ON CERTAIN INTEGRAL FUNCTIONALS OF SQUARED BESSEL PROCESSES

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ABSTRACT. For a squared Bessel process, X, the Laplace transforms of joint laws of $(U, \int_0^{R_y} X_s^p \, ds)$ are studied where R_y is the first hitting time of y by X and U is a random variable measurable with respect to the history of X until R_y. A subset of these results are then used to solve the associated small ball problems for $\int_0^{R_y} X_s^p \, ds$ and determine a Chung's law of iterated logarithm. $\left(\int_0^{R_y} X_s^p \, ds \right)$ is also considered as a purely discontinuous increasing Markov process and its infinitesimal generator is found. The findings are then used to price a class of exotic derivatives on interest rates and determine the asymptotics for the prices of some put options that are only slightly in-the-money.

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1. INTRODUCTION

Let X be a squared Bessel process which is the unique strong solution to

$$dX_t = 2(\nu + 1)\, dt + 2\sqrt{X_t}\, dB_t,$$

where $\nu \geq -1$ is a real constant and B is a standard Brownian motion. Letting $\delta = 2(\nu + 1)$, X is called a δ-dimensional squared Bessel process. We will denote such a process with $X_0 = z$ by $BESQ^\delta(z)$ and δ and ν will be related by $\delta = 2(\nu + 1)$ throughout the text. In this paper we are interested in the integral functional

$$\Sigma_{p,z,y}^\delta := \int_0^{R_y} X_s^p \, ds, \tag{1.1}$$

where $p > -1$ and $R_y := \inf\{t \geq 0 : X_t = y\}$ for $y \in [0, \infty)$ and X is $BESQ^\delta(z)$. (In the sequel, we will write R_y^δ only if we need to specify the dimension to avoid ambiguity.)

Squared Bessel processes have found wide applications especially in Finance Theory, see Chapter 6 in [8] for a recent account. They can, e.g., be used to model interest rates in a *Cox-Ingersoll-Ross* framework. In the above setting, if X^p models the spot interest rates, then $\exp\left(\Sigma_{p,z,y}^\delta\right)$ refers to the cumulative interest until the spot rate hits the barrier y^p. As such, this random variable is related to certain *exotic options* on interest rates (see [5] for some formulae regarding barrier options in a similar framework). Bessel processes also appear often in the study of *financial bubbles* since $1/\sqrt{X}$ is the prime example of a continuous (strict) local martingale when X is a $BESQ^3$ (see, e.g., [14], [17] and [18] for how strict local martingales, and in particular Bessel processes, appear in mathematical studies of bubbles).

In Section 2 we will determine the joint law of $(U, \Sigma_{p,z,y}^\delta)$ by martingale methods, where U is a random variable measurable with respect to the evolution of X until R_y. In particular we will obtain the joint distributions of $(R_y, \Sigma_{p,z,y}^\delta)$ and $(\max_{t \leq R_y} X_t, \Sigma_{p,z,y}^\delta)$. As a by-product of our findings, if $\frac{|\nu|}{p+1} = \frac{1}{2}$, we have a remarkable characterisation of the conditional law of $\Sigma_{p,z,y}^\delta$ given that the maximum (resp. minimum) of X at R_y is below (resp. above) a fixed level in terms of the first hitting time distributions of a 3-dimensional Bessel process when $z \geq y$ (resp. $z \leq y$).

We will use the results of Section 2 in order to study *small ball probabilities* for $\Sigma^\delta_{p,z,y}$ in Section 3. Solving the small ball problem for $\Sigma^\delta_{p,z,y}$ amounts to finding the asymptotic behaviour of $-\log \mathrm{Prob}(\Sigma^\delta_{p,z,y} < \varepsilon)$ as $\varepsilon \to 0$. We will then use this asymptotic form to determine a *law of iterated logarithm* for $(\Sigma^\delta_{p,0,y})_{y \geq 0}$ as $y \to \infty$.

Section 4 will analyse $(\Sigma^\delta_{p,0,y})_{y \geq 0}$ as a Markov process indexed by y and compute its infinitesimal generator when $\nu \geq 0$. We will also consider the process Z^δ which is obtained via a 'time reversal' from $(\Sigma^\delta_{p,0,y})_{y \geq 0}$. More precisely, we will find the generator of Z^δ defined by

$$Z^\delta_x = \int_{L_{1-x}}^{L_1} X^p_s \, ds \qquad \forall x \in [0,1),$$

where $L_x := \sup\{t \geq 0 : X_t = x\}$. In particular, we will obtain that Z^4 is identical in law to an increasing family of hitting times of a linear Brownian motion.

Finally, in Section 5 we will apply our findings to the pricing of some exotic derivatives on interest rates. The small ball probabilities will be used to find asymptotic behaviour of some put options with small strikes, the options that are only slightly *in-the-money*.

2. Preliminaries

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a stochastic base where $\mathcal{F}$ is completed with the $\mathbb{P}$-null sets. Let X be an $\mathbb{R}_+$-valued semimartingale which is the unique strong solution to

$$(2.1) \qquad\qquad dX_t = 2(\nu + 1) \, dt + 2\sqrt{X_t} \, dB_t,$$

where $\nu \geq -1$ is a real constant, B is a standard Brownian motion.

Let Q^δ_z be the measure on the path space, i.e. $C([0,\infty), [0,\infty))$, induced by X starting at z. It is well-known (see Section 1 of Chapter XI of [15]) that for $\nu \geq 0$ the set $\{0\}$ is polar, otherwise it is reached a.s.. Moreover, the process is transient for $\nu > 0$ and recurrent otherwise. We will denote the first hitting time of 0 for X with R. The scale function, s^ν, for $BESQ^\delta$ is given by

$$s^\nu(x) = -x^{-\nu} \quad \text{for } \nu > 0, \qquad s^0(x) = \log x, \qquad s^\nu(x) = x^{-\nu} \quad \text{for } \nu \in [-1, 0).$$

We refer the reader to [15] and [6] for a comprehensive study of Bessel processes and relevant bibliography.

In subsequent computations we will follow a Feynman-Kac type approach as in, e.g., [7].

Lemma 2.1. *Let $p > -1, \lambda > 0$ and suppose that $u \in C^2$, solves the following ordinary differential equation (ODE):*

$$(2.2) \qquad\qquad x^2 y'' + x y' - y[\nu^2 + \lambda x^{2(p+1)}] = 0,$$

and is strictly positive on $(0, \infty)$. Then, $(M^{(u)}_{t \wedge R})_{t \geq 0}$ is a local martingale where

$$M^{(u)}_t := u(\sqrt{X_t}) X_t^{-\frac{\nu}{2}} \exp\left(-\frac{\lambda}{2} \int_0^t X^p_s \, ds\right).$$

In particular,

$$\mathbf{1}_{[t<R]} dM^{(u)}_t = \mathbf{1}_{[t<R]} M^{(u)}_t \left(\frac{u'(\sqrt{X_t})}{u(\sqrt{X_t})} - \frac{\nu}{\sqrt{X_t}}\right) dB_t.$$

Proof. If we let $w(x) := u(\sqrt{x}) x^{-\frac{\nu}{2}}$, it is easily seen that w solves

$$(2.3) \qquad\qquad 2x w'' + 2(\nu + 1) w' - \frac{\lambda}{2} x^p w = 0.$$

Thus, on $[t < R]$

$$dM^{(u)}_t = 2 M^{(u)}_t \frac{w'}{w}(X_t) \sqrt{X_t} dB_t = M^{(u)}_t \left(\frac{u'(\sqrt{X_t})}{u(\sqrt{X_t})} - \frac{\nu}{\sqrt{X_t}}\right) dB_t.$$

$\square$

The solutions to (2.2) can easily be determined via the modified Bessel functions, I_α and K_α, of the first and second kind. We next summarise some properties of these functions and we refer the reader to Section 3.7 of [19] or Section 9.6.1 of [1] for further results and proofs. For $\alpha \geq 0$, I_α (resp. K_α) is a positive and increasing (resp. decreasing) solution to

$$(2.4) \qquad x^2 y'' + x y' - (x^2 + \alpha^2) y = 0$$

when x is restricted to $(0, \infty)$. Moreover, I_α has the following series representation:

$$(2.5) \qquad I_\alpha(x) = \sum_{m=0}^{\infty} \frac{1}{m! \Gamma(m + \alpha + 1)} \left(\frac{x}{2}\right)^{2m+\alpha}.$$

The above expansion can be used to define I_α for $\alpha < 0$ when α is not an integer less than -1 since the Gamma function is well defined and finite at non-integer negative values. For $\alpha \in \mathbb{Z} \cap (-\infty, 0)$ one can define it by taking limits and it turns out that for such an α $I_{-\alpha} = I_\alpha$. It is also a simple matter to check that I_α satisfies (2.3) for $\alpha < 0$ as well. Moreover,

$$(2.6) \qquad K_\alpha(x) = \frac{\pi}{2} \frac{I_{-\alpha}(x) - I_\alpha(x)}{\sin(\alpha \pi)}.$$

The above identity in particular entails $K_\alpha = K_{-\alpha}$.

The asymptotic behaviour of the modified Bessel functions can be described in terms of known functions:

$$(2.7) \qquad \begin{array}{ll} I_\alpha(x) \sim \frac{\left(\frac{x}{2}\right)^\alpha}{\Gamma(\alpha+1)} \text{ as } x \to 0, \, \alpha \neq -1, -2, \ldots; & I_\alpha(x) \sim \frac{e^x}{\sqrt{2\pi x}} \text{ as } x \to \infty. \\ K_\alpha(x) \sim \frac{\Gamma(\alpha)}{2} \left(\frac{x}{2}\right)^{-\alpha} \text{ as } x \to 0, \alpha > 0; & K_\alpha(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x} \text{ as } x \to \infty. \\ K_0(x) \sim -\log x \text{ as } x \to 0. \end{array}$$

Then, it is easy to check that the solutions to (2.2) is of the form

$$(2.8) \qquad C_1 K_{\frac{|\nu|}{p+1}} \left(\frac{1}{p+1} \sqrt{\lambda} x^{p+1}\right) + C_2 I_{\frac{|\nu|}{p+1}} \left(\frac{1}{p+1} \sqrt{\lambda} x^{p+1}\right),$$

where C_1 and C_2 are arbitrary constants.

Remark 1. *It is worth to observe some further monotonicity properties regarding modified Bessel functions that will be useful in the sequel.*

First consider $x^{-\nu} K_{\frac{|\nu|}{p+1}} \left(\frac{1}{p+1} \sqrt{\lambda} x^{p+1}\right)$ for $\nu \in \mathbb{R}$ and $p > -1$. We will now see that this function is decreasing. Indeed, differentiation with respect to x yields

$$-\nu x^{-\nu-1} K_{\frac{\nu}{p+1}} \left(\frac{1}{p+1} \sqrt{\lambda} x^{p+1}\right) + \sqrt{\lambda} x^{-\nu+p} K'_{\frac{\nu}{p+1}} \left(\frac{1}{p+1} \sqrt{\lambda} x^{p+1}\right).$$

When $\nu \geq 0$, utilising the recurrence elation (cf. Section 3.7 in [19])

$$K'_\alpha(x) = \frac{\alpha}{x} K_\alpha(x) - K_{\alpha+1}(x), \ \alpha \in \mathbb{R},$$

one can see that the above derivative is negative since K_α is positive for all $\alpha \in \mathbb{R}$. If $\nu < 0$, the other recurrence relation, i.e.

$$K'_\alpha(x) = -\frac{\alpha}{x} K_\alpha(x) - K_{\alpha-1}(x), \ \alpha \in \mathbb{R},$$

yields the conclusion.

Similarly, the function $x^{-\nu} I_{\frac{\nu}{p+1}} \left(\frac{1}{p+1} \sqrt{\lambda} x^{p+1}\right)$, has a positive derivative, i.e. is increasing, whenever $p > -1$ and $\frac{\nu}{p+1} > -2$. This follows from the recurrence relation

$$I'_\alpha(x) = \frac{\alpha}{x} I_\alpha(x) + I_{\alpha+1}(x), \ \alpha \in \mathbb{R},$$

and the positivity of I_α for $\alpha > -1$. Note that $\frac{\nu}{p+1} + 1 > -1$ under our hypotheses.

