlo paths.

In this case we observed some differences regarding the moneyness of the option in our numerical results. More precisely, the method performed relative poorly for the ATM option. For this reason we report below the results in all three cases and provide an explanation for the seemingly poor performance for the ATM option.

The discrepancies between benchmark prices and numerical methods for ITM, ATM and OTM double barrier call options are shown respectively in Figures 4, 5 and 6. Tables 1, 2 and 3 provide the 95% confidence intervals associated to each numerical method considered. For the same number of MC paths i.e 200K, recurrent transform shows tighter confidence intervals, which match the confidence intervals of the benchmark scheme with 1M paths. Overall, note that the size of the interval is about a few bps. We have not observed any stability issues with the recurrent transform scheme. Interestingly, our recurrent transformation has a much smaller error than the explicit Euler method with hitting probability correction when the number of discretisations is reasonably large. More importantly, this outperformance is still valid even if the number of Monte Carlo simulations for the explicit Euler method is increased five times. Having said that, one should still treat such a conclusion with caution as our benchmark price and hitting probabilities are calculated by applying a truncation and, thus, is subject to error. Nevertheless, the *outperformance* is still promising as our truncation is no coarser than the common industry practice.

Figure 7 shows the log-log plot of the discrepancy associated to the recurrent transform method for ITM, ATM and OTM double barrier call options. The numerical rate of convergence are respectively 0.91, 0.63 and 1, using 2×10^5 Monte Carlo simulations. Although the rate of convergence for the ATM option is far from the theoretical rate of 1, a closer look at Figure 5 reveals a clue. Note that the error of approximation converges very rapidly to zero after a few iterations and further discretisations do not significantly alter the already very small error term. This indicates that the observed error in this case can be mostly attributed to the statistical noise and the simple regression to obtain the convergence rate does not work well.

When we run the same experiment for the Euler scheme with hitting probability correction with 2×10^5 Monte Carlo simulations, we observe a similar drop in the performance and the convergence rates are found to be 0.50, 0.59 and 0.61, respectively. However, the convergence rates for the latter scheme increases to 0.83, 0.83 and 0.77, respectively, when the number of simulations are increased five-fold.

As complements, some variations of barrier levels are considered and its impact on numerical results quantified:

- Low barrier $b = 0.8$ and high barrier $B = 1.3$: widening of barrier levels;
- Low barrier $b = 0.8$ and high barrier $B = 1.15$: tightening of barrier levels;

Here, we increase the number of MC samples from 200,000 to 500,000 for BEM method and we keep the same number of MC paths at 1 million for the standard Euler scheme with series hitting probability. The 95% confidence interval for both methods are comparable and of order $2 - 3$ bps. For each barrier levels configuration, we have considered in-the-money ($K = 0.9$), at-the-money ($K = 1$) and out-the-money ($K = 1.05$) options. The discrepancies between benchmark prices and numerical methods w.r.t the number of time steps are presented in figures (8, 9, 10, 11, 12,

Table 1. ITM double barrier call: numerical prices and confidence intervals as a function of the number of time steps. Recurrent transform with 200K paths, Euler with series probability with 1 million paths (Euler+ (1M)) and Euler with series probability with 200K paths (Euler+ (200K)). Benchmark price = 0.0462.

Time steps	Recurrent transform	Euler+ (1M)	Euler+ (200K)
2	0.0489 ± 0.00012	0.0435 ± 0.00012	0.0438 ± 0.00026
6	0.0481 ± 0.00014	0.0451 ± 0.00014	0.0449 ± 0.00031
10	0.0475 ± 0.00014	0.0456 ± 0.00014	0.0455 ± 0.00032
16	0.0472 ± 0.00015	0.0458 ± 0.00015	0.0458 ± 0.00033
20	0.0471 ± 0.00015	0.0459 ± 0.00015	0.0460 ± 0.00033
30	0.0468 ± 0.00015	0.0460 ± 0.00015	0.0455 ± 0.00034
40	0.0468 ± 0.00015	0.0461 ± 0.00015	0.0459 ± 0.00034
60	0.0466 ± 0.00016	0.0462 ± 0.00015	0.0460 ± 0.00035
100	0.0463 ± 0.00016	0.0460 ± 0.00016	0.0457 ± 0.00035

Table 2. ATM double barrier call: numerical prices and confidence intervals as a function of the number of time steps. Recurrent transform with 200K paths, Euler with series probability with 1 million paths (Euler+ (1M)) and Euler with series probability with 200K paths (Euler+ (200K)). Benchmark price = 0.0193.

Time steps	Recurrent transform	Euler+ (1M)	Euler+ (200K)
2	0.0183 ± 0.00009	0.0176 ± 0.00006	0.0177 ± 0.00014
6	0.0194 ± 0.00010	0.0185 ± 0.00008	0.0185 ± 0.00017
10	0.0195 ± 0.00011	0.0189 ± 0.00008	0.0188 ± 0.00018
16	0.0195 ± 0.00011	0.0190 ± 0.00008	0.0190 ± 0.00018
20	0.0195 ± 0.00011	0.0191 ± 0.00008	0.0192 ± 0.00019
30	0.0194 ± 0.00011	0.0192 ± 0.00008	0.0189 ± 0.00019
40	0.0195 ± 0.00012	0.0192 ± 0.00009	0.0192 ± 0.00019
60	0.0194 ± 0.00012	0.0193 ± 0.00009	0.0192 ± 0.00019
100	0.0193 ± 0.00012	0.0192 ± 0.00009	0.0191 ± 0.00019

Table 3. OTM double barrier call: numerical prices and confidence intervals as a function of the number of time steps. Recurrent transform with 200K paths, Euler with series probability with 1 million paths (Euler+ (1M)) and Euler with series probability with 200K paths (Euler+ (200K)). Benchmark price = 0.0103.

