A SIMPLE MODEL FOR MARKET BOOMS AND CRASHES

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Abstract. Multiple equilibria models are one of the major categories of theoretical models for stock market crashes. The main objective of this paper is to model multiple equilibria and demonstrate how market prices move from one regime into another in a continuous time framework. As a consequence of this, a multiple jump structure is obtained with both booms and crashes, which are defined as points of discontinuity of the stock price process. For the constructed model, we prove that the stock price is a càdlàg semimartingale process, find the conditional distributions for the time of the next jump, the type of the next jump and the size of the next jump, given the public information available to market participants, and conduct a number of numerical studies.

1. Introduction

The main objective of this paper is to develop a simple quantitative framework for a financial market with multiple equilibria in order to analyse how booms and crashes could appear in a financial market. The presence of multiple equilibria amounts to different pricing regimes and for a complete analysis a model should also describe how the market switches from one regime to another. The connection between the multiple equilibria and market crashes or booms stems from the fact that a regime change typically produces a jump in the asset price, which can be associated to a boom or crash.

As a starting point for this study, we follow Gennotte and Leland [17]. In [17] Gennotte and Leland propose a one-period model to explain the market crash of 1987, which is commonly attributed to the presence of dynamic hedgers and information asymmetry about their hedging activity. In this rational expectations equilibrium model, two assets are traded: a single risky stock and a risk-free bond. The net supply of the risky asset consists of several components: a part observable by all market participants, an unobserved normally distributed liquidity shock, a normally distributed liquidity shock observed only by a class of informed investors, and the cumulative trades of dynamic hedgers,which was assumed to be deterministic. On the other hand, the total demand consists of uninformed, price-informed and supply-informed investors, who all maximise expected exponential utility of their wealth over a single period. While all the investors observe the equilibrium price, the price-informed investors also observe a personal unbiased signal on future price. According to this model, in the absence of dynamic hedgers no crashes can occur. However, if the hedging activity is sufficiently large, the market experiences multiple equilibria leading to crashes. Moreover, even if the hedging activity is relatively small, the market crashes could occur as a consequence of shifts in the information structure. We refer the reader to [17] for the precise description of their findings.

This paper aims to extend the model of Genotte and Leland to continuous time. In order to keep the model tractable we ignore the information asymmetry and assume all the traders have the same information. As such we describe the pricing mechanism among the following classes of traders: rational investors, dynamic hedgers and noise traders. In making their decisions, agents approximate the future stock price dynamics with an auxiliary Brownian motion with a drift process. As in [17] we find that a large amount of hedging activity is responsible for market crashes and booms manifested as big jumps in the price process. These jumps occur at random times when a certain

Brownian motion crosses a moving boundary. The magnitude of these jumps are also found to be random as one would expect from any meaningful model attempting to understand the impact of market crashes. Given our assumptions we are able to establish that the stock price process is a special semimartingale and find its explicit semimartingale decomposition.

Given such a model, one would naturally want to obtain expressions for the likelihood of booms or crashes along with their magnitudes. We compute the distributions for the time, the type and the size of the next jump of price process conditional on the market's filtration. These quantities are not obtained in closed form but rather in terms of integral equations. In order to complete the picture, we also undertake a numerical study to approximate these distributions and simulate some trajectories for possible price evolutions in this market with potential booms and crashes.

Our paper belongs to the class of papers on multiple equilibria and sunspot models in the literature (see, e.g., [24], [14], [4], [6], [40], [3], [5], [30], and [16]). The market crashes have been studied using other approaches as well. One can cite three categories in this respect: liquidity shortage models, bursting bubble models, and lumpy information aggregation models. In liquidity shortage models, the crashes occur when market price plummets due to a temporary reduction in liquidity (see, e.g., [18] and [20]). Bursting bubble models examine the scenarios when all market participants realise an asset price is greater than its fundamental value and, nevertheless, keep buying the asset as they believe there are others who do not know that the asset is overpriced, and to whom they expect to sell the asset at a higher price later. At some point everyone realises that everyone else is aware of the overpricing, which results in bubble bursts and corresponding market crashes (see, e.g., [1], [35], [29], [2], [9], [15], [23], and [10]). The papers in the final category follow an approach based on lumpy information aggregation. In these models the overpricing issue is not a common knowledge among the market participants. However, a revelation of a relevant information in the course of trading makes the less informed traders suddenly realise that an overpricing exists leading to sharp declines in prices (see, e.g., [34], [13] and [19]). We refer the reader to [8] for an excellent survey of these models. As an alternative to traditional approaches, more recently, [25] examined market crashes from the perspective of market microstructure invariance introduced in [26] and suggested that executions of large orders during volatile times at a rate which is too fast may induce dramatic price falls.

The outline of this paper is as follows. In Section 2, the market microstructure framework is defined. Section 3 is on model setup and analysis of its properties. Section 4 contains numerical studies.

2. Market microstructure framework

We will work on a filtered stochastic base $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ satisfying the usual conditions. It is assumed that the trading horizon, T, is finite and trading takes place continuously. In the model developed in this paper, there are two underlying assets in the economy: risky stock and risk-free bond. Risk-free bonds are in perfectly elastic supply and earn interest at the fixed rate $r > 0$. The risky stock is assumed to be in zero net supply. Three types of agents trade in this market: rational investors, dynamic hedgers, and noise traders.

2.1. **Rational investors' demand for the stock.** In order to obtain the demand process for a given investor we start with a discrete-time extension of the one-period model of [17] and take the limit of the corresponding demands in order to arrive at the demand process for continuous trading. To this end, suppose that the time between successive trades equals Δt and assume that the objective of the 'rational' investor at time t is to maximise the expected utility from wealth at time $t + \Delta t$. As such the 'rational' investor in this model has myopic preferences, which is enforced in order to obtain a more tractable framework. Rational investors believe that given the current price, P_t, the equilibrium price at time $t+\Delta$ is normally distributed with mean $P_t + \hat{\mu}\Delta t$ and variance $\hat{\sigma}^2\Delta t$. Assuming exponential utility function with the CARA coefficient $\rho^{-1} > 0$, the rational investors solve the following maximisation problem:

$$-e^{\frac{(e^{r\Delta t}-1)xP_t-\hat{\mu}x\Delta t}{\rho}}\mathbb{E}\Big(e^{-\frac{\hat{\sigma}x\sqrt{\Delta t}Z}{\rho}}\Big) \to \max_x,$$

where Z is a standard Normal random variable. Therefore, the form of the moment-generating function of a normal random variable yields the following individual rational investor's demand for stock in discrete time:

$$\frac{\rho(\hat{\mu}\Delta t-(e^{r\Delta t}-1)P_t)}{\hat{\sigma}^2\Delta t}.$$

As $\Delta t \downarrow 0$, it can be concluded that the cumulative demand for rational investors in the continuous framework is equal to

$$w^R \times \frac{\rho(\hat{\mu}-rP_t)}{\hat{\sigma}^2},$$

where w^R is the total number of rational investors, which is assumed to be constant.