We now return to determining the joint law of $(U, \Sigma^\delta_{p,z,y})$ for arbitrary positive $\mathcal{F}_{R_y}$-measurable random variables U, where $\Sigma^\delta_{p,z,y}$ is as defined in (1.1). We will analyse the cases of negative and positive ν separately.

2.1. **The case $\nu \in (-1,0)$.**

Theorem 2.1. *Suppose that $\nu \in (-1,0)$, $p > -1$ and let $u_0(x) := K_{\frac{-\nu}{p+1}}\left(\frac{1}{p+1}\sqrt{\lambda}x^{p+1}\right)$. Then,*

$$0 < \lim_{x\to 0} u_0(\sqrt{x})x^{-\frac{\nu}{2}} < \infty, \qquad \lim_{x\to\infty} u_0(\sqrt{z})x^{-\frac{\nu}{2}} < \infty.$$

Consequently, $(M_{t\wedge R}^{(u_0)})_{t\geq 0}$ is a strictly positive bounded martingale with

$$\mathbf{1}_{[t<R]}dM_t^{(u_0)} = \mathbf{1}_{[t<R]}M_t^{(u_0)}\left(\frac{u_0'(\sqrt{X_t})}{u_0(\sqrt{X_t})} - \frac{\nu}{\sqrt{X_t}}\right)dB_t.$$

Proof. Note that

$$\lim_{x\to 0} u_0(\sqrt{x})x^{-\frac{\nu}{2}} = \left(\frac{\sqrt{\lambda}}{p+1}\right)^{\frac{\nu}{p+1}}\lim_{x\to 0} x^{-\frac{\nu}{p+1}}K_{-\frac{\nu}{p+1}}(x) = \left(\frac{2\sqrt{\lambda}}{p+1}\right)^{\frac{\nu}{p+1}}\frac{\Gamma(-\frac{\nu}{p+1})}{2}$$

in view of the asymptotic relations from (2.7). Asymptotic behaviour of $K_{-\frac{\nu}{p+1}}$ for large x yields $u_0(\sqrt{x})x^{-\frac{\nu}{2}}$ has a finite limit at infinity, hence the boundedness of $(M_{t\wedge R}^{(u_0)})_{t\geq 0}$ in view of Lemma 2.1. Strict positivity of K_α on $(0,\infty)$ for all α completes the proof of that this martingale is strictly positive. □

It is well-known that (see, e.g. Section 2.8 in [13]) for $\nu > -1$

$$(2.9) \qquad Q_z^\delta\left[\exp\left(-\frac{\lambda}{2}R_y\right)\right] = \frac{z^{-\frac{\nu}{2}}K_\nu(\sqrt{\lambda z})}{y^{-\frac{\nu}{2}}K_\nu(\sqrt{\lambda y})}, \qquad y \leq z;$$

$$(2.10) \qquad\qquad\qquad\qquad = \frac{z^{-\frac{\nu}{2}}I_\nu(\sqrt{\lambda z})}{y^{-\frac{\nu}{2}}I_\nu(\sqrt{\lambda y})}, \qquad y \geq z.$$

The above formulae are still valid when $\nu \geq 0$ and note that $z^{-\frac{\nu}{2}}I_\nu(\sqrt{\lambda z})$ (resp. $z^{-\frac{\nu}{2}}K_\nu(\sqrt{\lambda z})$) is increasing (resp. decreasing) in view of Remark 1.

Since $R < \infty$, a.s. when $\nu < 0$, the following is a straightforward corollary to the theorem above for $\nu < 0$ and $y < z$.

Corollary 2.1. *Let u_0 be the function defined in Theorem 2.1 and suppose that $\nu \in (-1,0)$, $p > -1$ and $y < z$. If U is $\mathcal{F}_{R_y}$-measurable, then for $r \geq 0$,*

$$Q_z^\delta\left[\exp\left(-rU - \frac{\lambda}{2}\Sigma_{p,z,y}^\delta\right)\right] = \frac{u_0(\sqrt{z})}{u_0(\sqrt{y})}\left(\frac{z}{y}\right)^{-\frac{\nu}{2}}P_z^{\delta,u_0}\left[\exp(-rU)\right],$$

where P_z^{δ,u_0} is defined by $\frac{dP_z^{\delta,u_0}}{dQ_z^\delta} = M_R^{(u_0)}$. Moreover, under P_z^{δ,u_0}, X solves

$$(2.11) \qquad dX_t = 2\left(\frac{u_0'(\sqrt{X_t})\sqrt{X_t}}{u_0(\sqrt{X_t})} + 1\right)dt + 2\sqrt{X_t}\,d\beta_t, \qquad t \leq R,$$

for some P_z^{δ,u_0}-Brownian motion β.

Proof. Since $(M_{t\wedge R}^{(u_0)})_{t\geq 0}$ is a strictly positive martingale due to Theorem 2.1, Q_z^δ is an equivalent probability measure. The fact that X solves (2.11) follows from an application of Girsanov's theorem. □

Remark 2. *By taking $U \equiv 0$ Corollary 2.1 yields the law of $\Sigma_{p,z,y}^\delta$ for $y < z$. Comparing this Laplace transform with (2.9) shows that $\Sigma_{p,z,y}^\delta \stackrel{d}{=} R_{y^*}^{\delta^*}$ where $\delta^* = 2(\frac{\nu}{1+p} + 1)$, $y^* = \frac{y^{1+p}}{(1+p)^2}$ and $R_{y^*}^{\delta^*}$ is the first hitting time of y^* for some $BESQ^{\delta^*}\left(\frac{z^{1+p}}{(1+p)^2}\right)$. One can check by comparing the Laplace transforms that we will obtain later in this section that this equality in law would be valid for $y \geq z$, too. Moreover, the same identities in distribution will hold for $\nu \geq 0$, too. These facts also follow from the time-change result given in Proposition XI.1.11 in [15].*

Recall that if X is a one-dimensional regular diffusion on an interval (l_0, l_1) defined by its infinitesimal generator, $\mathcal{A}$, where

$$\mathcal{A} = \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx}$$

for some locally bounded functions σ and b such that $\sigma > 0$ on the open interval (l_0, l_1), then for any $r > 0$ there exists a positive, and strictly decreasing (resp. increasing) function Φ (resp. Ψ), which solves

$$\mathcal{A}u = r\Phi \text{ (resp. } r\Psi)$$

on (l_0, l_1) satisfying certain boundary requirements depending on the nature of the behaviour of the diffusion near l_0 and l_1. Moreover, any other solution of this equation with the above positivity and monotonicity assumptions is a fixed multiple of Φ (resp. Ψ) (see, e.g., Proposition V.50.3 in [16]). Then, if R_y is the first hitting time of y,

$$\begin{aligned} P_z\left[\exp\left(-rR_y\right)\right] &= \frac{\Phi(z)}{\Phi(y)}, && y \leq z; \\ &= \frac{\Psi(z)}{\Psi(y)}, && y \geq z, \end{aligned}$$

where P_z is the law of the diffusion that started at z at $t = 0$.

Thus, if $U = R_y$, we obtain the following in view of the above discussion.

Corollary 2.2. *Let u_0 be the function defined in Theorem 2.1 and suppose that $\nu \in (-1, 0)$ and $y < z$. Then for $r \geq 0$,*

$$Q_z^\delta\left[\exp\left(-rR_y - \frac{\lambda}{2}\Sigma_{p,z,y}^\delta\right)\right] = \frac{u_0(\sqrt{z})}{u_0(\sqrt{y})}\left(\frac{z}{y}\right)^{-\frac{\nu}{2}}\frac{\Phi(z)}{\Phi(y)},$$

where Φ is a positive and decreasing solution of

$$(2.12) \qquad 2xv'' + 2\left(\frac{u_0'(\sqrt{x})\sqrt{x}}{u_0(\sqrt{x})} + 1\right)v' = rv.$$

Proof. The ODE in (2.12) corresponds to the diffusion, X, which follows

$$(2.13) \qquad dX_t = 2\left(\frac{u_0'(\sqrt{X_t})\sqrt{X_t}}{u_0(\sqrt{X_t})} + 1\right)dt + 2\sqrt{X_t}\,d\beta_t,$$

where β is a Brownian motion, if a solution exists. Using the recursive relation

$$K_\alpha'(x) = -\frac{\alpha}{x}K_\alpha(x) - K_{\alpha-1}(x),$$

we obtain

$$\frac{u_0'(\sqrt{x})\sqrt{x}}{u_0(\sqrt{x})} = \nu - \sqrt{\lambda}x^{\frac{p+1}{2}}\frac{K_{\frac{-\nu-p-1}{p+1}}\left(\frac{1}{p+1}\sqrt{\lambda}x^{\frac{p+1}{2}}\right)}{K_{\frac{-\nu}{p+1}}\left(\frac{1}{p+1}\sqrt{\lambda}x^{\frac{p+1}{2}}\right)} \leq \nu.$$

This shows that the drift of the above SDE is less than that of the SDE solved by a $BESQ^\delta$. Thus, when a solution exists, it never explodes due to the standard comparison results for SDEs. Moreover, using the asymptotic relations for K as x tends to 0, we can directly verify that

$$\lim_{x\to 0}\frac{u_0'(\sqrt{x})\sqrt{x}}{u_0(\sqrt{x})} = \nu.$$

Thus, the solution is pushed towards the interior of $(0, \infty)$ as soon as it hits 0 since the diffusion coefficient vanishes at 0. Finally, the existence of a solution taking values in $[0, \infty)$ follows easily since the drift term is locally Lipschitz as K_α is strictly positive on $[0, \infty)$ and K_α is twice continuously differentiable for all α. The solution is also unique due to drift coefficient being locally Lipschitz.

Recall that in view of Corollary 2.1 X solves (2.13) until the first time it hits 0. Since it cannot hit 0 before hitting y, the formula is a direct consequence of the discussion preceding the corollary. $\square$

Corollary 2.1 also allows us to compute the law of $\Sigma^\delta_{p,z,y}$ on the event that a certain boundary is yet to be reached via the intimate relationship between the scale functions of diffusions and their exit probabilities (see Sect.3 of Chap. VII in [15] for details).

Corollary 2.3. *Let u_0 be the function defined in Theorem 2.1 and suppose that $\nu \in (-1,0)$, $p > -1$ and $y < z$. Then, a scale function of the diffusion defined by (2.11) is*

$$(2.14) \qquad \tilde{s}_0(x) = \int_1^x \frac{1}{y u_0^2(\sqrt{y})} dy, \qquad x \geq 0.$$

Thus, for any $a > x$

$$Q_z^\delta \left[\mathbf{1}_{[R_a > R_y]} \exp \left(-\frac{\lambda}{2} \Sigma^\delta_{p,z,y} \right) \right] = \frac{u_0(\sqrt{x})}{u_0(\sqrt{y})} \left(\frac{x}{y} \right)^{-\frac{\nu}{2}} \frac{\tilde{s}_0(x) - \tilde{s}_0(a)}{\tilde{s}_0(y) - \tilde{s}_0(a)}.$$

Proof. The representation of the scale function is due to the well-known formulas for the solutions of SDEs, see, e.g., Exercise VII.3.20 in [15]. Note that the function is well-defined at $x = 0$. Indeed, it follows from Theorem 2.1 that $\lim_{y \to 0} u_0(\sqrt{y}) y^{-\frac{\nu}{2}} > 0$. Thus, $y u_0^2(\sqrt{y}) > C y^{1+\nu}$ for some $C > 0$ for sufficiently small y. Since $y^{-(1+\nu)}$ is integrable for $\nu \in [-1,0)$, the claim holds.

The second assertion follows from Corollary 2.1 after taking $r = 1$ and $U = \log \mathbf{1}_{[R_a > R_y]}$ via the defining property of scale functions, see Definition VII.3.3 in [15]. □

Remark 3. *The above result in fact gives us the joint law of $(\max_{t \leq R_y} X_t, \Sigma^\delta_{p,z,y})$. Indeed, for any $a \geq z > y$*

$$[\max_{t \leq R_y} X_t < a] = [R_a > R_y].$$

Since $K_{\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}} e^{-x}$, we have more explicit formulas when $\frac{\nu}{p+1} = -\frac{1}{2}$.