Time steps	Recurrent transform	Euler+ (1M)	Euler+ (200K)
2	0.0085 ± 0.00006	0.0092 ± 0.00004	0.0093 ± 0.00009
6	0.0100 ± 0.00008	0.0098 ± 0.00005	0.0098 ± 0.00011
10	0.0101 ± 0.00008	0.0100 ± 0.00005	0.0100 ± 0.00012
16	0.0103 ± 0.00008	0.0101 ± 0.00005	0.0101 ± 0.00012
20	0.0103 ± 0.00008	0.0102 ± 0.00005	0.0102 ± 0.00012
30	0.0103 ± 0.00008	0.0102 ± 0.00006	0.0101 ± 0.00012
40	0.0104 ± 0.00009	0.0102 ± 0.00006	0.0102 ± 0.00013
60	0.0104 ± 0.00009	0.0103 ± 0.00006	0.0103 ± 0.00013
100	0.0103 ± 0.00009	0.0102 ± 0.00006	0.0101 ± 0.00013

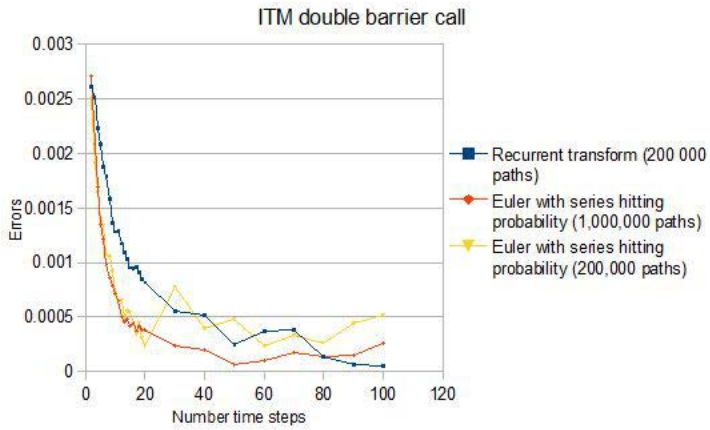

FIGURE 4. Absolute discrepancy between the benchmark price and those calculated by different numerical schemes for ITM double barrier call when $S_0 = 1$, $K = 0.9$, $\nu = 20\%$, $\beta = 0.5$, $T = 1$ year, $b = 0.85$, $B = 1.25$.

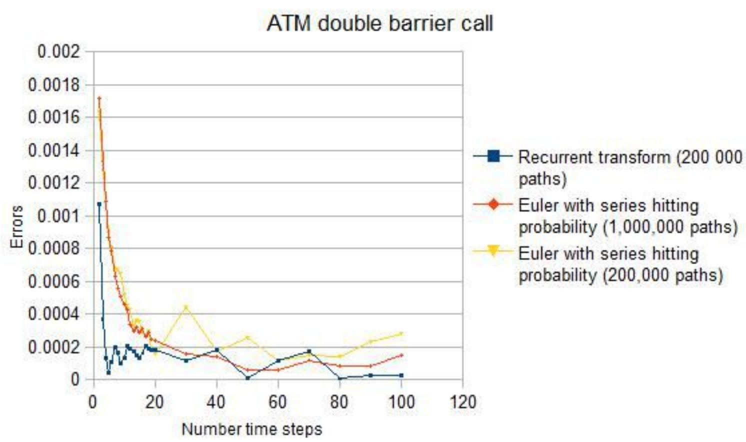

FIGURE 5. Absolute discrepancy between the benchmark price for and those calculated by different numerical schemes for ATM double barrier call when $S_0 = 1$, $K = 1$, $\nu = 20\%$, $\beta = 0.5$, $T = 1$ year, $b = 0.85$, $B = 1.25$.

13). Overall, both methods show comparable convergence results which are in accordance with the theoretical analysis.

6. Conclusion

We have introduced a novel backward Euler-Maruyama method to increase the weak convergence rate of approximations in the presence of killing. The numerical experiments confirm our theoretical result that the convergence rate is of order $1/N$, where N is the number of discretisations. This corresponds to an order-1 weak convergence rate, which is the best rate that one can achieve (see [21]).

Moreover, the numerical studies suggest that one does not need a large N to obtain a sufficiently close approximations as all numerical studies indicate errors terms diminishing very rapidly with a

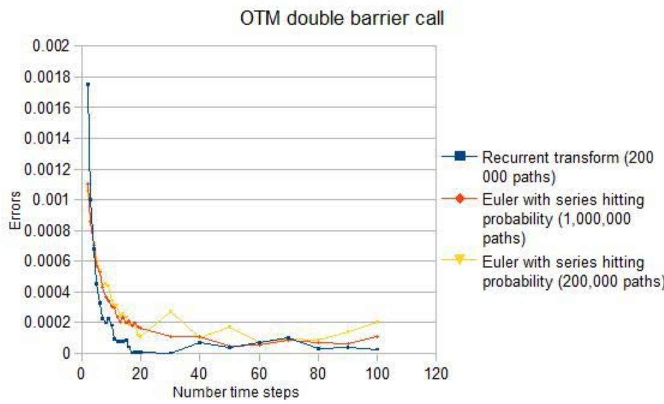

FIGURE 6. Absolute discrepancy between the benchmark price and those calculated by different numerical schemes for OTM double barrier call when $S_0 = 1$, $K = 1.05$, $\nu = 20\%$, $\beta = 0.5$, $T = 1$ year, $b = 0.85$, $B = 1.25$.