2.2. **Dynamic hedgers' demand for the stock.** It is assumed that the total number of dynamic hedgers is equal to some constant w^D with the sole objective to replicate contingent claims of the following type:

$$F(P_T) = \max(K-P_T, 0).$$

We normalise the total number of contingent claims for each hedger to 1 but assume that the number of contingent claims with strike $\in dK$ is equal to $\frac{1}{\sqrt{2\pi\sigma_\kappa^2}}e^{-\frac{(K-\kappa)^2}{2\sigma_\kappa^2}}dK$ for some typically small $\sigma_\kappa > 0$, where $\kappa = P_0e^{rT}$.

We suppose that the dynamic hedgers share the same belief as the rational investors that the equilibrium price will evolve as an arithmetic Brownian motion with drift $\hat{\mu}$ and volatility $\hat{\sigma}$. Thus, if the stock price at time t equals x, they value the claim at $P(t,x)$, where

$$\begin{aligned}P(t,x) &= \int_{-\infty}^{Ke^{-r(T-t)}}(Ke^{-r(T-t)}-y)\frac{1}{\sqrt{2\pi\Sigma^2(t)}}e^{-\frac{(y-x)^2}{2\Sigma^2(t)}}dy \\ &= \Sigma(t)\times\frac{1}{\sqrt{2\pi}}e^{-\frac{(Ke^{-r(T-t)}-x)^2}{2\Sigma^2(t)}} + (Ke^{-r(T-t)}-x)\Phi\Big(\frac{Ke^{-r(T-t)}-x}{\Sigma(t)}\Big),\end{aligned}$$

with

$$\Sigma(t) = \hat{\sigma}\sqrt{\frac{1-e^{-2r(T-t)}}{2r}} \text{ and } \Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{u^2}{2}}du.$$

Hence, the dynamic hedgers component of demand at time $t \in [0,T)$ is equal to $\pi(t, P_t)$, where

$$\begin{aligned}\pi(t,x;\kappa,\sigma_\kappa) &= w^D\int_{-\infty}^{\infty}\frac{\partial P(t,x)}{\partial x}\frac{1}{\sqrt{2\pi\sigma_\kappa^2}}e^{-\frac{(K-\kappa)^2}{2\sigma_\kappa^2}}dK \\ &= w^D\int_{-\infty}^{\infty}\Big[-\Phi\Big(\frac{Ke^{-r(T-t)}-x}{\Sigma(t)}\Big)\Big]\frac{1}{\sqrt{2\pi\sigma_\kappa^2}}e^{-\frac{(K-\kappa)^2}{2\sigma_\kappa^2}}dK \\ &= w^D\times\Big[\int_{-\infty}^{\infty}\Phi\Big(\frac{x-Ke^{-r(T-t)}}{\Sigma(t)}\Big)\frac{1}{\sqrt{2\pi\sigma_\kappa^2}}e^{-\frac{(K-\kappa)^2}{2\sigma_\kappa^2}}dK-1\Big].\end{aligned}$$

2.3. **Noise traders' demand for the stock.** Finally, it is assumed that the noise traders' cumulative demand is given by $w^N \times (\mu_N + \sigma_N B_t)$, $\sigma_N > 0$, where $(B_t, t \geq 0)$ is a standard Brownian motion starting at 0 and w^N is the total number of noise traders, which is assumed to be constant. Noise traders trade according to the rule that is independent of the stock price fundamental value and is exogenous to the model. The noise traders component of demand will make the dynamics of the stock price stochastic.

2.4. **The pricing equation.** The market clearing condition states that the total demand should be equal to 0:

$$w^R \times \frac{\rho(\hat{\mu} - rP_t)}{\hat{\sigma}^2} + w^D \times \Big[\int_{-\infty}^{\infty} \Phi\Big(\frac{P_t - Ke^{-r(T-t)}}{\Sigma(t)}\Big) \frac{1}{\sqrt{2\pi\sigma_\kappa^2}} e^{-\frac{(K-\kappa)^2}{2\sigma_\kappa^2}} dK - 1\Big] + w^N \times (\mu_N + \sigma_N B_t) = 0$$

Denote by

$$\gamma_1 = w^R \times \frac{\rho r}{\hat{\sigma}^2}, \quad \gamma_2 = w^R \times \frac{\rho\hat{\mu}}{\hat{\sigma}^2} - w^D + w^N \times \mu_N, \quad \gamma_3 = w^N \times \sigma_N,$$

and define function $h : [0, T) \times \mathbb{R} \to \mathbb{R}$ by

$$h(t,x) = \frac{\gamma_1 x - w^D \int_{-\infty}^{\infty} \Phi\Big(\frac{x - Ke^{-r(T-t)}}{\Sigma(t)}\Big) \frac{1}{\sqrt{2\pi\sigma_\kappa^2}} e^{-\frac{(K-\kappa)^2}{2\sigma_\kappa^2}} dK - \gamma_2}{\gamma_3}, \tag{2.1}$$

and observe that $h(t, x)$ is $C^{1,2}([0, T) \times \mathbb{R})$. Thus, the pricing equation is given by

$$h(t, P_t) = B_t. \tag{2.2}$$

Remark 1. *Since* $0 \leq w^D \int_{-\infty}^{\infty} \Phi\Big(\frac{x-Ke^{-r(T-t)}}{\Sigma(t)}\Big) \frac{1}{\sqrt{2\pi\sigma_\kappa^2}} e^{-\frac{(K-\kappa)^2}{2\sigma_\kappa^2}} dK \leq w^D$, *one can easily conclude that*

$$\lim_{x\to-\infty} h(t, x) = -\infty \quad \text{and} \quad \lim_{x\to\infty} h(t, x) = \infty.$$

The equilibrium price will be any solution to (2.2). In order to have a multiple equilibria it is necessary that h is not one-to-one. As h is continuously differentiable uniqueness of solution to (2.2) can be checked by analysing the behaviour of h_x.