Corollary 2.4. *Suppose that $\frac{\nu}{p+1} = -\frac{1}{2}$ and $p > -1$. Then, for $y \leq z$ we have the following:*

i)
$$Q_z^\delta \left[\exp \left(-r R_y - \frac{\lambda}{2} \Sigma^\delta_{p,z,y} \right) \right] = \exp \left(\sqrt{\lambda} \frac{z^{-\nu} - y^{-\nu}}{2\nu} \right) \frac{\Phi(z)}{\Phi(y)},$$

where Φ is a positive and decreasing solution of

$$(2.15) \qquad 2xv'' + 2 \left(\nu + 1 - \sqrt{\lambda} x^{-\nu} \right) v' = rv$$

on $(0, \infty)$.

ii) *The function $\tilde{s}_0$ is, up to an affine transformation, given by $\exp \left(-\sqrt{\lambda} \frac{x^{-\nu}}{\nu} \right)$.*

iii) *For $a > z$,*

$$Q_z^\delta \left[\mathbf{1}_{[R_a > R_y]} \exp \left(-\frac{\lambda}{2} \Sigma^\delta_{p,z,y} \right) \right] = \exp \left(\sqrt{\lambda} \frac{z^{-\nu} - y^{-\nu}}{2\nu} \right) \frac{\exp \left(-\sqrt{\lambda} \frac{z^{-\nu}}{\nu} \right) - \exp \left(-\sqrt{\lambda} \frac{a^{-\nu}}{\nu} \right)}{\exp \left(-\sqrt{\lambda} \frac{y^{-\nu}}{\nu} \right) - \exp \left(-\sqrt{\lambda} \frac{a^{-\nu}}{\nu} \right)}$$

$$(2.16) \qquad\qquad = \frac{\sinh \left(-\sqrt{\lambda} \frac{a^{-\nu} - z^{-\nu}}{2\nu} \right)}{\sinh \left(-\sqrt{\lambda} \frac{a^{-\nu} - y^{-\nu}}{2\nu} \right)}.$$

Note that the expression in (2.16) yields

$$Q_z^\delta \left[\exp \left(-\frac{\lambda}{2} \Sigma^\delta_{p,z,y} \right) \Big| R_a > R_y \right] = \frac{a^{-\nu} - y^{-\nu}}{a^{-\nu} - z^{-\nu}} \frac{\sinh \left(-\sqrt{\lambda} \frac{a^{-\nu} - z^{-\nu}}{2\nu} \right)}{\sinh \left(-\sqrt{\lambda} \frac{a^{-\nu} - y^{-\nu}}{2\nu} \right)}$$

using the scale function of X under Q_z^δ. Comparing this with (2.10) for $\nu = 1/2$ gives the following since $I_{\frac{1}{2}}(x) = \sqrt{\frac{2\pi}{x}} \sinh(x)$.

Corollary 2.5. *Suppose that $\frac{\nu}{p+1} = -\frac{1}{2}$, $\nu \in (-1,0)$, and $p > -1$. Then, for $y \leq z < a$ we have that the law of $\Sigma^{\delta}_{p,z,y}$ conditioned on the event $[R_a > R_y]$ is that of the first hitting time of $(a^{-\nu} - y^{-\nu})^2/4\nu^2$ by a 3-dimensional squared Bessel process started at $(a^{-\nu} - z^{-\nu})^2/4\nu^2$.*

Note that, since $\lim_{a\to 0} = \frac{1-x^a}{a} = -\log x$, when $y > 0$, we obtain that the above conditional laws converge as $\nu \to 0$ (and, thus, as $p \to -1$) to that of the first hitting time of $(\log \sqrt{a} - \log \sqrt{y})^2$ by a 3-dimensional squared Bessel process started at $(\log \sqrt{a} - \log \sqrt{z})^2$. This can be viewed as the analogous statement of the above corollary when $\nu = 0$ and $p = -1$.

Next we look at the case when $y > z \geq 0$. Observe that the function u_1 as defined in the theorem below is still well defined and finite at $z = 0$ in view of, e.g., the series representation of I_α in (2.5).

Theorem 2.2. *Suppose that $p \geq 0$, $y \geq z$, and $\nu \in (-1,0)$. Let $u_1(x) := I_{\frac{\nu}{p+1}}\left(\frac{\sqrt{\lambda}}{p+1}x^{p+1}\right)$. Then, $(M^{(u_1)}_{t \wedge R_y})_{t \geq 0}$ is a bounded martingale with*

$$\mathbf{1}_{[t<R_y]}dM^{(u_1)}_t = \mathbf{1}_{[t<R_y]}M^{(u_1)}_t\left(\frac{u_1'(\sqrt{X_t})}{u_1(\sqrt{X_t})} - \frac{\nu}{\sqrt{X_t}}\right)dB_t,$$

for any $y \geq 0$.

Proof. Note that $[R < R_y]$ has a positive probability. Thus, we have to pay attention to the behaviour of $w(x) = u_1(\sqrt{x})x^{-\frac{\nu}{2}}$ at $x = 0$. Observe that under our assumptions, $\frac{\nu}{p+1} > -1$, thus it follows from the series representation of I_α that $w(0) > 0$ and is finite since $I_\alpha(1) < \infty$ for any α. Next, we will show that w has an absolutely continuous derivative over $[0,\infty)$. Using the recurrence relations (see Section 3.7 in [19])

$$\begin{aligned}
I_{\alpha-1}(x) + I_{\alpha+1}(x) &= 2I_\alpha'(x), \text{ and} \\
\frac{x}{2}\left(I_{\alpha-1}(x) - I_{\alpha+1}(x)\right) &= \alpha I_\alpha(x),
\end{aligned}$$

we obtain that

$$(2.17) \qquad\qquad I_\alpha'(x) = I_{\alpha+1}(x) + \frac{\alpha}{x}I_\alpha(x).$$

Using this identity it follows from direct calculations that

$$w'(x) = \frac{\sqrt{\lambda}}{2}x^{\frac{p-\nu-1}{2}}I_\gamma\left(\frac{\sqrt{\lambda}}{p+1}x^{\frac{p+1}{2}}\right),$$

where $\gamma = 1 + \frac{\nu}{p+1}$. Since the leading term of $I_\gamma\left(\frac{\sqrt{\lambda}}{p+1}x^{\frac{p+1}{2}}\right)$ as $x \to 0$ is $x^{\frac{\nu+p+1}{2}}$, we see that $\lim_{x\to 0} w'(x) = 0$ when $p > 0$. Therefore, we obtain immediately from the ODE (2.3) that when $p > 0$ $\lim_{x\to 0} xw''(x) = 0$ for $\nu \in (-1,0)$. On the other hand, when $p = 0$ and $\nu > -1$,

$$\frac{w'(x)}{w(x)} = \frac{\lambda}{2}\frac{1}{\sqrt{\lambda x}}\frac{I_{\nu+1}(\sqrt{\lambda x})}{I_\nu(\sqrt{\lambda x})}$$

by another application of (2.17). Thus, in view of the asymptotics of I_α as $x \to 0$

$$\begin{aligned}
\lim_{x\to 0}\frac{w'(x)}{w(x)} &= \frac{\lambda}{2}\lim_{x\to 0}\frac{I_{\nu+1}(x)}{xI_\nu(x)} \\
&= \frac{\lambda}{4}\frac{\Gamma(\nu+1)}{\Gamma(\nu+2)} = \frac{\lambda}{4(\nu+1)}.
\end{aligned}$$

Consequently, $\lim_{x\to 0} 2(\nu+1)w' - \frac{\lambda}{2}w = 0$ since $w(x) > 0$ for all $x \geq 0$. Again, it follows from the ODE (2.3) that $\lim_{x\to 0} xw''(x) = 0$. However, this condition implies that $\int_0^x w''(y)\,dy$ exists and is finite. Since this integral equals $w'(x) - w'(0)$ for any $x \in [0,\infty)$, we conclude that w' is absolutely continuous on $[0,\infty)$

and $w'(x) = w'(0) + \int_0^x w''(y)\,dy$ for any $x \in [0, \infty)$. Then, in view of Problem 3.7.3 in [9] we immediately deduce that

$$
\begin{aligned}
w(X_t) &= w(X_0) + \int_0^t 2w'(X_s)\sqrt{X_s}\,dB_s + \int_0^t \{2w'(X_s)(\nu + 1) + 2w''(X_s)X_s\}\,ds \\
&= \int_0^t 2w'(X_s)\sqrt{X_s}\,dB_s + \frac{\lambda}{2}\int_0^t X_s^p w(X_s)\,ds.
\end{aligned}
$$

A simple application of integration by parts formula now shows that $M^{(u_1)}$ is a martingale with the claimed representation. □

Remark 4. *Observe that $u_1(x)x^{-\nu}$ is still monotone in view of Remark 1 since under our hypothesis $\frac{\nu}{p+1} > -2$.*

Corollary 2.6. *Let u_1 be the function defined in Theorem 2.2 and suppose that the hypotheses therein hold. Then, we have the following for all $z \leq y$:*

i) *If $r \geq 0$ and U is $\mathcal{F}_{R_y}$-measurable,*

$$
Q_z^\delta\left[\exp\left(-rU - \frac{\lambda}{2}\Sigma_{p,z,y}^\delta\right)\right] = \frac{u_1(\sqrt{z})}{u_1(\sqrt{y})}\left(\frac{z}{y}\right)^{-\frac{\nu}{2}} P_{z,y}^{\delta,u_1}\left[\exp\left(-rU\right)\right],
$$

where $P_{z,y}^{\delta,u_1}$ is defined by $\frac{dP_{z,y}^{\delta,u_1}}{dQ_z^\delta} = M_{R_y}^{(u_1)}$. Moreover, under $P_{z,y}^{\delta,u_1}$, X solves

(2.18)
$$
dX_t = 2\left(\frac{u_1'(\sqrt{X_t})\sqrt{X_t}}{u_1(\sqrt{X_t})} + 1\right)dt + 2\sqrt{X_t}\,d\beta_t, \qquad t \leq R_y,
$$

for some $P_{z,y}^{\delta,u_1}$-Brownian motion β.

ii) *For all $r \geq 0$*

$$
Q_z^\delta\left[\exp\left(-rR_y - \frac{\lambda}{2}\Sigma_{p,z,y}^\delta\right)\right] = \frac{u_1(\sqrt{z})}{u_1(\sqrt{y})}\left(\frac{z}{y}\right)^{-\frac{\nu}{2}}\frac{\Psi(z)}{\Psi(y)},
$$

where Ψ is a positive and increasing solution of

(2.19)
$$
2xv'' + 2\left(\frac{u_1'(\sqrt{x})\sqrt{x}}{u_1(\sqrt{x})} + 1\right)v' = rv
$$

on $(0, \infty)$.

iii) *A scale function of the diffusion defined in (2.18) is given by*

(2.20)
$$
\tilde{s}_1(x) = \int_1^x \frac{1}{yu_1^2(\sqrt{y})}\,dy, \qquad x \geq 0.
$$

Thus, for any $0 \leq a < z$

$$
Q_z^\delta\left[\mathbf{1}_{[R_a > R_y]}\exp\left(-\frac{\lambda}{2}\Sigma_{p,z,y}^\delta\right)\right] = \frac{u_1(\sqrt{z})}{u_1(\sqrt{y})}\left(\frac{z}{y}\right)^{-\frac{\nu}{2}}\frac{\tilde{s}_1(z) - \tilde{s}_1(a)}{\tilde{s}_1(y) - \tilde{s}_1(a)}.
$$

Proof. The proof follows the similar lines as in the proofs of analogous results for $y \leq z$. The only difference is that contrary to the previous case the drift term of the diffusion corresponding to the ODE in (2.19) is only locally bounded. In particular, it is now larger than $2(\nu + 1)$, which in turn implies that the solution is immediately pushed to $(0, \infty)$ as soon as it hits 0 by comparison results for SDEs. Moreover, being only locally bounded causes no concern for our purposes since the computations involve the law of the diffusion until the first time it reaches y. □