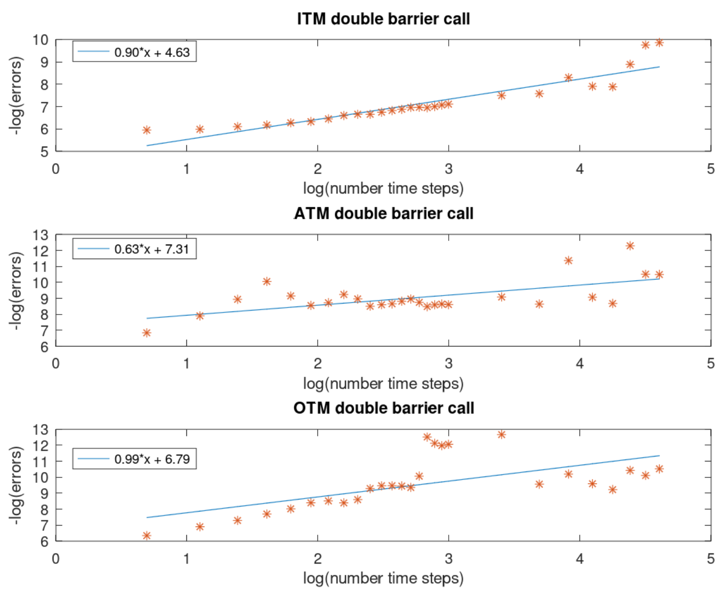

FIGURE 7. Log-log plot of the absolute discrepancy for double barrier call prices for ITM ($K = 0.9$), ATM ($K = 1$) and OTM ($K = 1.05$) with recurrent transform numerical scheme when $S_0 = 1$, $T = 1$ year, $b = 0.85$, $B = 1.25$, $\nu = 20\%$ and $\beta = 0.5$.

small number of iterations. The numerical experiments also suggested our method outperforming the *Brownian bridge method* in certain cases although such a statement does not currently have any theoretical backing.

We suggest a couple of interesting avenues for future research in addition to the extension of the BEM scheme to higher dimensions as discusses in the Introduction:

- The transform method with BEM has the potential to achieve higher order of weak convergence. Indeed one possibility of improvement is to combine BEM scheme with the Romberg

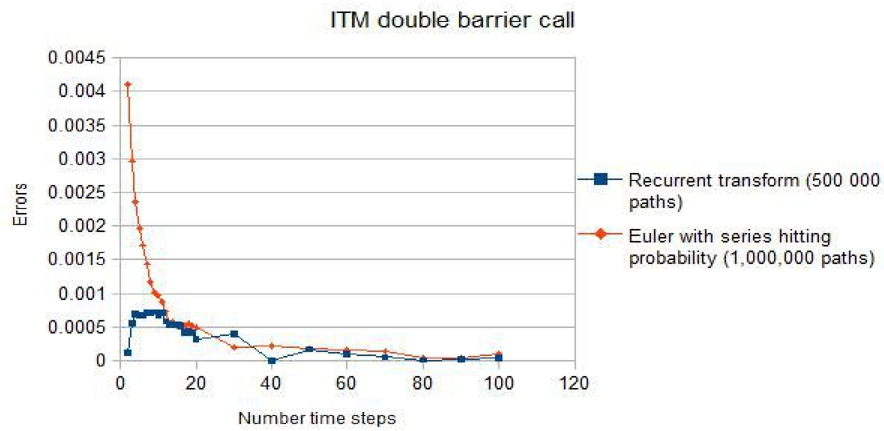

Figure 8. Absolute discrepancy between the benchmark price and those calculated by different numerical schemes for ITM double barrier call when $S_0 = 1$, $K = 0.9$, $\nu = 20\%$, $\beta = 0.5$, $T = 1$ year, $b = 0.8$, $B = 1.3$.

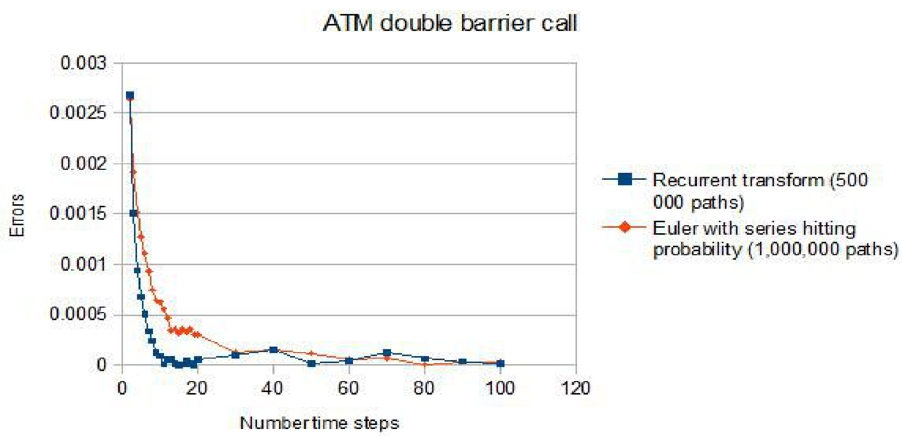

Figure 9. Absolute discrepancy between the benchmark price and those calculated by different numerical schemes for ATM double barrier call when $S_0 = 1$, $K = 1$, $\nu = 20\%$, $\beta = 0.5$, $T = 1$ year, $b = 0.8$, $B = 1.3$.

extrapolation method (see e.g Section 6.2.4 in Glasserman [20]) . Some preliminary tests are encouraging, showing that Romberg method generates significant lower errors compared to Euler hitting probability method and BEM. We believe we can expand further the weak error to justify the use the Romberg method and higher order weak convergence analysis is currently under investigation.