Differentiating $h(t, x)$ with respect to x, we can see that

$$\begin{aligned} h_x(t,x) &= \frac{1}{\gamma_3}\Big(\gamma_1 - \frac{w^D}{\sqrt{2\pi\sigma_\kappa^2\Sigma^2(t)}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(Ke^{-r(T-t)}-x)^2}{2\Sigma^2(t)}} e^{-\frac{(K-\kappa)^2}{2\sigma_\kappa^2}} dK\Big) \\ &= \frac{1}{\gamma_3}\Big(\gamma_1 - \frac{w^D}{\sqrt{2\pi\Big(\sigma_\kappa^2 e^{-2r(T-t)} + \Sigma^2(t)\Big)}} e^{-\frac{(x-\kappa e^{-r(T-t)})^2}{2(\sigma_\kappa^2 e^{-r(T-t)} + \Sigma^2(t))}}\Big). \end{aligned} \tag{2.3}$$

To ensure h_x changes sign we impose the following assumption on the number of dynamic hedgers, w^D, and the variance of the distribution of claims of each hedger, σ_κ.

Assumption 1. *w^D and σ_κ obey the following bounds:*

$$\begin{aligned} w^D &> \gamma_1 \sqrt{2\pi\Big(\sigma_\kappa^2 e^{-2rT} + \Sigma^2(0)\Big)} \\ 0 &< \sigma_\kappa^2 \leq \frac{\hat{\sigma}^2}{2r}. \end{aligned}$$

The above assumption immediately yields

$$w^D > \max_{t\in[0,T)} \left(\gamma_1 \sqrt{2\pi\left(\sigma_\kappa^2 e^{-2r(T-t)} + \Sigma^2(t)\right)}\right),$$

and $h_x(t,x)$ as a function of x changes its sign at $p_1(t)$ and $p_2(t)$ as follows:

$$h_x(t,x) \begin{cases} > 0 & \text{if } x < p_1(t) \text{ or } x > p_2(t) \\ = 0 & \text{if } x = p_1(t) \text{ or } x = p_2(t) \\ < 0 & \text{if } p_1(t) < x < p_2(t), \end{cases} \tag{2.4}$$

where

$$p_1(t) = \kappa e^{-r(T-t)} - \sqrt{-2(\sigma_\kappa^2 e^{-r(T-t)} + \Sigma^2(t)) \ln\Big(\frac{\gamma_1}{w^D}\sqrt{2\pi(\sigma_\kappa^2 e^{-2r(T-t)} + \Sigma^2(t))}\Big)}, \text{ and} \tag{2.5}$$

$$p_2(t) = \kappa e^{-r(T-t)} + \sqrt{-2(\sigma_\kappa^2 e^{-r(T-t)} + \Sigma^2(t)) \ln\Big(\frac{\gamma_1}{w^D}\sqrt{2\pi(\sigma_\kappa^2 e^{-2r(T-t)} + \Sigma^2(t))}\Big)}. \tag{2.6}$$

Figure 1 exhibits the shape of h when the number of dynamic hedgers is small or large.

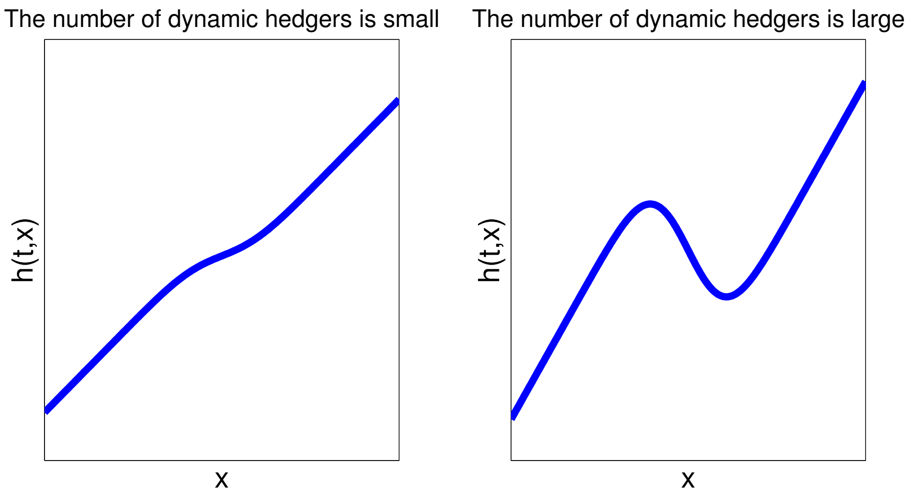

FIGURE 1. The shape of $h(t,x)$ as it depends on the magnitude of the number of dynamic hedgers. The value of w^D for the left plot violates Assumption 1.

Let's denote the local maximum and local minimum values by

$$h^h(t) = h(t, p_1(t)) \quad \text{and} \quad h^l(t) = h(t, p_2(t)). \tag{2.7}$$

Observe that if B_t is between $h^l(t)$ and $h^h(t)$, the equilibrium price may take 3 different values under Assumption 1, which we will assume henceforth. Indeed, as the dynamic hedgers' demand, $\pi(t, P_t)$, is an increasing function of P_t, while the rational investors' demand, $w^R \times \frac{\rho(\hat{\mu}-rP_t)}{\hat{\sigma}^2}$ is decreasing in P_t, if the total number of dynamic hedgers w^D is large enough, then the roots of the pricing equation (2.2) have the following structure:

$$P_t \in \begin{cases} \{p^l(t,B_t)\} & \text{if } B_t \le h^l(t) \\ \{p^l(t,B_t), p^m(t,B_t), p^h(t,B_t)\} & \text{if } h^l(t) < B_t < h^h(t) \\ \{p^h(t,B_t)\} & \text{if } B_t \ge h^h(t), \end{cases} \tag{2.8}$$

where

$$p^l(t,y) = \min\{x : h(t,x) = y\},$$
$$p^h(t,y) = \max\{x : h(t,x) = y\},$$

and $p^m(t,x)$ is defined to be the middle root of the equation $h(t,y) = x$ for $x \in (h^l(t), h^h(t))$. As such, p^l (resp. p^h) will determine the equilibrium price when the state of the economy is *low* (resp. *high*), while p^m will be the pricing function when the economy is in the *middle* state.

To model how market prices move from one root to another we define a state process S_t taking values in a state space $\mathbb{S}$ consisting of three different states: low level equilibrium s_l, middle level equilibrium s_m and high level equilibrium s_h. Thus, when B falls between h^l and h^h, the equilibrium price will be determined by the state process, S. The state process will also be responsible for the jumps in the stock price. We will call a negative jump a *crash* and positive jumps will be named *booms*.

Remark 2. *Observe that h_x is an affine function of the gamma of the portfolio of dynamic hedgers. In particular, a higher gamma might result in a more negative h_x. This would imply, e.g., when the equilibrium switches from a higher level to a middle level, the drop in the stock price will be more pronounced.*

In the next section we will make precise how the state of the economy changes. We end this section with a useful result on h^h and h^l.