Again, since $I_{-\frac{1}{2}}(x) = \sqrt{\frac{2\pi}{x}}\cosh(x)$ we have

Corollary 2.7. *$p \geq 0$, $\nu \in (-1, 0)$ and $\frac{\nu}{p+1} = -\frac{1}{2}$. Then, for $y \geq z$ we have the following:*

i)
$$Q_z^\delta \left[\exp\left(-rR_y - \frac{\lambda}{2}\Sigma_{p,z,y}^\delta \right) \right] = \frac{\cosh\left(-\frac{\sqrt{\lambda}}{2\nu}z^{-\nu} \right)}{\cosh\left(-\frac{\sqrt{\lambda}}{2\nu}y^{-\nu} \right)} \frac{\Psi(z)}{\Psi(y)},$$

where Ψ is a positive and increasing solution of

(2.21)
$$2xv'' + 2\left(\nu + 1 + \sqrt{\lambda}x^{-\nu}\tanh\left(-\frac{\sqrt{\lambda}}{2\nu}x^{-\nu} \right) \right)v' = rv$$

on $(0,\infty)$.

ii) *The function $\tilde{s}_1$ is, up to an affine transformation, given by* $\tanh\left(-\sqrt{\lambda}\frac{x^{-\nu}}{2\nu} \right)$.

iii) *For $0 \le a < z$,*

(2.22)
$$Q_z^\delta \left[\mathbf{1}_{[R_a > R_y]} \exp\left(-\frac{\lambda}{2}\Sigma_{p,z,y}^\delta \right) \right] = \frac{\sinh\left(-\sqrt{\lambda}\frac{z^{-\nu} - a^{-\nu}}{2\nu} \right)}{\sinh\left(-\sqrt{\lambda}\frac{y^{-\nu} - a^{-\nu}}{2\nu} \right)}.$$

Proof. Only part iii) needs proof. Note that

$$Q_z^\delta \left[\mathbf{1}_{[R_a > R_y]} \exp\left(-\frac{\lambda}{2}\Sigma_{p,z,y}^\delta \right) \right] = \frac{\cosh\left(-\frac{\sqrt{\lambda}}{2\nu}z^{-\nu} \right)}{\cosh\left(-\frac{\sqrt{\lambda}}{2\nu}y^{-\nu} \right)} \frac{\tanh\left(-\frac{\sqrt{\lambda}}{2\nu}x^{-\nu} \right) - \tanh\left(-\frac{\sqrt{\lambda}}{2\nu}a^{-\nu} \right)}{\cosh\left(-\frac{\sqrt{\lambda}}{2\nu}y^{-\nu} \right) - \tanh\left(-\frac{\sqrt{\lambda}}{2\nu}a^{-\nu} \right)}.$$

On the other hand,

$$\cosh\left(-\frac{\sqrt{\lambda}}{2\nu}z^{-\nu} \right) \left\{ \tanh\left(-\frac{\sqrt{\lambda}}{2\nu}x^{-\nu} \right) - \tanh\left(-\frac{\sqrt{\lambda}}{2\nu}a^{-\nu} \right) \right\}$$

$$= \frac{\sinh\left(-\frac{\sqrt{\lambda}}{2\nu}z^{-\nu} \right)\cosh\left(-\frac{\sqrt{\lambda}}{2\nu}a^{-\nu} \right) - \cosh\left(-\frac{\sqrt{\lambda}}{2\nu}z^{-\nu} \right)\sinh\left(-\frac{\sqrt{\lambda}}{2\nu}a^{-\nu} \right)}{\cosh\left(-\frac{\sqrt{\lambda}}{2\nu}a^{-\nu} \right)}$$

$$= \frac{\sinh\left(-\frac{\sqrt{\lambda}}{2\nu}z^{-\nu} \right)\cosh\left(\frac{\sqrt{\lambda}}{2\nu}a^{-\nu} \right) + \cosh\left(-\frac{\sqrt{\lambda}}{2\nu}z^{-\nu} \right)\sinh\left(\frac{\sqrt{\lambda}}{2\nu}a^{-\nu} \right)}{\cosh\left(-\frac{\sqrt{\lambda}}{2\nu}a^{-\nu} \right)}$$

$$= \frac{\sinh\left(\frac{\sqrt{\lambda}}{2\nu}(a^{-\nu} - z^{-\nu}) \right)}{\cosh\left(-\frac{\sqrt{\lambda}}{2\nu}a^{-\nu} \right)},$$

which yields the claimed representation. $\qquad\square$

Note that in fact we do not need to assume $p > 0$ for part iii) of the above result to hold, since X is never 0 before R_a for $0 < a < z$. Moreover, (2.22) and (2.16) are the same. Thus,

Corollary 2.8. *Suppose that $\frac{\nu}{p+1} = -\frac{1}{2}$, $\nu \in (-1,0)$ and $p > -1$. Then, for $y \ge z > a$ we have that the law of $\Sigma_{p,z,y}^\delta$ conditioned on the event $[R_a > R_y]$ is that of the first hitting time of $(a^{-\nu} - y^{-\nu})^2/4\nu^2$ by a 3-dimensional squared Bessel process started at $(a^{-\nu} - z^{-\nu})^2/4\nu^2$.*

We end this section with a scaling property which will be useful in the subsequent section. It is a direct consequence of the scaling property of $BESQ^\delta$ applied to the definition of $\Sigma_{p,0,y}^\delta$.

Proposition 2.1. *Suppose that $p \ge 0, \nu \in (-1,0)$. Then, we have the following identity in law for any $y \ge 0$:*

$$y^{p+1}\Sigma_{p,0,1}^\delta \overset{d}{=} \Sigma_{p,0,y}^\delta.$$

2.2. **The case $\nu \geq 0$.** Recall that when $\nu \geq 0$ the point 0 is polar for X. Thus, one can prove without any difficulty that $M^{(u_1)}$, where u_1 is as defined in Theorem 2.2, is a martingale stopped at R_y. Note that for $\nu \geq 0$ the random variable $\Sigma_{p,0,y}^\delta$ is well-defined and finite even in the case $p \in (-1, 0)$. Indeed, Corollary XI.1.12 in [15] show that

$$\int_0^1 X_s^p ds \stackrel{d}{=} \left(\int_0^1 Y_s^{-\frac{2p}{2+p}} ds \right)^{\frac{2+p}{p}},$$

where X is a $BESQ^{2(\nu+1)}(0)$ and Y is a $BESQ^{2\left(\nu \frac{2}{2+p}+1\right)}(0)$. Since $-\frac{2p}{2+p} > 0$ whenever $p \in (-1, 0)$ we easily deduce that the random variable on the left-hand side of the above identity is finite. Since a Bessel process with a positive dimension never comes back to 0 again this yields the finiteness of $\int_0^\varepsilon X_s^p ds$ for all $\varepsilon > 0$, which in turn implies the finiteness of $\Sigma_{p,0,y}^\delta$.

Theorem 2.3. *Suppose that $p > -1, y \geq z$, and $\nu \geq 0$. Let u_1 be the function defined in Theorem 2.2. Then, $(M_{t \wedge R_y}^{(u_1)})_{t \geq 0}$ is a bounded martingale.*

Recall that $BESQ^\delta$ is transient when $\nu > 0$, thus $Q_z^\delta(R_y < \infty) = 1$ whenever $y \geq z$. Consequently, we can deduce the following.

Corollary 2.9. *Let u_1 be the function defined in Theorem 2.2 and suppose that $p > -1, \nu \geq 0$. Then, we have the following for all $z \leq y$:*

i) *If $r \geq 0$ and U is $\mathcal{F}_{R_y}$-measurable,*

$$Q_z^\delta \left[\exp \left(-rU - \frac{\lambda}{2} \Sigma_{p,z,y}^\delta \right) \right] = \frac{u_1(\sqrt{z})}{u_1(\sqrt{y})} \left(\frac{z}{y} \right)^{-\frac{\nu}{2}} P_{z,y}^{\delta, u_1} \left[\exp(-rU) \right],$$

where $P_{z,y}^{\delta, u_1}$ is defined by $\frac{dP_{z,y}^{\delta,u_1}}{dQ_z^\delta} = M_{R_y}^{(u_1)}$. Moreover, under $P_{z,y}^{\delta, u_1}$, X satisfies (2.18).

ii) *For all $r \geq 0$*

$$Q_z^\delta \left[\exp \left(-rR_y - \frac{\lambda}{2} \Sigma_{p,z,y}^\delta \right) \right] = \frac{u_1(\sqrt{z})}{u_1(\sqrt{y})} \left(\frac{z}{y} \right)^{-\frac{\nu}{2}} \frac{\Psi(z)}{\Psi(y)},$$

where Ψ is a positive and increasing solution of (2.19) on $(0, \infty)$.

iii) *For any $a < z$*

$$Q_z^\delta \left[\mathbf{1}_{[R_a > R_y]} \exp \left(-\frac{\lambda}{2} \Sigma_{p,z,y}^\delta \right) \right] = \frac{u_1(\sqrt{z})}{u_1(\sqrt{y})} \left(\frac{z}{y} \right)^{-\frac{\nu}{2}} \frac{\tilde{s_1}(z) - \tilde{s_1}(a)}{\tilde{s_1}(y) - \tilde{s_1}(a)},$$

where $\tilde{s_1}$ is as defined in (2.20).

Analogous to Proposition 2.1 we have the following scaling property.

Proposition 2.2. *Suppose that $p > -1, \nu \geq 0$. Then, we have the following identity in law for any $y \geq 0$:*

$$y^{p+1} \Sigma_{p,0,1}^\delta \stackrel{d}{=} \Sigma_{p,0,y}^\delta.$$

We now return to the case $y \leq z$.

Proposition 2.3. *Let u_0 be the function defined in Theorem 2.1 and suppose that $\nu \geq 0, p > -1$. Then, we have the following for all $z \geq y$:*

i) *If $r \geq 0$ and U is $\mathcal{F}_{R_y}$-measurable,*

$$Q_z^\delta \left[\exp \left(-rU - \frac{\lambda}{2} \Sigma_{p,z,y}^\delta \right) \right] = \frac{u_0(\sqrt{z})}{u_0(\sqrt{y})} \left(\frac{z}{y} \right)^{-\frac{\nu}{2}} P_{z,y}^{\delta, u_0} \left[\exp(-rU) \right],$$

where $P_{z,y}^{\delta, u_0}$ is defined by $\frac{dP_{z,y}^{\delta,u_0}}{dQ_z^\delta} = M_{R_y}^{(u_0)}$. Moreover, under $P_{z,y}^{\delta, u_0}$, X satisfies (2.11) until R_y.

ii) *For all $r \geq 0$*

$$Q_z^\delta \left[\exp \left(-rR_y - \frac{\lambda}{2}\Sigma_{p,z,y}^\delta \right) \right] = \frac{u_0(\sqrt{z})}{u_0(\sqrt{y})} \left(\frac{z}{y} \right)^{-\frac{\nu}{2}} \frac{\Phi(z)}{\Phi(y)},$$

where Φ is a positive and decreasing solution of (2.12) on $(0,\infty)$.

iii) *For any $a > z$*

$$Q_z^\delta \left[\mathbf{1}_{[R_a > R_y]} \exp \left(-\frac{\lambda}{2}\Sigma_{p,z,y}^\delta \right) \right] = \frac{u_0(\sqrt{z})}{u_0(\sqrt{y})} \left(\frac{z}{y} \right)^{-\frac{\nu}{2}} \frac{\tilde{s}_0(z) - \tilde{s}_0(a)}{\tilde{s}_0(y) - \tilde{s}_0(a)},$$

where $\tilde{s}_0$ is as defined in (2.14).