- Exact simulation of diffusions: Especially in a multi-dimensional setting, it is a challenge to keep the simulated values of X in its domain even thought it should not touch its boundaries in theory. The implicit scheme considered in this paper is one way out. Exact simulation method, introduced in [5] for the one-dimensional case, may be another way to resolve this issue. It involves a rejection-sampling algorithm and, when applicable, returns *exact* draws from any finite-dimensional distribution of the solution of SDE. The method has been

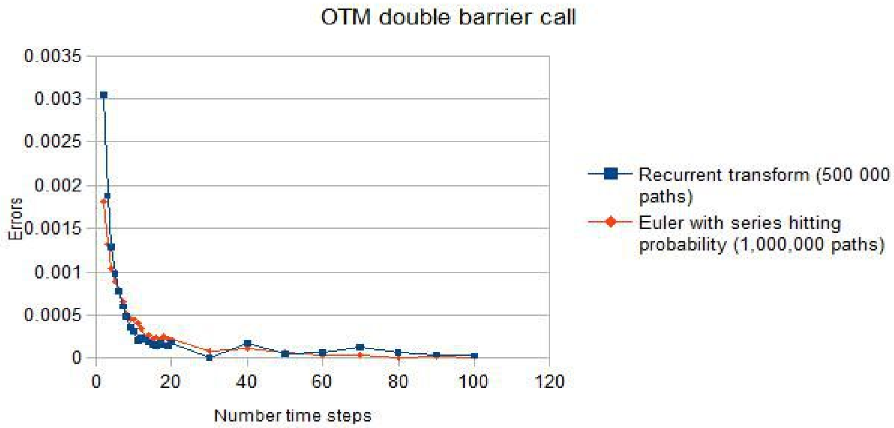

FIGURE 10. Absolute discrepancy between the benchmark price and those calculated by different numerical schemes for OTM double barrier call when $S_0 = 1$, $K = 1.05$, $\nu = 20\%$, $\beta = 0.5$, $T = 1$ year, $b = 0.8$, $B = 1.3$.

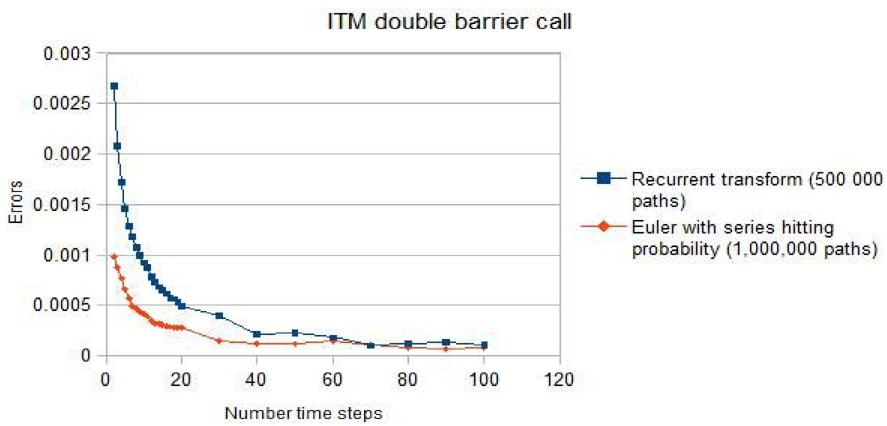

FIGURE 11. Absolute discrepancy between the benchmark price and those calculated by different numerical schemes for ITM double barrier call when $S_0 = 1$, $K = 0.9$, $\nu = 20\%$, $\beta = 0.5$, $T = 1$ year, $b = 0.8$, $B = 1.15$.

further extended to the multivariate diffusions in [6] although some open questions remain regarding the speed of convergence of the algorithm. It will be interesting to study exact simulations for the recurrent transformations that may lead to a decrease in computation time by avoiding implicit schemes, especially in higher dimensions.

APPENDIX A. PROOF OF LEMMA 2.1

Proof. (1) Since h is concave, $H'(x) > 1$, which shows the desired strict monotonicity.

If $r = \infty$ and $h(x) = x$, that $H((0, \infty)) = \mathbb{R}$ is immediate.

Next, suppose $h \in \mathcal{H}_0$. Then, the dominated convergence theorem implies that $h(0) = 0$ as well as $h(r) = 0$ if $r < \infty$ since the potential density vanishes at finite endpoints. Moreover, as

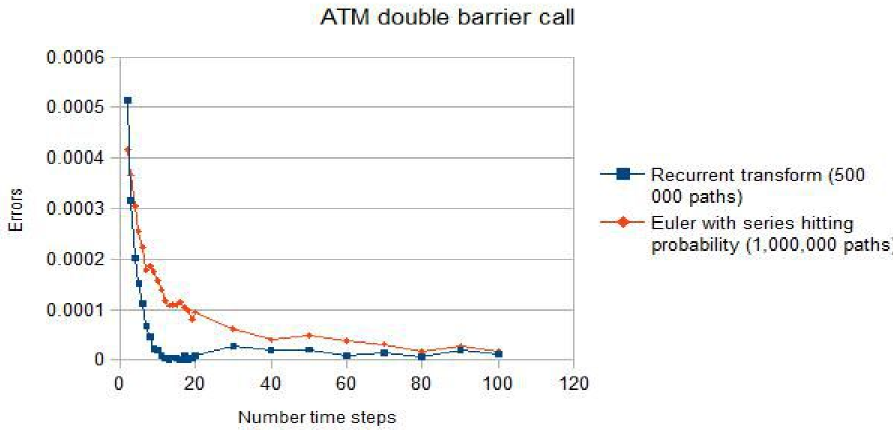

FIGURE 12. Absolute discrepancy between the benchmark price and those calculated by different numerical schemes for ATM double barrier call when $S_0 = 1$, $K = 1$, $\nu = 20\%$, $\beta = 0.5$, $T = 1$ year, $b = 0.8$, $B = 1.15$.