Theorem 2.1. *There exists some $\Delta > 0$ such that*

$$h^h(t) - h^l(t) \geq \Delta, \ \forall t \in [0,T).$$

Proof. The proof is provided in the Appendix. □

3. Exogenous shocks

The state of the economy will change according to some exogenous shocks, i.e. *sunspots*. If $B_t \leq h^l(t)$ (resp. $B_t \geq h^h(t)$), i.e. when there is no possibility of multiple equilibria, then $S_t = s_l$ (resp. $S_t = s_h$), for all $t \in [0,T)$. If $h^l(t) < B_t < h^h(t)$, the system stays in its current state until there is a new arrival in an exogenous sunspot shock process which is assumed to be a Poisson process independent of B_t. The shock switches the state of the system to one of the other two states for no fundamental reason, and the new state is determined according to the value of an independent Bernoulli random variable with probability of success depending on the current state of the state process.

3.1. **Model setup.** The sunspot process, $(Z_t, t \geq 0)$, is an adapted homogeneous Poisson process with intensity λ_Z and independent of B. When a shock occurs the state of the system will change in accordance with the values of certain Bernoulli random variables. To this end, let $(\zeta_i^{lh}, i = 0,1,...)$, $(\zeta_i^{mh}, i = 0,1,...)$ and $(\zeta_i^{hl}, i = 0,1,...)$ be independent sequences of independent Bernoulli random variables with success probabilities p^{lh}, p^{mh}, and p^{hl}, respectively. The success event corresponding to the probability p^{ij} indicates a switch from level i to j, while the failure corresponds to a switch from i to k, where i, j and k are distinct states in $\mathbb{S}$.

The processes S and P will be defined according to the following algorithm.

Step 1: Set $i = 0$, $\tau_0 = 0$ and the starting value of the state process

$$S_{\tau_0} = \begin{cases} s_l & \text{if } B_{\tau_0} \le h^l(\tau_0) \\ s_h & \text{if } B_{\tau_0} \ge h^h(\tau_0) \\ s & \text{if } h^l(\tau_0) < B_{\tau_0} < h^h(\tau_0), \end{cases}$$

where $s \in \mathbb{S}$ is an arbitrary constant.

Step 2: Set

$$\tau_{i+1} = \begin{cases} \inf\left\{t > \tau_i : B_t \ge h^h(t)\right\} \wedge \hat{\tau}_i \wedge T & \text{if } S_{\tau_i} = s_l \\ \inf\left\{t > \tau_i : B_t \ge h^h(t) \text{ or } B_t \le h^l(t)\right\} \wedge \hat{\tau}_i \wedge T & \text{if } S_{\tau_i} = s_m \\ \inf\left\{t > \tau_i : B_t \le h^l(t)\right\} \wedge \hat{\tau}_i \wedge T & \text{if } S_{\tau_i} = s_h, \end{cases}$$

where $\hat{\tau}_i$ is the first arrival of Z after τ_i, while $\inf \emptyset = \infty$ by convention.

Step 3: Set $S_t = S_{\tau_i}, \forall t \in [\tau_i, \tau_{i+1})$.

Step 4: If $\tau_{i+1} = T$, stop the algorithm.

Step 5: Set

$$S_{\tau_{i+1}} = \begin{cases} s_l & \text{if } B_{\tau_{i+1}} \le h^l(\tau_{i+1}) \\ s_h & \text{if } B_{\tau_{i+1}} \ge h^h(\tau_{i+1}) \\ s_h & \text{if } h^l(\tau_{i+1}) < B_{\tau_{i+1}} < h^h(\tau_{i+1}) \text{ and } S_{\tau_i} = s_l \text{ and } \zeta_i^{lh} = 1 \\ s_m & \text{if } h^l(\tau_{i+1}) < B_{\tau_{i+1}} < h^h(\tau_{i+1}) \text{ and } S_{\tau_i} = s_l \text{ and } \zeta_i^{lh} = 0 \\ s_l & \text{if } h^l(\tau_{i+1}) < B_{\tau_{i+1}} < h^h(\tau_{i+1}) \text{ and } S_{\tau_i} = s_h \text{ and } \zeta_i^{hl} = 1 \\ s_m & \text{if } h^l(\tau_{i+1}) < B_{\tau_{i+1}} < h^h(\tau_{i+1}) \text{ and } S_{\tau_i} = s_h \text{ and } \zeta_i^{hl} = 0 \\ s_h & \text{if } h^l(\tau_{i+1}) < B_{\tau_{i+1}} < h^h(\tau_{i+1}) \text{ and } S_{\tau_i} = s_m \text{ and } \zeta_i^{mh} = 1 \\ s_l & \text{if } h^l(\tau_{i+1}) < B_{\tau_{i+1}} < h^h(\tau_{i+1}) \text{ and } S_{\tau_i} = s_m \text{ and } \zeta_i^{mh} = 0. \end{cases}$$

Step 6: Set $i = i + 1$ and go to Step 2.

In order for the above algorithm to produce meaningful results, we need to make sure that it ends after finitely many steps.

Theorem 3.1. *Let $(\tau_i)_{i\ge 0}$ be the sequence of stopping times defined above. Then,*

(1) *For all $i \ge 0$, if $\tau_i < T$, $\mathbb{P}$-a.s., then $\tau_i < \tau_{i+1}$, $\mathbb{P}$-a.s., too.*

(2) $\sup\{i : \tau_i < T\} < \infty$, *$\mathbb{P}$-a.s..*

Proof. The first statement holds due to Theorem 2.1 in view of the continuity of B and the strict positivity of the exponential random variable.

To show the second statement suppose there is an infinite number of τ_i in $[0, T)$ with positive probability. Then, either there are infinitely many i.i.d. exponential random variables such that their sum is less than T, or, there exists an interval of length δ in $[0, T)$ in which B crosses the moving interval $(h^l(t), h^h(t))$ infinitely many times. However, the first option is not possible since

$$P(\sum_{i=1}^{\infty} X_i < T) \le P(\sum_{i=1}^{n} X_i < T) = 1 - \sum_{i=0}^{n-1} \frac{(\lambda T)^i}{i!} e^{-\lambda T} \to 0, \quad n \to \infty.$$

The second scenario is also not possible. Indeed, since h^l is continuously differentiable, there exists a probability measure $\mathbb{Q} \sim \mathbb{P}$ on $\mathcal{F}$ under which $\beta_t := B_t - h^l(t)$ is a Brownian motion and the second scenario entails that β crosses $(0, h^h(t) - h^l(t))$ in a finite interval infinitely often. As