Proof. We will only give the details for the parts of the proof that differ from the analogous results of the previous section. Observe that $(M_{t \wedge R_y}^{(u_0)})_{t \geq 0}$ is a uniformly integrable martingale when X starts at a value larger than y. Thus, using the Optional Stopping Theorem, we obtain

$$
\begin{aligned}
u_0(\sqrt{z})z^{-\frac{\nu}{2}} P_{z,y}^{\delta,u_0} \left[\exp\left(-rU\right) \right] &= Q_z^\delta \left[\exp\left(-rU\right) M_{R_y}^{(u_0)} \right] \\
&= Q_z^\delta \left[\mathbf{1}_{[R_y < \infty]} u_0(\sqrt{y})y^{-\frac{\nu}{2}} \exp\left(-rU - \frac{\lambda}{2}\Sigma_{p,z,y}^\delta \right) \right] \\
&\quad + Q_z^\delta \left[\mathbf{1}_{[R_y = \infty]} M_\infty^{(u_0)} \right].
\end{aligned}
$$

For $\nu = 0$, R_y is finite a.s., hence the claim. In case of $\nu > 0$, we still obtain the formula since, on the set $[R_y = \infty]$, $\Sigma_{p,z,y}^\delta = \infty$ as well as $u_0(\infty) = 0$ due to the transience of X.

Using the recurrence relations for K_α as in Corollary 2.2, we can deduce that

$$\frac{u_0'(\sqrt{x})\sqrt{x}}{u_0(\sqrt{x})} \leq -\nu,$$

and the left-hand side of the above inequality converging to the right-hand side as x tends to 0. Thus, solution of the SDE given by (2.11) until the first time it hits 0 is less than that of

$$dX_t = 2(-\nu + 1)dt + 2\sqrt{|X_t|}d\beta_t.$$

Note that if $\nu > 1$ the drift of the above SDE is negative hence its solution corresponds to a squared Bessel process of negative dimension. This SDE has a unique strong solution and it stays $(-\infty, 0]$ as soon as it hits 0 (see Section 3 of [6]). Thus, we conclude by means of comparison results for SDEs that there exists a solution to (2.11) until R_y for every $y > 0$.

The ODE (2.12) has increasing and decreasing solutions which can be determined as before via the Laplace transforms of the first hitting times of the the diffusion on $(0, \infty)$ with the generator

$$\mathcal{A} = 2x\frac{d^2}{dx^2} + 2\left(\frac{u_0'(\sqrt{x})\sqrt{x}}{u_0(\sqrt{x})} + 1 \right) \frac{d}{dx},$$

which is killed as soon as it reaches 0. $\qquad \square$

As before, using the explicit form of $K_{\frac{1}{2}}$, one gets

Corollary 2.10. *Suppose that $\frac{\nu}{p+1} = \frac{1}{2}$ and $p > -1$. Then, for $y \leq z$ we have the following:*

i)

$$Q_z^\delta \left[\exp \left(-rR_y - \frac{\lambda}{2}\Sigma_{p,z,y}^\delta \right) \right] = \frac{y^\nu}{z^\nu} \exp \left(-\sqrt{\lambda}\frac{z^\nu - y^\nu}{2\nu} \right) \frac{\Phi(z)}{\Phi(y)},$$

where Φ is a positive and decreasing solution of

(2.23)
$$2xv'' + 2\left(-\nu + 1 - \sqrt{\lambda}x^\nu \right) v' = rv$$

on $(0, \infty)$.

ii) *The function $\tilde{s}_0$ is, up to an affine transformation, given by* $\exp\left(\sqrt{\lambda}\frac{x^\nu}{\nu}\right)$.

iii) *For $a > z$*

$$Q_z^\delta\left[\mathbf{1}_{[R_a > R_y]}\exp\left(-\frac{\lambda}{2}\Sigma_{p,z,y}^\delta\right)\right] = \frac{y^\nu}{z^\nu}\exp\left(-\sqrt{\lambda}\frac{z^\nu - y^\nu}{2\nu}\right)\frac{\exp\left(\sqrt{\lambda}\frac{z^\nu}{\nu}\right) - \exp\left(\sqrt{\lambda}\frac{a^\nu}{\nu}\right)}{\exp\left(\sqrt{\lambda}\frac{y^\nu}{\nu}\right) - \exp\left(\sqrt{\lambda}\frac{a^\nu}{\nu}\right)}$$

$$(2.24) \qquad\qquad = \frac{y^\nu}{z^\nu}\frac{\sinh\left(\sqrt{\lambda}\frac{a^\nu - z^\nu}{2\nu}\right)}{\sinh\left(\sqrt{\lambda}\frac{a^\nu - y^\nu}{2\nu}\right)}.$$

As in the case with $\frac{\nu}{p+1} = -\frac{1}{2}$ we get

$$Q_z^\delta\left[\exp\left(-\frac{\lambda}{2}\Sigma_{p,z,y}^\delta\right)\middle| R_a > R_y\right] = \frac{y^\nu}{z^\nu}\frac{y^{-\nu} - a^{-\nu}}{z^{-\nu} - a^{-\nu}}\frac{\sinh\left(\sqrt{\lambda}\frac{a^\nu - z^\nu}{2\nu}\right)}{\sinh\left(\sqrt{\lambda}\frac{a^\nu - y^\nu}{2\nu}\right)}$$

$$= \frac{1 - \left(\frac{y}{a}\right)^\nu}{1 - \left(\frac{z}{a}\right)^\nu}\frac{\sinh\left(\sqrt{\lambda}\frac{a^\nu - z^\nu}{2\nu}\right)}{\sinh\left(\sqrt{\lambda}\frac{a^\nu - y^\nu}{2\nu}\right)}$$

$$= \frac{a^\nu - y^\nu}{a^\nu - z^\nu}\frac{\sinh\left(\sqrt{\lambda}\frac{a^\nu - z^\nu}{2\nu}\right)}{\sinh\left(\sqrt{\lambda}\frac{a^\nu - y^\nu}{2\nu}\right)},$$

and hence

Corollary 2.11. *Suppose that $\frac{\nu}{p+1} = \frac{1}{2}$ and $p > -1$. Then, for $y \le z < a$ we have that the law of $\Sigma_{p,z,y}^\delta$ conditioned on the event $[R_a > R_y]$ is that of the first hitting time of $(a^\nu - y^\nu)^2/4\nu^2$ by a 3-dimensional squared Bessel process started at $(a^\nu - z^\nu)^2/4\nu^2$.*

Similarly, since $I_{\frac{1}{2}}(x) = \sqrt{\frac{2\pi}{x}}\sinh(x)$ we have

Corollary 2.12. *$p > -1$ and $\frac{\nu}{p+1} = \frac{1}{2}$. Then, for $y \ge z$ we have the following:*

i)

$$Q_z^\delta\left[\exp\left(-rR_y - \frac{\lambda}{2}\Sigma_{p,z,y}^\delta\right)\right] = \frac{y^\nu \sinh\left(\frac{\sqrt{\lambda}}{2\nu}z^\nu\right)}{z^\nu \sinh\left(\frac{\sqrt{\lambda}}{2\nu}y^\nu\right)}\frac{\Psi(z)}{\Psi(y)},$$

where Ψ is a positive and increasing solution of

$$(2.25) \qquad 2xv'' + 2\left(-\nu + 1 + \sqrt{\lambda}x^\nu\coth\left(\frac{\sqrt{\lambda}}{2\nu}x^\nu\right)\right)v' = rv$$

on $(0, \infty)$.

ii) *The function $\tilde{s}_1$ is, up to an affine transformation, given by* $\coth\left(\sqrt{\lambda}\frac{x^\nu}{2\nu}\right)$.

iii) *For $0 \le a < z$,*

$$(2.26) \qquad Q_z^\delta\left[\mathbf{1}_{[R_a > R_y]}\exp\left(-\frac{\lambda}{2}\Sigma_{p,z,y}^\delta\right)\right] = \frac{y^\nu}{z^\nu}\frac{\sinh\left(\sqrt{\lambda}\frac{z^\nu - a^\nu}{2\nu}\right)}{\sinh\left(\sqrt{\lambda}\frac{y^\nu - a^\nu}{2\nu}\right)}.$$

Remark 5. *Comparing parts i) of Corollary 2.4 and 2.10 immediately gives us that, for $z \ge y$, the distributions of $\Sigma_{p,z,y}^\delta$ are different when ν has different signs. On the other hand, Corollaries 2.5 and 2.11 imply that they have the same distribution once they are conditioned on the event that the maximum of the underlying squared Bessel process is less than a by time R_y. Same conclusion holds when $z \le y$.*

3. Small ball problem and Chung's law of iterated logarithm

The small ball problem (also called small deviations) for a stochastic process $Z = (Z_t)_{t \in \mathcal{T}}$ consists in finding the probability

$$\mathbb{P}[\|Z\| < \varepsilon] \qquad \text{as } \varepsilon \to 0,$$

$\|\cdot\|$ is a given norm, usually L^p or L^∞. It is connected to many other questions, such as the law of the iterated logarithm of Chung's type (Chung's LIL for short), strong limit laws in statistics, metric entropy properties of linear operators and several approximation quantities for stochastic processes. The determination of the above probability is not feasible other than in a very few cases and one is inclined to consider the asymptotic behaviour of

$$- \log \mathbb{P}[\|Z\| < \varepsilon] \qquad \text{as } \varepsilon \to 0.$$

The solution to the latter problem is also not available in full generality. However, one can get this asymptotic behaviour for Gaussian processes (see, e.g., [12] and [11]) or real-valued Lévy processes (see [2]). There is a large amount of literature on small ball probabilities in the Gaussian setting and one can consult the survey article [12].

As one can expect from the computations made in the previous section, we will be interested in the small ball probabilities for the stochastic process $(X_t)_{t \geq 0}$, and the "norm"

$$\|Z\|_{p,y} = \left(\int_0^{R_y} |Z_t|^p \, dt \right)^{\frac{1}{p}},$$

where $p \in (0, \infty)$ and $R_y = \inf\{t > 0 : X_t = y\}$. Observe that the above definition is not a real norm unless $p \geq 1$, however, as the results in this section does not depend on whether $\|\cdot\|_{p,y}$ is a true norm, this is not a problem. Our results and proofs are close in nature to the results of [10].

Interestingly, the small ball probabilities for X under the above norm does not depend on its index, ν, as seen from the next theorem.

Theorem 3.1. *Let X be a $BESQ^\delta$ as defined by (2.1) with $\delta > 0$, and $R_y = \inf\{t > 0 : X_t = y\}$. Then, one has, for $z \geq 0$ and $y \geq 0$,*

$$\lim_{\lambda \to \infty} \lambda^{-\frac{1}{2}} \log Q_z^\delta \left[\exp\left(-\lambda \|X\|_{p,y}^p \right) \right] = -\frac{\sqrt{2}}{p+1} \left| z^{\frac{p+1}{2}} - y^{\frac{p+1}{2}} \right|$$

$$\lim_{\varepsilon \to 0} \varepsilon^p \log Q_z^\delta \left[\|X\|_{p,y} < \varepsilon \right] = -\frac{1}{2(p+1)^2} \left(z^{\frac{p+1}{2}} - y^{\frac{p+1}{2}} \right)^2.$$

Proof. Let $w(x) = u(\sqrt{x}) x^{-\frac{\nu}{2}}$ where $u = u_0$ (resp. $u = u_1$) for $y \leq z$ (resp. $y > z$) and u_0 and u_1 are as defined in Theorems 2.1 and 2.2, respectively. Then, it follows from the results of the previous section that

$$\sqrt{2} \lambda^{-\frac{1}{2}} \log Q_z^\delta \left[\exp\left(-\frac{\lambda}{2} \|X\|_{p,y}^p \right) \right] = \sqrt{2} \frac{\log w(z)}{\sqrt{\lambda}} - \sqrt{2} \frac{\log w(y)}{\sqrt{\lambda}}.$$

Moreover, when $u = u_0$,

$$\begin{aligned}
\lim_{\lambda \to \infty} \frac{\log w(x)}{\sqrt{\lambda}} &= \lim_{\lambda \to \infty} \frac{\log u_0(\sqrt{x})}{\sqrt{\lambda}} \\
&= \frac{x^{\frac{p+1}{2}}}{p+1} \lim_{\lambda \to \infty} \frac{\log K_{\frac{|\nu|}{p+1}} \left(\frac{\sqrt{\lambda}}{p+1} x^{\frac{p+1}{2}} \right)}{\sqrt{\lambda}} \frac{p+1}{x^{\frac{p+1}{2}}} \\
&= \frac{x^{\frac{p+1}{2}}}{p+1} \lim_{\lambda \to \infty} \frac{\log K_{\frac{|\nu|}{p+1}} (\lambda)}{\lambda}.
\end{aligned}$$

However, using the asymptotic expansions in (2.7) we obtain that for any $\alpha \geq 0$,

$$\lim_{\lambda \to \infty} \frac{\log K_\alpha(\lambda)}{\lambda} = -1.$$

This shows that when $y \le z$

$$\lim_{\lambda \to \infty} \lambda^{-\frac{1}{2}} \log Q_z^\delta \left[\exp \left(-\lambda \|X\|_{p,y}^p \right) \right] = -\frac{\sqrt{2}}{p+1} \left| z^{\frac{p+1}{2}} - y^{\frac{p+1}{2}} \right|.$$

In order to show the above limit when $y > x$, it suffices to show that $\lim_{x \to \infty} \frac{\log I_{\frac{\nu}{p+1}}(x)}{x} = 1$, which again follows from (2.7). This completes the proof of the first assertion of the theorem.