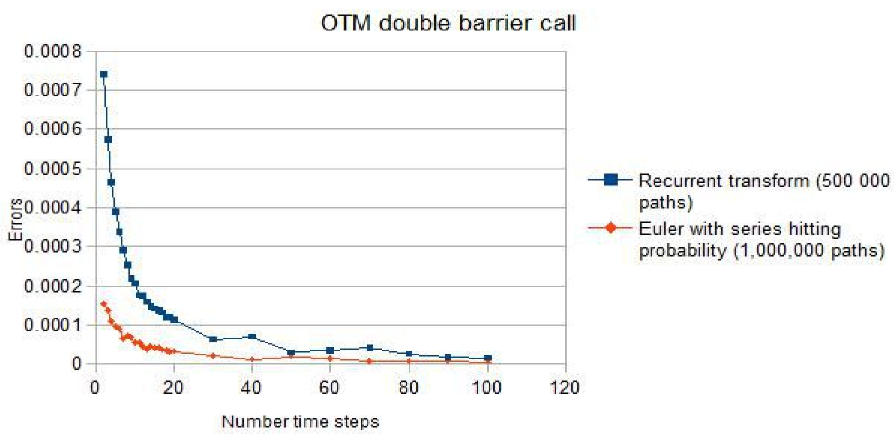

FIGURE 13. Absolute discrepancy between the benchmark price and those calculated by different numerical schemes for OTM double barrier call when $S_0 = 1$, $K = 1.05$, $\nu = 20\%$, $\beta = 0.5$, $T = 1$ year, $b = 0.8$, $B = 1.15$.

h is strictly concave and never vanishes in the interior of the state space, $h'(0) > 0$. Thus,

$$\lim_{x \to 0} \frac{h'(x)}{h(x)} = \infty.$$

This proves the desired range for H when $r = \infty$. Indeed, in this case h is increasing, which in turn yields

$$\frac{h'(x)}{h(x)} \leq \frac{h'(1)}{h(1)} \qquad x \geq 1.$$

If $r < \infty$, similar considerations imply $h'(r) < 0$, and therefore

$$\lim_{x \to r} \frac{h'(x)}{h(x)} = -\infty.$$

This completes proof of the first assertion.

(2) If $r=\infty$ and $h(x)=x$, $h'(x)=1$ for all $x\geq 0$. If $h\in\mathcal{H}_0$,

$$h'(x)=\int_x^\infty f(y)m(dy),$$

which is non-negative and finite by the assumption on f. In particular, $h'(0)=\int_0^\infty f(y)m(dy)$ and $h'(\infty)=0$.

If $r<\infty$,

$$h(x)=\frac{r-x}{r}\int_0^x yf(y)m(dy)+x\int_x^r\frac{r-y}{r}f(y)m(dy).$$

Thus,

$$h'(x)=\int_x^r f(y)m(dy)-\frac{1}{r}\int_0^r yf(y)m(dy).$$

This yields the desired boundedness and the boundary levels for the derivatives.

(3) First suppose $r<\infty$. Since h' does not vanish at the boundaries, $u(y,y)/h(y)$ is bounded. Moreover,

$$-\int_{(0,r)}h''(y)dy=\int_{(0,r)}f(y)m(dy).$$

This proves the claim when $r<\infty$ since h' is clearly integrable by the finiteness of h and that h' changes sign only once.

Now suppose that $r=\infty$. Thus, $u(y,y)=y$ and

$$-\int_{(0,\infty)}y\frac{h''(y)}{h(y)}dy=\int_{(0,1)}y\frac{f(y)}{h(y)}m(dy)+\int_{(1,\infty)}y\frac{f(y)}{h(y)}m(dy).$$

The first integral on the right hand side is finite since f is m-integrable and $y/h(y)$ is bounded on $[0,1]$ as $h'(0)>0$. The second integral is also finite since $h(\infty)>0$ and $\int yf(y)m(dy)<\infty$ by assumption.

Moreover, noting that $h'\geq 0$,

$$\int_0^\infty(y\wedge 1)\frac{h'(y)}{h(y)}dy=\int_0^1\frac{yh'(y)}{h(y)}dy+\int_1^\infty\frac{h'(y)}{h(y)}dy\leq K\int_0^1 h'(y)dy+\log\frac{h(\infty)}{h(1)}<\infty.$$

□

Appendix B. Proof of Theorem 4.1

Proof will be divided into several steps considering first the case of $r=\infty$ and making use of the comparison Lemma 4.2. In what follows K denotes a generic constant independent of N.

(1) First suppose $r=\infty$ and recall that $1/h$ is decreasing. Then Lemma 4.2 and Theorem 2.2 imply $\sup_{t\leq T,N}E^{h,X_0}[\frac{1}{h}(\widehat{X}_t)]<\infty$. Moreover, Lemma 4.2 also yields

$$E^{h,X_0}\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}}\frac{\sigma^2(\widehat{X}_{t_n})h^{-2-p}(\widehat{X}_t)}{H_x^2(t_n,\widehat{X}_{t_n};t,\widehat{X}_t)}dt\leq E^{h,X_0}\int_0^{A_T}\frac{1}{h^{2+p}(Y_t)}dt\leq E^{h,X_0}\int_0^{\|\sigma\|_\infty^2 T}\frac{1}{h^{2+p}(Y_t)}dt<\infty,$$

where Y is a process that shares the same law with the process in Theorem 2.2 with $c=c_1$ and the last inequality follows from Theorem 2.2.