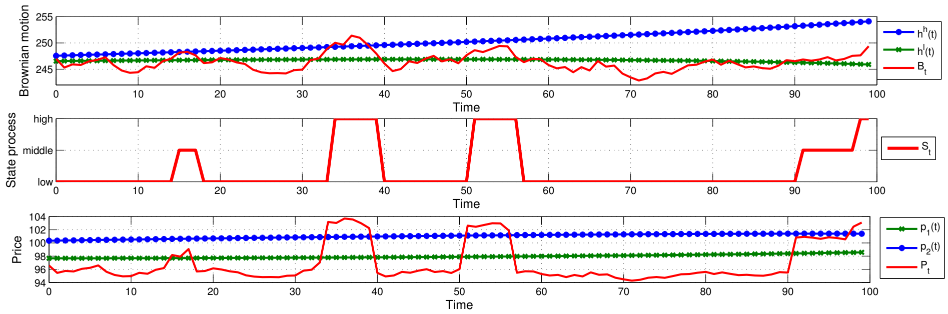

FIGURE 2. Simulated stock price dynamics for the following set of parameters: $r = 0.0001$, $T = 100$, $\alpha_1 = 0.2$, $\sigma_\kappa = 1$, $\kappa = 100$, $\gamma_1 = 3$, $\gamma_2 = 40$, $\gamma_3 = 1$, $w^D = 20$. Initial value of S_t is assumed to be equal to s_l. The shocks occur at times $t = 10$, $t = 15$, $t = 25$, $t = 51$, $t = 68$, $t = 91$ and $t = 98$; stock price jumps at $t = 15$, $t = 18$, $t = 34$, $t = 40$, $t = 51$, $t = 57$, $t = 91$ and $t = 98$.

$h^h(t) - h^l(t) \geq \Delta$ by Theorem 2.1, such an infinite crossing has zero probability due to Doob's upcrossing inequality (see, e.g., Theorem 1.3.8 (iii) in [22]). $\square$

Remark 3. *Note that, according to the construction of the stock price process, for all $t \in [0, T)$, P_t can not be equal to $p_1(t)$ or $p_2(t)$ in view of the pricing algorithm outlined above. Indeed, if P_t is equal to $p_1(t)$, then $B_t = h^h(t)$ and either $S_t = s_l$ or $S_t = s_m$, but by construction if $B_t \geq h^h(t)$, then $S_t = s_h$, which is a contradiction. The same argument applies to $p_2(t)$.*

Remark 4. *In view of the previous remark, there is one-to-one correspondence between P_t and (B_t, S_t). More precisely, given P_t,*

$$B_t = h(t, P_t) \quad \textit{and} \quad S_t = \begin{cases} s_l & \textit{if } P_t < p_1(t) \\ s_m & \textit{if } p_1(t) < P_t < p_2(t) \\ s_h & \textit{if } P_t > p_2(t). \end{cases}$$

Conversely, if B_t and S_t are known, P_t can be determined via (2.8).

By the virtue of Theorem 3.1, there is no infinite price oscillation and $(\tau_i < T, i = 1, 2, ...)$ are the only jump points on $[0, T)$. We denote the value of the i-th jump in price by $J_i = \Delta P_{\tau_i} = P_{\tau_i} - P_{\tau_i -}$. Note that by construction P is càdlàg .

Definition 3.1. *A big market crash (resp. a big market boom) is a transition of S_t from state s_h (respectively s_l) to state s_l (respectively s_h). Similarly, a small market crash (resp. a small market boom) is a transition from state s_h (resp. s_l) to state s_m, or from state s_m to state s_l (resp. s_h).*

The following lists the alternative values that J_i can take:

$$J_i = \begin{cases} J^h(\tau_i) & \text{if } B_{\tau_i} = h^h(\tau_i) \\ J^l(\tau_i) & \text{if } B_{\tau_i} = h^l(\tau_i) \\ J^{lh}(\tau_i, B_{\tau_i}) & \text{if } h^l(\tau_i) < B_{\tau_i} < h^h(\tau_i), S_{\tau_i} = s_l \text{ and } S_{\tau_{i+1}} = s_h \\ J^{lm}(\tau_i, B_{\tau_i}) & \text{if } h^l(\tau_i) < B_{\tau_i} < h^h(\tau_i), S_{\tau_i} = s_l \text{ and } S_{\tau_{i+1}} = s_m \\ J^{mh}(\tau_i, B_{\tau_i}) & \text{if } h^l(\tau_i) < B_{\tau_i} < h^h(\tau_i), S_{\tau_i} = s_m \text{ and } S_{\tau_{i+1}} = s_h \\ J^{ml}(\tau_i, B_{\tau_i}) & \text{if } h^l(\tau_i) < B_{\tau_i} < h^h(\tau_i), S_{\tau_i} = s_m \text{ and } S_{\tau_{i+1}} = s_l \\ J^{hl}(\tau_i, B_{\tau_i}) & \text{if } h^l(\tau_i) < B_{\tau_i} < h^h(\tau_i), S_{\tau_i} = s_h \text{ and } S_{\tau_{i+1}} = s_l \\ J^{hm}(\tau_i, B_{\tau_i}) & \text{if } h^l(\tau_i) < B_{\tau_i} < h^h(\tau_i), S_{\tau_i} = s_h \text{ and } S_{\tau_{i+1}} = s_m. \end{cases}, \tag{3.1}$$

where

$$\begin{aligned}
J^h(\tau_i) &= p^h(\tau_i, h^h(\tau_i)) - p_1(\tau_i) \\
J^l(\tau_i) &= p^l(\tau_i, h^l(\tau_i)) - p_2(\tau_i) \\
J^{lh}(\tau_i, B_{\tau_i}) &= p^h(\tau_i, B_{\tau_i}) - p^l(\tau_i, B_{\tau_i}) \\
J^{lm}(\tau_i, B_{\tau_i}) &= p^m(\tau_i, B_{\tau_i}) - p^l(\tau_i, B_{\tau_i}) \\
J^{mh}(\tau_i, B_{\tau_i}) &= p^h(\tau_i, B_{\tau_i}) - p^m(\tau_i, B_{\tau_i}) \\
J^{ml}(\tau_i, B_{\tau_i}) &= p^l(\tau_i, B_{\tau_i}) - p^m(\tau_i, B_{\tau_i}) \\
J^{hl}(\tau_i, B_{\tau_i}) &= p^l(\tau_i, B_{\tau_i}) - p^h(\tau_i, B_{\tau_i}) \\
J^{hm}(\tau_i, B_{\tau_i}) &= p^m(\tau_i, B_{\tau_i}) - p^h(\tau_i, B_{\tau_i}).
\end{aligned}$$

Recall that an increase in the number of dynamic hedgers, w^D, leads to an increase in the magnitude of booms and crashes. We next obtain an upper bound on the jump sizes of the price process.