The second assertion follows by applying de Bruijn's exponential Tauberian theorem (see Theorem 4.12.9 in [4]) to $\alpha = -1$ and $\beta = \frac{1}{2(p+1)^2} \left(z^{\frac{p+1}{2}} - y^{\frac{p+1}{2}} \right)^2$. $\qquad\qquad\square$

Observe that $\Sigma_{p,0,y}^\delta$ is an increasing process when indexed by y. We will next use the above theorem to obtain Chung's LIL for $(\Sigma_{p,0,y}^\delta)_{y \ge 0}$.

Theorem 3.2. *Let* $\phi(y) := \frac{y^{p+1}}{\log\log y}$. *Then, for any* $\nu > -1$ *and* $p > 0$ *one has*

$$\liminf_{y \to \infty} \frac{\Sigma_{p,0,y}^\delta}{\phi(y)} = \frac{1}{2(p+1)^2}, \ Q_0^\delta\text{-}a.s..$$

Proof. It follows form Theorem 3.1 that

$$\lim_{\varepsilon \to 0} \varepsilon \log Q_0^\delta \left[\Sigma_{p,0,1}^\delta < \varepsilon \right] = -\frac{1}{2(p+1)^2},$$

thus, for sufficiently small ε,

$$Q_0^\delta \left[\Sigma_{p,0,1}^\delta < \varepsilon \right] \le \exp\left(-\frac{K}{\varepsilon} \right),$$

where K is a fixed, but arbitrary, constant in $(0, \frac{1}{2(p+1)^2})$. Fix $C > 1$ and set $y_n = C^n$. Next choose $k > 0$ so that $kC^{p+1} < K$, In view of Propositions 2.1 and 2.2, we get for all large n

$$Q_0^\delta \left[\Sigma_{p,0,y_n}^\delta < k\phi(y_{n+1}) \right] = Q_0^\delta \left[\Sigma_{p,0,1}^\delta < \frac{kC^{p+1}}{\log\log y_{n+1}} \right]$$

$$\le \exp\left(-\frac{K}{kC^{p+1}} \log\left((n+1) \log C \right) \right) = (\log C)^{-\frac{K}{kC^{p+1}}} (n+1)^{-\frac{K}{kC^{p+1}}},$$

which is summable in n. Therefore, by the first Borel-Cantelli lemma, we have that, a.s. for large n, $\frac{\Sigma_{p,0,y_n}^\delta}{\phi(y_{n+1})} \ge k$. On the other hand, for $y \in [y_n, y_{n+1}]$,

$$\Sigma_{p,0,y}^\delta \ge \Sigma_{p,0,y_n}^\delta \ge k\phi(y_{n+1}) \ge k\phi(y),$$

which shows that

$$\liminf_{y \to \infty} \frac{\Sigma_{p,0,y}^\delta}{\phi(y)} \ge k, \ Q_0^\delta\text{-a.s.},$$

and thus

$$\liminf_{y \to \infty} \frac{\Sigma_{p,0,y}^\delta}{\phi(y)} \ge \frac{1}{2(p+1)^2}, \ Q_0^\delta\text{-a.s.},$$

by the arbitrariness of C, K and k.

We now turn to prove the reverse inequality. First, let's observe that

$$(3.1) \qquad\qquad \liminf_{y \to \infty} \frac{\Sigma_{p,0,y}^\delta}{y^{p+1}} < \infty, \ Q_0^\delta\text{-a.s..}$$

The above claim follows from a direct application of Fatou's lemma since $\Sigma_{p,0,y}^\delta \overset{d}{=} y^{p+1} \Sigma_{p,0,1}^\delta$.

Next, fix an $\varepsilon > 0$, let $y_n = n^n$ and consider the events

$$E_n := \left[\int_{R_{y_{n-1}}}^{R_{y_n}} X_s^p \, ds \le (1 + 2\varepsilon) \frac{1}{2(p+1)^2} \phi(y_n) \right].$$

It follows from the strong Markov property of X that E_ns are independent, and we will now see that E_ns occur infinitely often due to the second Borel-Cantelli lemma. Indeed, by the definition of $\Sigma_{p,0,y_n}^\delta$, we obtain

$$
\begin{aligned}
Q_0^\delta(E_n) &\geq Q_0^\delta\left[\Sigma_{p,0,y_n}^\delta \leq (1+2\varepsilon)\frac{1}{2(p+1)^2}\phi(y_n)\right] \\
&= Q_0^\delta\left[\Sigma_{p,0,1}^\delta \leq (1+2\varepsilon)\frac{1}{2(p+1)^2\log\log y_n}\right] \\
&\geq \exp\left(-\frac{1+\varepsilon}{1+2\varepsilon}\log\log y_n\right) \geq \frac{1}{\log y_n},
\end{aligned}
$$

where the second inequality is due to the fact that, for a given $\varepsilon > 0$, $Q_0^\delta\left[\Sigma_{p,0,1}^\delta \leq \eta\right] \geq \exp\left(-(1+\varepsilon)\frac{1}{2(p+1)^2}\frac{1}{\eta}\right)$ for sufficiently small η in view of the convergence result of Theorem 3.1. Since $\frac{1}{n\log n}$ is not summable, it follows from the Borel-Cantelli lemma that E_n occurs infinitely often. As ε was arbitrary this allows us to conclude, a.s.,

$$
\liminf_{n\to\infty} \frac{\int_{R_{y_{n-1}}}^{R_{y_n}} X_s^p\, ds}{\phi(y_n)} \leq \frac{1}{2(p+1)^2}.
$$

Thus, Q_0^δ-a.s.,

$$
(3.2) \qquad \liminf_{n\to\infty}\frac{\Sigma_{p,0,y_n}^\delta}{\phi(y_n)} \leq \liminf_{n\to\infty}\frac{\Sigma_{p,0,y_{n-1}}^\delta}{\phi(y_n)} + \frac{1}{2(p+1)^2} = \liminf_{n\to\infty}\frac{\Sigma_{p,0,y_{n-1}}^\delta}{y_{n-1}^{p+1}}\frac{y_{n-1}^{p+1}}{\phi(y_n)} + \frac{1}{2(p+1)^2}.
$$

On the other hand,

$$
\frac{y_{n-1}^{p+1}}{\phi(y_n)} \leq \frac{\log\log n^{n(p+1)}}{n^{p+1}},
$$

which converges to 0 as $n \to \infty$. Therefore, in view of (3.1) and (3.2), we obtain

$$
\liminf_{n\to\infty}\frac{\Sigma_{p,0,y_n}^\delta}{\phi(y_n)} \leq \frac{1}{2(p+1)^2}, \quad Q_0^\delta\text{-a.s..}
$$

$\square$

Remark 6. *We can in fact extend the above result so that for all $z \geq 0$*

$$
\liminf_{y\to\infty}\frac{\Sigma_{p,z,y}^\delta}{\phi(y)} = \frac{1}{2(p+1)^2}, \quad Q_z^\delta\text{-a.s..}
$$

Indeed, using the strong Markov property of X we have the decomposition

$$
\Sigma_{p,0,y}^\delta = \Sigma_{p,0,z}^\delta + \Sigma_{p,z,y}^\delta
$$

where $\Sigma_{p,0,z}^\delta$ and $\Sigma_{p,z,y}^\delta$ are independent. Dividing both sides by $\phi(y)$ and letting $y \to \infty$ yields the result.
Another formal check to this result can be performed by observing

$$
\Sigma_{p,z,y}^\delta \overset{d}{=} y^{p+1}\Sigma_{p,\frac{z}{y},1}^\delta \overset{d}{=} y^{p+1}\Sigma_{p,0,1}^\delta\frac{\Sigma_{p,\frac{z}{y},1}^\delta}{\Sigma_{p,0,1}^\delta} \overset{d}{=} \Sigma_{p,0,y}^\delta\frac{\Sigma_{p,\frac{z}{y},1}^\delta}{\Sigma_{p,0,1}^\delta},
$$

and that $\dfrac{\Sigma_{p,\frac{z}{y},1}^\delta}{\Sigma_{p,0,1}^\delta}$ converges to 1 as $y \to \infty$ once we identify $\Sigma_{p,\frac{z}{y},1}^\delta$ with

$$
\int_{R_{\frac{z}{y}}}^{R_1} X_s^p\, ds,
$$

where X is $BESQ^\delta(0)$.

4. Feller property and 'time reversal'

In the previous section we have proved a law of iterated logarithm for $\Sigma_{p,0,y}^\delta$ by considering it as a process indexed by y. In this section we will see, for $\nu \geq 0$, that it is in fact an *inhomogeneous* Feller process and find its infinitesimal generator.

First of all, it immediately follows from the strong Markov property of X that $(\Sigma_{p,0,y}^\delta, \mathcal{F}_{R_y})_{y \geq 0}$ is Markov. Suppose $P_{z,y}$ is the associated semigroup, i.e. $P_{z,y} f(a) = Q_0^\delta[f(\Sigma_{p,0,y}^\delta)|\Sigma_{p,0,z}^\delta = a]$. Since the increments of $(\Sigma_{p,0,y}^\delta)_{y \geq 0}$ are independent, we have for any bounded measurable f

$$(4.1) \qquad P_{z,y}f(a) = \int_0^\infty f(a+b) Q_z^\delta(\Sigma_{p,z,y}^\delta \in db).$$

Let C_0 denote the class of continuous functions on $\mathbb{R}_+$ that vanish at ∞. (4.1) readily implies that when $f \in C_0$, $P_{z,y}f \in C_0$ as well. Moreover, it follows from Corollary 2.9, and the observation that u_1 is finite at 0, that for each $z \geq 0$ the measure $Q_z^\delta(\Sigma_{p,z,y}^\delta \in db)$ converges weakly to the Dirac point mass at 0 as $y \downarrow z$ since its Laplace transform converges to 1. Therefore, $\lim_{y \downarrow z} P_{z,y}f(a) = f(a)$ and consequently $(\Sigma_{p,0,y}^\delta, \mathcal{F}_{R_y})_{y \geq 0}$ is Feller.

The form of the infinitesimal generator of $(\Sigma_{p,0,y}^\delta, \mathcal{F}_{R_y})_{y \geq 0}$ will follow from the following theorem.