Similarly, by considering instead the time change

$$dA_t = \frac{\sigma^2(\widehat{X}_{t_n})}{H_x^2(t_n, \widehat{X}_{t_n}; t, \widehat{X}_t)} dt, \qquad t \in (t_n, t_{n+1}), \qquad A_{t_n} = t_n,$$

we obtain $h^{-p}(\widehat{X}_\tau) \leq h^{-p}(Y_{A_\tau})$, where Y is a process such that $Y_{t_n} = \widehat{X}_{t_n}$ and

$$dY_t = d\beta_t + \left(\frac{h'}{h}(Y_t) + c_1\right) dt, \quad t \geq t_n,$$

with β being a standard Brownian motion. Consequently, Theorem 2.2 yields

$$\text{ess sup}_{\tau \in \mathcal{T}_n} E^{h,X_0}\left[\frac{1}{h}(\widehat{X}_\tau)\Big|\mathcal{F}_{t_n}\right] < \infty$$

since $A_\tau \leq t_n + \|\sigma\|_\infty^2 \frac{T}{N}$, a.s. for $\tau \in \mathcal{T}_n$.

(2) For the case $r < \infty$, we define $x_1 := \inf\{x \geq 0 : h'(x) = 0\}$ and $x_2 := \inf\{x \geq x_1 : h'(x) < 0\}$. Then, there exist functions h_1 and h_2 such that $h = h_1 h_2$, h_1 (resp. h_2) is non-decreasing (resp. non-increasing) and constant on (x_1, r) (resp. $(0, x_2)$).

Let's define the processes $\widehat{Y}^i$, where $\widehat{Y}^i_0 = X_0$ and

$$d\widehat{Y}^i_t = \frac{\sigma(\widehat{X}_{t_n})}{H_x(t_n, \widehat{X}_{t_n}; t, \widehat{Y}^i_t)} dW_t + \frac{\sigma^2(\widehat{X}_{t_n})}{H_x^2(t_n, \widehat{X}_{t_n}; t, \widehat{Y}^i_t)} \left(\frac{h_i'}{h_i}(\widehat{Y}^i_t) + c_i\right) dt, \quad t \in (t_n, t_{n+1}].$$

Applying Ito formula to $((x_2 - \widehat{Y}^1_t)^+)^2$ and $((x_2 - \widehat{X}_t)^+)^2$, the comparison theorem employed in Lemma 4.2 shows that

$$P^{h,X_0}[\widehat{Y}^1_t \wedge x_2 \leq \widehat{X}_t \wedge x_2, t \leq T] = 1.$$

An analogous argument also shows that

$$P^{h,X_0}[\widehat{Y}^2_t \vee x_1 \geq \widehat{X}_t \vee x_1, t \leq T] = 1$$

as well.

As h_1 is non-decreasing, h_2 is non-increasing and $\frac{h}{h_1}$ (resp. $\frac{h}{h_2}$) is constant on $(0, x_2)$ (resp. (x_1, r)), the above comparisons imply that $\frac{1}{h(\widehat{X}_t \wedge x_2)} \leq \frac{1}{h(\widehat{Y}^1_t \wedge x_2)}$ and $\frac{1}{h(\widehat{X}_t \vee x_1)} \leq \frac{1}{h(\widehat{Y}^2_t \vee x_1)}$.

Thus, the same time change argument from Lemma 4.2 yields that

$$\sup_{t \leq T, N} E^{h,X_0}\left[\frac{1}{h}(\widehat{X}_t)\right] < \infty$$

by another application of Theorem 2.2. Although h_2 does not quite satisfy the condition therein, we obtain the result that we need by a change of scale and considering instead the function h defined by $h(x) = h_2((r-x)^+)$. Note that h is still continuously differentiable. This readily implies

$$\sup_{t \leq T, N} E^{h,X_0}\left[\frac{1}{h(\widehat{X}_t)}\right] \leq \frac{K'}{h(X_0)},$$

for some K' that depends only on T.

Similarly,

$$\sup_N E^{h,X_0} \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \frac{\sigma^2(\widehat{X}_{t_n}) h^{-2-p}(\widehat{X}_t)}{H_x^2(t_n, \widehat{X}_{t_n}; t, \widehat{X}_t)} dt < \infty,$$

in view of Theorem 2.2 again.

Analogous considerations also yield $\text{ess sup}_{\tau \in \mathcal{T}_n} E^{h,X_0}\left[\frac{1}{h}(\widehat{X}_\tau)\Big|\mathcal{F}_{t_n}\right] < \infty$.

(3) We shall now show the boundedness of the moments. Note that there is nothing to show when $r < \infty$. So, let's assume that $r = \infty$. Recall that

$$\widehat{X}_t = \widehat{X}_{t_n} + \sigma^2(\widehat{X}_{t_n})(t - t_n)\frac{h'}{h}(\widehat{X}_t) + \sigma(\widehat{X}_{t_n})(W_t - W_{t_n}).$$

Thus,

$$E^{h,X_0}[\widehat{X}_t] \le E^{h,X_0}[\widehat{X}_{t_n}] + K(t - t_n)$$

for some K due to the boundedness of σ and h' as well as the uniform bound on the inverse moment of $h(\widehat{X}_t)$. This shows that $\sup_{t \le T, M} E^{h,X_0}[\widehat{X}_t] \le X_0 + KT$. Now, suppose that $E(m) := \sup_{t \le T, N} E^{h,X_0}[\widehat{X}_t^m] < \infty$ and deduce from (4.1) that

$$\begin{aligned} d\widehat{X}_t^{m+1} &= dZ_t + (m+1)\frac{\widehat{X}_t^m \sigma^2(\widehat{X}_{t_n})}{H_x^2(t_n, \widehat{X}_{t_n}; t, \widehat{X}_t)} \left\{ \frac{h'}{h}(\widehat{X}_t) + \mu(t_n, \widehat{X}_{t_n}; t, \widehat{X}_t) \right\} dt \\ &\quad + \frac{1}{2}m(m+1)\frac{\widehat{X}_t^{m-1}\sigma^2(\widehat{X}_{t_n})}{H_x^2(t_n, \widehat{X}_{t_n}; t, \widehat{X}_t)} dt, \end{aligned}$$

where Z is a local martingale. Next observe that for $m \ge 1$

$$x^m/h \le K(1 + x^{m-1}) \tag{B.1}$$

as $h(0) = 0$, $h'(0) > 0$ and $h'/h \le \frac{1}{x}$. The last identity follows from the fact that

$$h(x) = -\int_0^x y h''(y) dy + x h'(x).$$

Moreover, the representation of μ from (4.2) and (4.3) show that

$$|\mu| \le K(H_x + 1)\frac{1}{h} \tag{B.2}$$

since the term in front of the parentheses in (4.2) is bounded.