Proposition 3.1. *Jump sizes, $\mid \Delta P_{\tau_i} \mid$, of the price process are uniformly bounded such that*

$$\mid \Delta P_{\tau_i} \mid \leq \frac{w^D}{\gamma_1}.$$

Proof. The pricing equation (2.2) and the continuity of Brownian motion yield that $h(\tau_i, P_{\tau_i}) = B_{\tau_i}$ and $h(\tau_i, P_{\tau_i -}) = B_{\tau_i}$. Thus,

$$h(\tau_i, P_{\tau_i}) = h(\tau_i, P_{\tau_i -}),$$

i.e.

$$\gamma_1 \Delta P_{\tau_i} - w^D \int_{-\infty}^{\infty} \Big[\Phi\Big(\frac{P_{\tau_i} - K e^{-r(T-\tau_i)}}{\Sigma(\tau_i)} \Big) - \Phi\Big(\frac{P_{\tau_i -} - K e^{-r(T-\tau_i)}}{\Sigma(\tau_i)} \Big) \Big] \frac{1}{\sqrt{2\pi\sigma_\kappa^2}} e^{-\frac{(K-\kappa)^2}{2\sigma_\kappa^2}} dK = 0.$$

As a consequence,

$$|\Delta P_{\tau_i}| \leq \frac{w^D}{\gamma_1} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_\kappa^2}} e^{-\frac{(K-\kappa)^2}{2\sigma_\kappa^2}} dK = \frac{w^D}{\gamma_1}.$$

□

Let $\mathcal{F}_t^P$ be the usual augmentation of the natural filtration of P. We call this filtration the *market filtration* since this is the public information available to all market agents.

Proposition 3.2. *The sequence $(\tau_i < T, i = 1, 2, ...)$ is a sequence of $\mathcal{F}_t^P$-stopping times.*

Proof. Since P is adapted to $(\mathcal{F}_t^P)$, the result follows from Proposition 1.32 in [21]. □

Theorem 3.2. *Stock price process is a special semimartingale such that*

$$P_t = P_0 + \int_0^t \theta_1(s, P_s) ds + \int_0^t \theta_2(s, P_s) dB_s + \sum_{i=1}^{N_t} \Delta P_{\tau_i}, \quad \textit{for } t \in [0, T), \tag{3.2}$$

where $N_t = \sum_{i \geq 1} \mathbf{1}_{(\tau_i \leq t)}$ is the total number of jumps on $[0, t]$,

$$\theta_1(s, P_s) = -\frac{h_s(s, P_s) + \frac{1}{2} h_{xx}(s, P_s) (\frac{1}{h_x(s, P_s)})^2}{h_x(s, P_s)} \tag{3.3}$$

and

$$\theta_2(s, P_s) = \frac{1}{h_x(s, P_s)}. \tag{3.4}$$

Proof. Consider the decomposition

$$P_t - P_0 = P_t - P_{\tau_{N_t}} + \sum_{i=1}^{N_t}(P_{\tau_i-} - P_{\tau_{i-1}}) + \sum_{i=1}^{N_t} \Delta P_{\tau_i}. \tag{3.5}$$

In view of Ito's formula and the implicit function theorem,

$$P_t - P_{\tau_{N_t}} = \int_{\tau_{N_t}}^t \theta_1^{(N_t)}(s, P_s)ds + \int_{\tau_{N_t}}^t \theta_2^{(N_t)}(s, P_s)dB_s,$$

for some functions $\theta_1^{(N_t)}$ and $\theta_2^{(N_t)}$. Applying again Ito's formula to (2.2) yields

$$h_t(t, P_t)dt + h_x(t, P_t)\theta_1^{(N_t)}(t, P_t)dt + h_x(t, P_t)\theta_2^{(N_t)}(t, P_t)dB_t + \frac{1}{2}h_{xx}(t, P_t)(\theta_2^{(N_t)}(t, P_t))^2dt = dB_t.$$

Thus,

$$\theta_2^{(N_t)}(s, P_s) = \frac{1}{h_x(s, P_s)}, \quad \theta_1^{(N_t)}(s, P_s) = -\frac{h_s(s, P_s) + \frac{1}{2}h_{xx}(s, P_s)(\frac{1}{h_x(s,P_s)})^2}{h_x(s, P_s)},$$

and

$$P_t - P_{\tau_{N_t}} = -\int_{\tau_{N_t}}^t \frac{h_s(s, P_s) + \frac{1}{2}h_{xx}(s, P_s)(\frac{1}{h_x(s,P_s)})^2}{h_x(s, P_s)}ds + \int_{\tau_{N_t}}^t \frac{1}{h_x(s, P_s)}dB_s.$$

Similarly,

$$P_{\tau_i-} - P_{\tau_{i-1}} = -\int_{\tau_{i-1}}^{\tau_i-} \frac{h_s(s, P_s) + \frac{1}{2}h_{xx}(s, P_s)(\frac{1}{h_x(s,P_s)})^2}{h_x(s, P_s)}ds + \int_{\tau_{i-1}}^{\tau_i-} \frac{1}{h_x(s, P_s)}dB_s, \quad i = 1, 2, ..., N_t.$$

Next, define the processes $(P_t^{(k)}, k = 1, 2, ...)$ by

$$P_t^{(k)} = P_0 + \int_0^{t\wedge\tau_k} \theta_1(s, P_s)ds + \int_0^{t\wedge\tau_k} \theta_2(s, P_s)dB_s + \sum_{i=1}^{N_t\wedge k} \Delta P_{\tau_i}.$$

By Theorem 32 in Chap. II of [33],

$$P_0 + \int_0^{t\wedge\tau_k} \theta_1(s, P_s)ds + \int_0^{t\wedge\tau_k} \theta_2(s, P_s)dB_s$$

is a semimartingale. Since by Proposition 3.1 the jumps of P are bounded, each $P_t^{(k)}$ is a semimartingale as well. Proposition 1.4.25c in [21] yields P is a semimartingale. Since it has bounded jumps it is a special semimartingale by Proposition 1.4.24 in [21]. □

3.2. **Canonical decomposition of the stock price process.** Theorem 3.2 states that the stock price process is a special semimartingale. In this section, we will find its canonical decomposition as a sum of a local martingale and predictable process of finite variation. To this end we need to compute certain conditional distributions. We start with deriving the joint conditional distribution of the time of the next jump, the type of the next jump and the size of the next jump given the evolution of stock prices.