Theorem 4.1. *Suppose that $\nu \geq 0$. Then, for every $x \geq 0$ there exists a decreasing function $\overline{\pi}(x, \cdot)$ satisfying*

$$\int_0^\infty e^{-\frac{\lambda}{2}b} \overline{\pi}(x,b) db = \frac{2}{\lambda} \frac{w'(x)}{w(x)}, \qquad \lambda \in (0, \infty),$$

where $w(x) := x^{-\frac{\nu}{2}} I_{\frac{\nu}{p+1}} \left(\frac{\sqrt{\lambda}}{p+1} x^{\frac{p+1}{2}} \right)$. Moreover,

$$(4.2) \qquad \lim_{y \downarrow z} \frac{Q_0^\delta \left[\exp\left(-\lambda \Sigma_{p,0,y}^\delta\right) \Big| \Sigma_{p,0,z}^\delta = a \right] - e^{-\lambda a}}{y-z} = \int_0^\infty \left\{ e^{-\lambda(a+b)} - e^{-\lambda a} \right\} \pi(z, db),$$

where $\pi(z, db) := -\overline{\pi}(z, db)$ for $b \geq 0$. In particular,

$$\int_0^1 b\pi(z,db) < \infty.$$

Proof. In view of the strong Markov property of X,

$$Q_0^\delta \left[\exp\left(-\frac{\lambda}{2} \Sigma_{p,0,y}^\delta\right) \Big| \Sigma_{p,0,z}^\delta = a \right] = e^{-\frac{\lambda}{2}a} Q_z^\delta \left[\exp\left(-\frac{\lambda}{2} \Sigma_{p,z,y}^\delta\right) \right].$$

Thus, it follows from Corollary 2.9 that

$$\lim_{y \downarrow z} \frac{Q_0^\delta \left[\exp\left(-\frac{\lambda}{2} \Sigma_{p,0,y}^\delta\right) \Big| \Sigma_{p,0,z}^\delta = a \right] - e^{-\frac{\lambda}{2}a}}{y-z} = e^{-\frac{\lambda}{2}a} \lim_{y \downarrow z} \frac{Q_z^\delta \left[\exp\left(-\frac{\lambda}{2} \Sigma_{p,z,y}^\delta\right) \right] - 1}{y-z}$$

$$(4.3) \qquad\qquad\qquad\qquad = e^{-\frac{\lambda}{2}a} \lim_{y \downarrow z} \frac{\frac{w(z)}{w(y)} - 1}{y-z} = -e^{-\frac{\lambda}{2}a} \frac{w'(z)}{w(z)}.$$

On the other hand, using integration by parts, we obtain

$$Q_z^\delta \left[\exp\left(-\frac{\lambda}{2} \Sigma_{p,z,y}^\delta\right) \right] - 1 = \int_0^\infty e^{-\frac{\lambda}{2}b} Q_z^\delta(\Sigma_{p,z,y}^\delta \in db) - 1$$

$$(4.4) \qquad\qquad\qquad\qquad = -\frac{\lambda}{2} \int_0^\infty e^{-\frac{\lambda}{2}b} Q_z^\delta(\Sigma_{p,z,y}^\delta > b) \, db.$$

It is well-known (see Corollary 3.8 in Chap. VII of [15]) that

$$Q_z^\delta[\Sigma_{p,z,y}^\delta] = \int_0^y x^p G(z,x) m(dx),$$

where m and G are the associated speed measure and Green's function, respectively. In our case, these are given by

$$m(dx) = \frac{x^\nu}{2\nu}dx; \qquad G(z,x) = -y^{-\nu} + (x \vee z)^{-\nu}.$$

Consequently, the above formula yields

$$Q_z^\delta[\Sigma_{p,z,y}^\delta] = \frac{y^{p+1} - z^{p+1}}{2(p+1)(p+\nu+1)},$$

which in particular implies that $\Pi(z,y,db) := \frac{2(p+1)(p+\nu+1)}{y^{p+1}-z^{p+1}}Q_z^\delta(\Sigma_{p,z,y}^\delta > b)db$ is a probability measure on $[0,\infty)$ for each (z,y). However, (4.3) and (4.4) imply that $\Pi(z,y,\cdot)$ converges weakly as y tends to z to some probability measure, $\overline{\Pi}(z,\cdot)$ on $[0,\infty)$ which satisfies[1]

$$(4.5) \qquad L\left(\frac{\lambda}{2}\right) := \int_0^\infty e^{-\frac{\lambda}{2}b}\overline{\Pi}(z,db) = \frac{4(p+\nu+1)}{\lambda}z^{-p}\frac{w'(z)}{w(z)}.$$

Moreover, since $Q_z^\delta(\Sigma_{p,z,y}^\delta > b)$ is decreasing in b for each (z,y), the limiting measure $\overline{\Pi}$ is necessarily of the form

$$c\varepsilon_0(db) + 2(p+\nu+1)z^{-p}\overline{\pi}(z,b)\,db,$$

where ε_0 is the Dirac point mass at 0, c a nonnegative constant, and $\overline{\pi}(z,\cdot)$ is a decreasing function for each x. In particular, $\overline{\pi}(z,\infty) = 0$. In order to find the constant c, it suffices to check the value of the function L at ∞. However, using the explicit form of w,

$$c = \lim_{\lambda\to\infty} \frac{4(p+\nu+1)}{\lambda}z^{-p}\frac{w'(z)}{w(z)} = \lim_{\lambda\to\infty} 2\frac{p+\nu+1}{\sqrt{\lambda}z^{\frac{p+1}{2}}}\frac{I'_{\frac{\nu}{p+1}}\left(\frac{\sqrt{\lambda}}{p+1}z^{\frac{p+1}{2}}\right)}{I_{\frac{\nu}{p+1}}\left(\frac{\sqrt{\lambda}}{p+1}z^{\frac{p+1}{2}}\right)} = 0$$

since

$$\lim_{x\to\infty} \frac{1}{x}\frac{I'_\alpha(x)}{I_\alpha(x)} = 2\lim_{x\to\infty} \frac{\log I_\alpha(x)}{x^2} = 0$$

due to (2.7).

Thus, we have shown that

$$-\frac{w'(z)}{w(z)} = \lim_{y\downarrow z} \frac{Q_z^\delta\left[\exp\left(-\frac{\lambda}{2}\Sigma_{p,z,y}^\delta\right)\right] - 1}{y-z} = -\frac{\lambda}{2}\int_0^\infty e^{-\frac{\lambda}{2}b}\overline{\pi}(z,b)\,db.$$

Since $\overline{\pi}$ is decreasing with $\overline{\pi}(z,\infty) = 0$, we obtain by integrating by parts

$$\lim_{y\downarrow z} \frac{Q_0^\delta\left[\exp\left(-\frac{\lambda}{2}\Sigma_{p,0,y}^\delta\right)\Big|\Sigma_{p,0,z}^\delta = a\right] - e^{-\frac{\lambda}{2}a}}{y-z} = e^{-\frac{\lambda}{2}a}\int_0^\infty \left\{e^{-\frac{\lambda}{2}b} - 1\right\}\pi(z,db)$$

where $\pi(z,db) = -\overline{\pi}(z,db)$. Finally, note that one necessarily has

$$\int_0^1 b\pi(z,db) < \infty,$$

since otherwise L, as defined in (4.5), would have been infinite. $\qquad \square$

The above theorem yields that the sequence of measures $\left(\frac{Q_z^\delta(\Sigma_{p,z,y}^\delta > b)}{y-z}db\right)$ converges vaguely to a finite measure on $(0,\infty)$ as $y \to z$. Thus, for any $f \in C_K^1(\mathbb{R}_+,\mathbb{R})$, i.e. the space of continuously differentiable functions with a compact support, we have

$$(4.6) \qquad \lim_{y\downarrow z} \frac{Q_0^\delta\left[f\left(\Sigma_{p,0,y}^\delta\right)|\Sigma_{p,0,z}^\delta = a\right] - f(a)}{y-z} = \int_0^\infty \{f(a+b) - f(a)\}\pi(z,db).$$

---

[1]Using the integral representation of I_α for $\alpha > -\frac{1}{2}$, it is tedious but straightforward to check that this representation holds for $z = 0$ as well by taking the limit as $z \to 0$ and showing that $L(\lambda) < \infty$ for $\lambda > 0$.

In other words, letting $B(\mathbb{R}_+)$ denote the bounded Borel functions defined on $\mathbb{R}_+$, if we define the operator $\mathcal{A}_z : B(\mathbb{R}_+) \mapsto B(\mathbb{R}_+)$ by setting

$$\mathcal{A}_z f(a) := \int_0^\infty \{f(a+b) - f(a)\}\, \pi(z, db) \qquad \text{for } f \in C_K^1(\mathbb{R}_+, \mathbb{R}),$$

then we see that the process $(M_y^f)_{y \geq 0}$ defined by

$$M_y^f := f(\Sigma_{p,0,y}^\delta) - \int_0^y \mathcal{A}_z f(\Sigma_{p,0,z}^\delta)\, dz$$

is a martingale with respect to the filtration $(\mathcal{F}_{R_y})_{y \geq 0}$ whenever f belongs to the domain of $\mathcal{A}_z$ for all $z \geq 0$. The form of the infinitesimal generator also reveals the fact that the increasing process $(\Sigma_{p,0,y}^\delta)_{y \geq 0}$ is purely discontinuous, i.e. there is no interval (a, b) in which it is continuous.

Remark 7. *It follows from the fact that $(R_y)_{y \geq 0}$ is left continuous that $(\Sigma_{p,0,y}^\delta)_{y \geq 0}$ is a left-continuous process. However, in view of the above Feller property one can obtain a càdlàg version of it when we augment the filtration with the null sets. Existence of a right-continuous version can also be independently verified by observing that*

$$\lim_{y \downarrow z} Q_0^\delta \left[\exp\left(-\lambda \Sigma_{p,0,y}^\delta \right) \right] = Q_0^\delta \left[\exp\left(-\lambda \Sigma_{p,0,z}^\delta \right) \right].$$

We will end this section by analysing a specific 'time reversal' example. To this end let $L_x := \sup\{t \geq 0 : X_t = x\}$ and suppose that $\nu > 0$ so that $Q_0^\delta(L_x < \infty) = 1$ for all $x \geq 0$. We will consider the process Z^δ defined by

$$(4.7) \qquad\qquad Z_x^\delta := \int_{L_{1-x}}^{L_1} X_s^p\, ds \qquad \forall x \in [0, 1).$$

In view of the well-known time reversal results for diffusions, see, e.g., Exercise 1.23 in Chap. XI of [15], the law of the process $(X_{L_1-t}, t < L_1)$ under Q_0^δ is identical to that of $(X_t, t < R_0)$ under $Q_1^{2-2\nu}$. Recall that $Q_1^{2-2\nu}(R_0 < \infty) = 1$. Thus, we can write

$$(4.8) \qquad\qquad Z_x^\delta = \int_0^{R_{1-x}} X_s^p\, ds, \qquad X = BESQ^{2-2\nu}(1).$$

Note that the above equality is to be understood in the sense of equality between the laws of the processes. Due to the strong Markov property of X we again have that $(Z_x^\delta)_{x \in [0,1)}$ is a Markov process with respect to the filtration $(\overleftarrow{\mathcal{F}}_x)_{x \in [0,1)}$ where $\overleftarrow{\mathcal{F}}_x := \sigma(X_s; L_{1-x} \leq s \leq L_1)$. Observe, more easily in view of (4.8), that Z^δ, too, has independent increments rendering its Feller property in view of the arguments that led to the Feller property of $(\Sigma_{p,0,z}^\delta)_{z \geq 0}$ at the beginning of this section. The next theorem will yield the form of the infinitesimal generator. Its proof will follow similar lines of the proof of Theorem 4.1 so we will only give the details when it differs.