Observe that $(\tau_k)_{k \ge 1}$, where $\tau_k := \inf\{t \ge t_n : \widehat{X}_t \ge k\}$ is a localising sequence for Z. Therefore, a standard localisation argument, (B.1) and (B.2) together imply for $t \in (t_n, t_{n+1}]$

$$E^{h,X_0}[\widehat{X}_t^{m+1}] \le E^{h,X_0}[\widehat{X}_{t_n}^{m+1}] + (t - t_n)KE(m-1),$$

in view of the Fatou's lemma for some constant K, which in turn yields

$$E(m+1) \le X_0^{m+1} + KTE(m-1).$$

Finally, note that this in particular implies that Z is a true martingale. Thus, for $\tau \in \mathcal{T}_n$ and $m \ge 2$

$$\widehat{X}_\tau^m \le \widehat{X}_{t_n}^m + M_\tau + K\int_{t_n}^{t_{n+1}} \widehat{X}_t^{m-1} dt.$$

Taking conditional expectations show

$$E^{h,X_0}\big[\widehat{X}_\tau^m \big| \mathcal{F}_{t_n}\big] \le \widehat{X}_{t_n}^m + KE^{h,X_0}\Big[\int_{t_n}^{t_{n+1}} \widehat{X}_t^{m-1} dt \Big| \mathcal{F}_{t_n}\Big], \tag{B.3}$$

yielding (4.6).

To establish (4.7) we need the following lemma.

Lemma B.1. *Suppose that h satisfies the conditions of Lemma 4.1, σ is bounded, and consider the BEM scheme defined by (2.4). For any $p \in [0,1)$, any n and $t_n \leq s \leq t < t_{n+1}$ we have*

$$E^{h,X_0}\big[h^{-p}(\widehat{X}_t)\big|\mathcal{F}_s\big] \leq h^{-p}(\widehat{X}_s)\exp(K(t-s)),$$

for some constant $K>0$ that is independent of n.

Proof. Let $\mu_t := \mu(t_n, \widehat{X}_{t_n}; t, \widehat{X}_t)$. A straightforward application of Ito's formula yields

$$\begin{aligned} dh^{-p}(\widehat{X}_t) &= dM_t - \frac{\sigma^2(\widehat{X}_{t_n})}{H_x^2(t_n,\widehat{X}_{t_n};t,\widehat{X}_t)} ph^{-p}(\widehat{X}_t)\left(\frac{2\mu_t h'(\widehat{X}_t)+h''(\widehat{X}_t)}{2h(\widehat{X}_t)} + \frac{1-p}{2}\left(\frac{h'}{h}(\widehat{X}_t)\right)^2\right)dt \\ &\leq dM_t - \frac{\sigma^2(\widehat{X}_{t_n})}{H_x^2(t_n,\widehat{X}_{t_n};t,\widehat{X}_t)} ph^{-p}(\widehat{X}_t)\left(-\frac{\alpha_1}{h(\widehat{X}_t)} + \frac{(1-p)\alpha_2}{h^2(\widehat{X}_t)}\right)dt \end{aligned}$$

where M is a local martingale and α_1 and α_2 are positive constants depending on the bounds on h' and h'' since $\mu_t > c_1$ (resp. $\mu_t < c_2$) whenever $h'(\widehat{X}_t) > 0$ (resp. $h'(\widehat{X}_t) < 0$) by Lemma 4.1 and h' never vanishes at the same time as h. Thus, there exists a constant K that depends only on h, p and c_1 and c_2 such that

$$dh^{-p}(\widehat{X}_t) \leq dM_t + K\frac{\sigma^2(\widehat{X}_{t_n})h^{-p}(\widehat{X}_t)}{H_x^2(t_n,\widehat{X}_{t_n};t,\widehat{X}_t)}dt$$

since $-\alpha_1 x + (1-p)\alpha_2 x^2$ is bounded from below.

Next note that $\tau_k := \inf\{t \geq t_n : \widehat{X}_t < 1/k\}$ is a localising sequence for M. Thus, using the optional stopping theorem and Fatou's lemma and monotone convergence we arrive at

$$E^{h,X_0}\left[h^{-p}(\widehat{X}_t)\Big|\mathcal{F}_s\right] \leq h^{-p}(\widehat{X}_s) + KE^{h,X_0}\left[\int_s^t h^{-p}(\widehat{X}_u)du\Big|\mathcal{F}_s\right], \quad t_n \leq s \leq t \leq t_{n+1},$$

for some constant K in view of the boundedness of σ. We deduce the claim by Gronwall's lemma. □

Now we return to the proof of the estimate (4.7).