Theorem 3.3. *Let C_1 be any combination of elements in $\mathbb{S}$ and $C_2 \in \mathcal{B}(\mathbb{R})$, a Borel subset of $\mathbb{R}$. The joint distribution of the time of the next jump, the type of the next jump and the size of the next jump conditional on $\mathcal{F}_t^P$ is given by*

$$\mathbb{P}(\tau_{N_t+1} < u, S_{\tau_{N_t+1}} \in C_1, J_{N_t+1} \in C_2 \mid \mathcal{F}_t^P) = \begin{cases} F_1(t, B_t, u, C_1, C_2) & \text{if } S_t = s_l \\ F_2(t, B_t, u, C_1, C_2) & \text{if } S_t = s_m \\ F_3(t, B_t, u, C_1, C_2) & \text{if } S_t = s_h \end{cases}$$

for $u \in (t, T]$, where expressions for F_1, F_2 and F_3 are given in (A.3), (A.4), and (A.6), respectively.

Proof. See the Appendix. $\square$

Remark 5. *By integrating the above densities we can in particular obtain*

$$\mathbb{P}(\tau_{N_t+1} < u \mid \mathcal{F}_t^P) = \begin{cases} F_4(t, B_t, u) & \text{if } S_t = s_l \\ F_5(t, B_t, u) & \text{if } S_t = s_m \\ F_6(t, B_t, u) & \text{if } S_t = s_h, \end{cases}$$

where F_4, F_5 and F_6 satisfy

$$\begin{aligned} F_4(t, B_t, u) &= e^{-\lambda_Z(u-t)}\left(1 - D_1(u, t, B_t)\right) + \int_0^{u-t} \lambda_Z e^{-\lambda_Z r}\Big[(1 - D_1(t + r, t, B_t)) \\ &\quad + \int_{-\infty}^{h^l(t+r)} q_1(x; r, t, B_t) F_4(t + r, x, u)dx + \Phi_1(t + r, t, B_t)\Big]dr, \\ F_5(t, B_t, u) &= 1 - e^{-\lambda_Z(u-t)} D_m(u, t, B_t), \\ F_6(t, B_t, u) &= e^{-\lambda_Z(u-t)}\left(1 - D_2(u, t, B_t)\right) + \int_0^{u-t} \lambda_Z e^{-\lambda_Z r}\Big[(1 - D_2(t + r, t, B_t)) \\ &\quad + \int_{h^h(t+r)}^{\infty} q_2(x; r, t, B_t) F_6(t + r, x, u)dx + \Phi_2(t + r, t, B_t)\Big]dr. \end{aligned}$$

In above representation,

$$\begin{aligned} D_m(u, t, y) &= \mathbb{P}\Big(h^l(t + s) - y < B_s < h^h(t + s) - y, \forall s \in [0, u - t]\Big), \\ \Phi_1(u, t, y) &= \mathbb{P}\Big(B_s < h^h(t + s) - y, 0 \le s \le u - t; B_{u-t} > h^l(u) - y\Big), \\ \Phi_2(u, t, y) &= \mathbb{P}\Big(B_s > h^l(t + s) - y, 0 \le s \le u - t; B_{u-t} < h^h(u) - y\Big). \end{aligned}$$

Recall that D_1 and D_2 are as defined in (A.2) and (A.7), $q_1(x; r, t, y)$ is the density of B_r on the set $\Big[B_s < h^h(t + s) - y, \forall s \in [0, r]\Big]$ whereas $q_2(x; r, t, y)$ is the density of B_r on the set $\Big[B_s > h^l(t + s) - y, \forall s \in [0, r]\Big]$.

Let

$$J_0 = 0 \quad \text{and} \quad Z_i^P = (P_{\tau_i}, J_i), i = 0, 1, ..., \tag{3.6}$$

and

$$g^{(i+1)}(u, C) = \frac{\partial \mathbb{P}(\tau_{i+1} \le u, Z_{i+1}^P \in C \mid \mathcal{F}_{\tau_i}^{Z^P})}{\partial u}, \quad u \in [\tau_i, T), \tag{3.7}$$

where $C = (C_1, C_2)$, $C_1 \in \mathcal{B}(\mathbb{R})$ and $C_2 \in \mathcal{B}(\mathbb{R})$.

Note that (τ_i, Z_i^P) is an $\mathbb{R}^2$-marked point process. In the sequel we will denote by $\mathcal{F}_i^{Z^P}$ the σ-algebra $\sigma\{(\tau_j, Z_j^P), 0 \le j \le i\}$ completed by the $\mathbb{P}$-null sets.

Lemma 3.1. *Assume that $u \in [\tau_i, T)$, $i = 0, 1, ...$, $C = (C_1, C_2)$, $C_1 \in \mathcal{B}(\mathbb{R})$ and $C_2 \in \mathcal{B}(\mathbb{R})$. Then conditional distribution for the marked point process (τ_i, Z_i^P) given $\mathcal{F}_i^{Z^P}$ is equal to*

$$\mathbb{P}(\tau_{i+1} \le u, Z_{i+1}^P \in C \mid \mathcal{F}_i^{Z^P}) = \begin{cases} F_7(u, \tau_i, B_{\tau_i}, C) & \text{if } S_{\tau_i} = s_l \\ F_8(u, \tau_i, B_{\tau_i}, C) & \text{if } S_{\tau_i} = s_m \\ F_9(u, \tau_i, B_{\tau_i}, C) & \text{if } S_{\tau_i} = s_h, \end{cases}$$

where F_7, F_8 and F_9 are as defined in (A.8).

Proof. See the Appendix. $\square$

Lemma 3.2. *Assume that $u \in [\tau_i, T)$, $i = 0, 1, ...$, $C = (C_1, C_2)$, $C_1 \in \mathcal{B}(\mathbb{R})$ and $C_2 \in \mathcal{B}(\mathbb{R})$. Then we have*

$$g^{(i+1)}(u, C) = \begin{cases} F_{10}(u, \tau_i, B_{\tau_i}, C) & \text{if } S_{\tau_i} = s_l \\ F_{11}(u, \tau_i, B_{\tau_i}, C) & \text{if } S_{\tau_i} = s_m \\ F_{12}(u, \tau_i, B_{\tau_i}, C) & \text{if } S_{\tau_i} = s_h, \end{cases}$$

where expressions for F_{10}, F_{11} and F_{12} are defined in (A.9), (A.10), and (A.11), respectively. In particular, F_{10}, F_{11} and F_{12} satisfy