Theorem 4.2. *Let Z^δ be as defined in (4.7) and suppose $\nu \in (0, 1]$. Then, for every $x \in [0, 1)$ there exists a decreasing function $\tilde{\pi}(x, \cdot)$ satisfying*

$$\int_0^\infty e^{-\frac{\lambda}{2} b} \tilde{\pi}(x, b)\, db = -\frac{2}{\lambda} \frac{w'(1-x)}{w(1-x)}, \qquad \lambda \in (0, \infty),$$

where $w(x) := x^{\frac{\nu}{2}} K_{\frac{\nu}{p+1}}\left(\frac{\sqrt{\lambda}}{p+1} x^{\frac{p+1}{2}} \right)$. Moreover,

$$(4.9) \qquad \lim_{y \downarrow x} \frac{Q_0^\delta \left[\exp\left(-\lambda Z_y^\delta \right) | Z_x^\delta = a \right] - e^{-\lambda a}}{y - x} = \int_0^\infty \left\{ e^{-\lambda(a+b)} - e^{-\lambda a} \right\} \rho(x, db),$$

where $\rho(x, db) := -\tilde{\pi}(x, db)$ for $b \geq 0$. In particular,

$$\int_0^1 b \rho(x, db) < \infty.$$

Proof. First observe that in view of (4.8) and the aforementioned independent increments property

$$Q_0^\delta\left[\exp\left(-\frac{\lambda}{2}Z_y^\delta\right)\middle|Z_x^\delta=a\right]=e^{-\frac{\lambda}{2}a}Q_{1-x}^\beta\left[\exp\left(-\frac{\lambda}{2}\Sigma_{p,1-x,1-y}^\beta\right)\right],$$

where $\beta=2-2\nu$. Therefore, the same arguments in the beginning of the proof of Theorem 4.1 yields that the measures $\frac{Q_{1-x}^\beta(\Sigma_{p,1-x,1-y}^\beta>b)db}{y-x}$ converges vaguely to some probability measure, $\tilde{\Pi}(x,\cdot)$ on $[0,\infty)$ which satisfies

$$(4.10) \qquad L\left(\frac{\lambda}{2}\right):=\int_0^\infty e^{-\frac{\lambda}{2}b}\tilde{\Pi}(x,db)=-\frac{1}{\lambda}\frac{w'(1-x)}{w(1-x)}.$$

Moreover, the limiting measure is necessarily of the form

$$c\varepsilon_0(db)+\tilde{\pi}(x,b)\,db,$$

where ε_0 is the Dirac point mass at 0, c a nonnegative constant, and $\tilde{\pi}(x,\cdot)$ is a decreasing function for each x. Observe that since $Q_{1-x}^\beta(\Sigma_{p,1-x,1-y}^\beta)=\infty$, the measures $\frac{Q_{1-x}^\beta(\Sigma_{p,1-x,1-y}^\beta>b)db}{y-x}$ are not finite and neither is their vague limit. On the other hand, we can still conclude that $c=0$ since, in view of (2.7), one has

$$\lim_{x\to\infty}\frac{1}{x}\frac{K_\alpha'(x)}{K_\alpha(x)}=2\lim_{x\to\infty}\frac{\log K_\alpha(x)}{x^2}=0.$$

Next, integrating (4.10) by parts yields

$$(4.11) \qquad \tilde{\pi}(x,\infty)-\int_0^\infty(1-e^{-\frac{\lambda}{2}b})\tilde{\pi}(x,db)=-\frac{w'(1-x)}{w(1-x)}.$$

Thus, due to the dominated convergence theorem, taking the limit of the above as λ tends to 0 yields

$$\tilde{\pi}(x,\infty)=-\lim_{\lambda\to 0}\frac{w'(1-x)}{w(1-x)}.$$

Also note that using the asymptotic behaviour of K near 0 (2.7) one has that

$$\lim_{x\to 0}x\frac{K_\alpha'(x)}{K_\alpha(x)}=\lim_{x\to 0}\frac{\log K_\alpha(x)}{\log x}=-\alpha.$$

for any $\alpha\geq 0$. This in turn yields that

$$\lim_{\lambda\to 0}\frac{w'(x)}{w(x)}=\frac{\nu}{2x}+\frac{p+1}{2x}\lim_{\lambda\to 0}\frac{K_{\frac{\nu}{p+1}}'\left(\frac{\sqrt{\lambda}}{p+1}x^{\frac{p+1}{2}}\right)}{K_{\frac{\nu}{p+1}}\left(\frac{\sqrt{\lambda}}{p+1}x^{\frac{p+1}{2}}\right)}\frac{\sqrt{\lambda}}{p+1}x^{\frac{p+1}{2}}=0,$$

and thus $\tilde{\pi}(x,\infty)=0$. As in the proof of Theorem 4.1

$$e^{-\frac{\lambda}{2}a}\frac{w'(1-x)}{w(1-x)}=\lim_{y\downarrow x}\frac{Q_0^\delta\left[\exp\left(-\frac{\lambda}{2}Z_y^\delta\right)\middle|Z_x^\delta=a\right]-e^{-\frac{\lambda}{2}a}}{y-x}.$$

Thus, combining above with (4.11) and plugging in the value of $\tilde{\pi}(x,\infty)$ yield

$$\lim_{y\downarrow x}\frac{Q_0^\delta\left[\exp\left(-\frac{\lambda}{2}Z_y^\delta\right)\middle|Z_x^\delta=a\right]-e^{-\lambda a}}{y-x}=\int_0^\infty\left\{e^{-\frac{\lambda}{2}(a+b)}-e^{-\frac{\lambda}{2}a}\right\}\rho(x,db).$$

$\square$

Example 4.1. *As an application of the above theorem consider the case when $\nu\in(0,1]$ and $\frac{\nu}{p+1}=\frac{1}{2}$. Then, the associated $\tilde{\pi}$ is defined by*

$$\int_0^\infty e^{-\frac{\lambda}{2}b}\tilde{\pi}(x,b)db=\frac{1}{\sqrt{\lambda}}(1-x)^{\nu-1}$$

in view of the explicit form for $K_{\frac{1}{2}}$. Thus, by inverting the above transform, we have

$$\tilde{\pi}(x,b)=(1-x)^{\nu-1}\frac{1}{\sqrt{2\pi b}}.$$

This reveals that the infinitesimal generator, $\tilde{A}_x$ of Z^δ is defined by

$$\tilde{A}_x = (1-x)^{\nu-1} \int_0^\infty \{f(a+b) - f(a)\} \frac{1}{2\sqrt{2\pi b^3}} \, db$$

for any f in C^1 with a compact support. Consequently,

$$f(Z_x^\delta) - \int_0^x (1-y)^{\nu-1} \int_0^\infty \{f(Z_y^\delta + b) - f(Z_y^\delta)\} \frac{1}{2\sqrt{2\pi b^3}} \, db \, dy$$

is an $\overleftarrow{\mathcal{F}}$-martingale for such f.

Example 4.2. *Observe that although Z^δ, or $(\Sigma_{p,0,y}^\delta)$, is an increasing process with independent increments, it is not a subordinator (see [3] for a definition and further properties) since the increments are not stationary. However, in the above example, if one takes $\nu = 1$, then one sees that Z^δ becomes time homogeneous, i.e. Z^δ is a subordinator. Observe that $\nu = 1$ implies $p = 1$ in this framework. More precisely,*

$$Z_x^\delta = Z_x^4 = \int_{L_{1-x}}^{L_1} X_s \, ds,$$

where X is $BESQ^4(1)$, is a subordinator. Moreover, Corollary 2.4 and (4.8) yield

$$Q_0^\delta \left(\exp\left(-\lambda Z_x^4\right)\right) = \exp\left(-\sqrt{\frac{\lambda}{2}} x\right).$$

Thus, $Z_x^4 \stackrel{d}{=} T_{\frac{x}{2}}$, where T_x is the first hitting time of x for a Brownian motion starting at 0.

5. Applications to finance

   Our aim in this section is to give some examples arising from some financial models and discuss how the results from previous sections can be used to obtain prices of certain financial products.

   As explained in Introduction the process X is commonly used in the finance literature to model interest rates. Suppose the spot interest rate is given by X^p where $p > -1$ and X is $BESQ^\delta(z)$ and consider the following *exotic* derivative on interest rates which pays one unit of a currency at time R_y if the accumulated interest is less than k, i.e. $\Sigma_{p,z,y}^\delta \leq \log k$. This is an example of a digital option and its price, as usual in the Finance Theory, is given as an expectation of its discounted payoff:

$$D(k; \delta, p, z, y) = Q_x^\delta \left[\mathbf{1}_{[\Sigma_{p,z,y}^\delta \leq k]} \exp\left(-\Sigma_{p,z,y}^\delta\right) \right].$$

On the other hand, if one computes the Laplace transform of $D(k; \delta, p, z, y)$, one obtains

$$\int_0^\infty e^{-\mu k} D(k; \delta, p, z, y) \, dk = Q_z^\delta \left[\int_{\Sigma_{p,z,y}^\delta}^\infty e^{-\mu k} \exp\left(-\Sigma_{p,z,y}^\delta\right) \, dk \right] = Q_z^\delta \left[\exp\left(-(\mu+1)\Sigma_{p,z,y}^\delta\right) \right],$$

which is at our disposal due to the results from Section 2. In particular, the above identity implies

$$\int_0^\infty e^{-\mu k} e^k D(k; \delta, p, z, y) \, dk = \int_0^\infty e^{-\mu u} Q_z^\delta \left[\Sigma_{p,z,y}^\delta \in du \right],$$

hence

(5.1) $$\int_0^u e^k D(k; \delta, p, x, y) \, dk = Q_z^\delta \left[\Sigma_{p,z,y}^\delta \leq u \right], \qquad \forall u \geq 0.$$

Moreover, once the function D is determined by inverting the corresponding Laplace transform, one can also obtain the prices for put options written on the accumulated interest. Indeed, for any $K > 1$

$$
\begin{aligned}
P(\delta; z, y, K) &:= Q_z^\delta \left[\left(K - \exp\left(\Sigma_{p,z,y}^\delta\right) \right)^+ \exp\left(-\Sigma_{p,z,y}^\delta\right) \right] \\
&= Q_z^\delta \left[\int_1^K \mathbf{1}_{[\Sigma_{p,z,y}^\delta \leq \log k]} \, dk \exp\left(-\Sigma_{p,z,y}^\delta\right) \right] \\
&= \int_1^K D(\log k; \delta, p, z, y) \, dk = \int_0^{\log K} e^u D(u; \delta, p, z, y) \, du.
\end{aligned}
$$
(5.2)

Note that for $K = 1$, i.e when the option is *at-the-money* since the cumulative interest at $t = 0$ is defined to be 1, the put option is worthless. We can in fact get how fast the option becomes worthless as K approaches to 1. Indeed, comparing (5.2) to (5.1) yields the following asymptotics for the option value in view of Theorem 3.1:

$$
\lim_{K \downarrow 1} \log K \log P(\delta; z, y, K) = -\frac{1}{2(p+1)^2} \left(z^{\frac{p+1}{2}} - y^{\frac{p+1}{2}} \right)^2 .
$$
(5.3)

The above expression tells us how small the option price becomes when the option is slightly *in-the-money*, i.e. when K is very close to 1.

Next, assume $y < z$ and consider another type of a put option on the maximum of the short rate with maturity R_y and payoff $(K - \max_{t \leq R_y} X_t)^+$ for $K > z$. The price of this option equals

$$
\begin{aligned}
Q_z^\delta \left[\exp\left(-\Sigma_{p,z,y}^\delta\right) \int_0^K \mathbf{1}_{[\max_{t \leq R_y} X_t < a]} \, da \right] &= Q_z^\delta \left[\exp\left(-\Sigma_{p,z,y}^\delta\right) \int_x^K \mathbf{1}_{[R_a > R_y]} \, da \right] \\
&= \frac{u_0(\sqrt{z})}{u_0(\sqrt{y})} \left(\frac{z}{y} \right)^{-\frac{\nu}{2}} \int_x^K \frac{\tilde{s}_0(z) - \tilde{s}(a)}{\tilde{s}_0(y) - \tilde{s}_0(a)} \, da,
\end{aligned}
$$

in view of Corollary 2.3 and Proposition 2.3, where the pair $(u_0, \tilde{s}_0)$ is computed by setting $\lambda = 2$.

Finally, if one is interested in pricing an Asian option on the short rate until time R_y, it suffices to use the identity

$$
Q_z^\delta \left[\left(K - \frac{\Sigma_{p,z,y}^\delta}{R_y} \right)^+ \exp\left(-\Sigma_{p,z,y}^\delta\right) \right] = Q_z^\delta \left[\exp\left(-\Sigma_{p,z,y}^\delta\right) \int_0^K \mathbf{1}_{[\Sigma_{p,z,y}^\delta < k R_y]} \, dk \right]
$$

and invert the joint Laplace transform of R_y and $\Sigma_{p,z,y}^\delta$ obtained in Section 2.

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