Observe that the hypothesis on h'' implies $1-\exp((s-t_n)\sigma^2(\widehat{X}_{t_n})\frac{h''}{2h}(\widehat{X}_{t_n})) \leq K\frac{T}{N}\frac{1}{h^p}(\widehat{X}_{t_n})$ for some $K>0$. Without loss of generality let's also suppose that $h \leq 1$. Thus,

$$\begin{aligned} &E^{h,X_0}\left[\int_{t_n}^{t_{n+1}}\left(1-\exp\left((s-t_n)\sigma^2(\widehat{X}_{t_n})\frac{h''}{2h}(\widehat{X}_{t_n})\right)\right)\frac{\sigma^2(\widehat{X}_{t_n})h^{-p}(\widehat{X}_s)}{H_x^2(t_n,\widehat{X}_{t_n};s,\widehat{X}_s)}ds\right] \\ &\leq K\frac{T}{N}E^{h,X_0}\left[\int_{t_n}^{t_{n+1}} h^{-p}(\widehat{X}_{t_n})h^{-p}(\widehat{X}_s)ds\right] \\ &\leq K\frac{T}{N}E^{h,X_0}\left[\int_{t_n}^{t_{n+1}} h^{-2p}(\widehat{X}_{t_n})ds\right] \leq K\frac{T^2}{N^2}E^{h,X_0}[h^{-1}(\widehat{X}_{t_n})], \end{aligned}$$

where the second line follows from Lemma B.1 and that $H_x \geq 1$.

Next suppose $m \geq 1$. Note that the calculations similar to the ones leading to (B.3) imply that

$$E^{h,X_0}\big[\widehat{X}_t^m\big|\mathcal{F}_{t_n}\big] \leq \widehat{X}_{t_n}^m + KE^{h,X_0}\Big[\int_{t_n}^{t_{n+1}}\widehat{X}_s^{m-1}ds\Big|\mathcal{F}_{t_n}\Big],$$

Thus, the elementary inequality $x^{m-1} \leq 1 + x^m$ and Gronwall's lemma yield

$$E^{h,X_0}[\widehat{X}_t^m|\mathcal{F}_{t_n}] \leq K(\widehat{X}_{t_n} + \frac{T}{N}).$$

Therefore,

$$E^{h,X_0}\left[\int_{t_n}^{t_{n+1}}\left(1-\exp\left((s-t_n)\sigma^2(\widehat{X}_{t_n})\frac{h''}{2h}(\widehat{X}_{t_n})\right)\right)\frac{\sigma^2(\widehat{X}_{t_n})\widehat{X}_s^m}{H_x^2(t_n,\widehat{X}_{t_n};s,\widehat{X}_s)}ds\right]$$
$$\leq K\frac{T}{N}E^{h,X_0}\left[(\widehat{X}_{t_n}+\frac{T}{N})\left(1-\exp\left(-\frac{T}{N}\frac{a}{h}(\widehat{X}_{t_n})\right)\right)\right]\leq K\frac{T^2}{N^2}E^{h,X_0}\left[(\widehat{X}_{t_n}+\frac{T}{N})\frac{1}{h(\widehat{X}_{t_n})}\right]$$
$$\leq K\frac{T^2}{N^2}E^{h,X_0}[\widehat{X}_{t_n}+h^{-1}(\widehat{X}_{t_n})],$$

where the last line follows from (4.8).

Combining above estimates, we arrive at the claimed result via (4.5).

Appendix C. Relationship with Alfonsi and Neunkirch & Szpruch

Purpose of this short section is to show that the results of [3] and [35] do not apply to the simulation of the recurrent diffusion process following (1.2). We assume Assumptions 2.1-2.3 below.

Proposition C.1. *Let X be given by* (1.2) *where h is a function satisfying the condition of Theorem 3.2 in* [9]. *Suppose further that $b\equiv 0$, and $\sigma\equiv 1$. Then*

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T\left\{\left(\frac{h'}{h}\right)'(X_t)\right\}^2dt\right]=\infty.$$

Proof. Suppose

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T\left\{\left(\frac{h'}{h}\right)'(X_t)\right\}^2dt\right]<\infty. \tag{C.1}$$

First observe that

$$d\frac{1}{h}(X_t)=-\frac{h'}{h^2}(X_t)dW_t+dC_t, \tag{C.2}$$

where C is an adapted, continuous and increasing process.

Moreover, h is concave since X is on natural scale under $\mathbb{P}$, and we also have

$$\left(\frac{h'}{h}\right)'=\frac{h''}{h}-\left(\frac{h'}{h}\right)^2.$$

In particular, $\left|\left(\frac{h'}{h}\right)'\right|>\left(\frac{h'}{h}\right)^2$. Thus,

$$\mathbb{E}^{\mathbb{Q}}\left[\left(\frac{h'}{h}\right)^4(X_t)\right]\leq\mathbb{E}^{\mathbb{Q}}\left[\left\{\left(\frac{h'}{h}\right)'(X_t)\right\}^2\right]$$

Also note that h' never vanishes at the boundary points where h does, and it does vanish at r if $r=\infty$, where h does not. Consequently,

$$\frac{(h')^2}{h^4}\leq K_1\left(\frac{h'}{h}\right)^4+K_2,$$

for some K_1 and K_2, which in conjunction with (C.1) implies

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T\left(\frac{h'}{h^2}\right)^2(X_t)dt\right]<\infty.$$

Therefore, the local martingale term in (C.2) is a true $\mathbb{Q}$-martingale, which in turn will yield $\frac{1}{h}\exp(-A)$ is a true $\mathbb{Q}$-martingale, where $dA_t = -\frac{1}{2}\frac{h''(X_t)}{h(X_t)}dt$. In particular, $\mathbb{P} \ll \mathbb{Q}$ when restricted to $\mathcal{F}_t$.

This further implies that $\mathbb{P} \sim \mathbb{Q}$ when restricted to $\mathcal{F}_t$ since $\mathbb{Q} \ll \mathbb{P}$ on the same restriction by Theorem 3.2 in [9]. However, this is a contradiction since, for any $t > 0$, $\mathbb{P}(\zeta < t) > 0$ while $\mathbb{Q}(\zeta < t) = 0$ for X is a recurrent process under $\mathbb{Q}$. □

Competing Interests

The authors declare no competing interests.

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