$$\begin{aligned} F_{10}(u, t, y, \mathbb{R}^2) &= e^{-\lambda_Z(u-t)}\phi_1(u, t, y) + \lambda_Z e^{-\lambda_Z(u-t)}\Phi_1(u, t, y) \\ &\quad + \int_0^{u-t} \lambda_Z e^{-\lambda_Z r}\Big[\int_{-\infty}^{h^l(t+r)} q_1(x; r, t, y) F_{10}(u, t+r, x, \mathbb{R}^2)dx\Big]dr \\ F_{11}(u, t, y, \mathbb{R}^2) &= e^{-\lambda_Z(u-t)}\phi_m(u, t, y) + \lambda_Z e^{-\lambda_Z(u-t)} \\ F_{12}(u, t, y, \mathbb{R}^2) &= e^{-\lambda_Z(u-t)}\phi_2(u, t, y) + \lambda_Z e^{-\lambda_Z(u-t)}\Phi_2(u, t, B_t) \\ &\quad + \int_0^{u-t} \lambda_Z e^{-\lambda_Z r}\Big[\int_{h^h(t+r)}^{\infty} q_2(x; t, y, r) F_{12}(u, t+r, x, \mathbb{R}^2)dx\Big]dr, \end{aligned}$$

and

$$\phi_m(u, t, y) = -\frac{\partial D_m(u, t, y)}{\partial u},$$

and $\phi_1, \phi_2, D_1, D_2, D_m, q_1$ and q_2 are as defined in (A.2), (A.7) and Remark 5.

Proof. See the Appendix. □

In view of above results we are now ready to obtain the canonical decomposition of the stock price process.

Corollary 3.1. *The canonical decomposition of P is given by*

$$P_t = P_0 + M_t + A_t, \quad M_0 = 0, \quad A_0 = 0,$$

where

$$M_t = \int_0^t \theta_2(s, P_s)dB_s + \sum_{i=1}^{N_t} \Delta P_{\tau_i} - \int_0^t \theta_3(s, \tau_{N_s}, B_{\tau_{N_s}})ds$$

is a local martingale,

$$A_t = \int_0^t \theta_1(s, P_s)ds + \int_0^t \theta_3(s, \tau_{N_s}, B_{\tau_{N_s}})ds$$

is a predictable process with finite variation for θ_1 and θ_2 defined by (3.3) *and* (3.4)*, respectively. Moreover,*

$$\theta_3(s, \tau_{N_s}, B_{\tau_{N_s}}) = \begin{cases} \frac{F_{13}(s, \tau_{N_s}, B_{\tau_{N_s}})}{1 - \int_0^{s-\tau_{N_s}} F_{13}(\tau_{N_s}+y, \tau_{N_s}, B_{\tau_{N_s}})dy} & \text{if } S_{\tau_{N_s}} = s_l \\ \frac{F_{14}(s, \tau_{N_s}, B_{\tau_{N_s}})}{1 - \int_0^{s-\tau_{N_s}} F_{14}(\tau_{N_s}+y, \tau_{N_s}, B_{\tau_{N_s}})dy} & \text{if } S_{\tau_{N_s}} = s_m \\ \frac{F_{15}(s, \tau_{N_s}, B_{\tau_{N_s}})}{1 - \int_0^{s-\tau_{N_s}} F_{15}(\tau_{N_s}+y, \tau_{N_s}, B_{\tau_{N_s}})dy} & \text{if } S_{\tau_{N_s}} = s_h, \end{cases}$$

with

$$\begin{aligned}
F_{13}(u,t,y) &= e^{-\lambda_Z(u-t)}J^h(u)\phi_1(u,t,y) + \lambda_Z e^{-\lambda_Z(u-t)}\Big[\int_{h^l(u)}^{h^h(u)} q_1(x;u-t,t,y)\Big(p_{lh}J^{lh}(u,x) \\
&\quad + p_{lm}J^{lm}(u,x)\Big)dx\Big] + \int_0^{u-t}\lambda_Z e^{-\lambda_Z r}\Big[\int_{-\infty}^{h^l(t+r)} q_1(x;r,t,y)F_{13}(u,t+r,x)dx\Big]dr, \\
F_{14}(u,t,y) &= e^{-\lambda_Z(u-t)}\Big[J^h(u)\phi_{m,1}(u,t,y) + J^l(u)\phi_{m,2}(u,t,y)\Big] \\
&\quad + \lambda_Z e^{-\lambda_Z(u-t)}\Big[\int_{h^l(u)}^{h^h(u)} q^m(x;u-t,t,y)\Big(p_{mh}J^{mh}(u,x) + p_{ml}J^{ml}(u,x)\Big)dx\Big], \\
F_{15}(u,t,y) &= e^{-\lambda_Z(u-t)}J^l(u)\phi_2(u,t,y) + \lambda_Z e^{-\lambda_Z(u-t)}\Big[\int_{h^l(u)}^{h^h(u)} q_2(x;u-t,t,y)\Big(p_{hl}J^{hl}(u,x) \\
&\quad + p_{hm}J^{hm}(u,x)\Big)dx\Big] + \int_0^{u-t}\lambda_Z e^{-\lambda_Z r}\Big[\int_{h^h(t+r)}^{\infty} q_2(x;r,t,y)F_{15}(u,t+r,x)dx\Big]dr,
\end{aligned}$$

and J^i *and* J^{ij} *are as defined by (3.1). Recall that* $q^m(x;r,t,y)$ *is the density of* B_r *on the set* $\Big[h^l(t+s) - y < B_s < h^h(t+s) - y, \forall s \in [0,r]\Big]$.

Proof. Applying Theorem T7 in Chapter VIII of [7] to the counting process $N_t^Z(C)$ defined by

$$N_t^Z(C) = \sum_{i\geq 1}\mathbb{I}(Z_i^P \in C)\mathbb{I}(\tau_i \leq t),$$

we conclude that the process $\int_0^t l_s(C)ds$ with

$$l_s(C) = \frac{g^{(i+1)}(s,C)}{1 - \int_0^{s-\tau_i} g^{(i+1)}(\tau_i + y, \mathbb{R}^2)dy} \quad \text{for } s \in [\tau_i, \tau_{i+1}), \quad i = 0, 1\ldots,$$

is the compensator of $N^Z(C)$.

In view of Lemma 3.2, we have

$$l_s(C) = \begin{cases} \frac{F_{13}(\tau_i, B_{\tau_i}, s, C)}{1 - \int_0^{s-\tau_i} F_{13}(\tau_i, B_{\tau_i}, \tau_i + y, \mathbb{R}^2)dy} & \text{if } S_{\tau_i} = s_l \\ \frac{F_{14}(\tau_i, B_{\tau_i}, s, C)}{1 - \int_0^{s-\tau_i} F_{14}(\tau_i, B_{\tau_i}, \tau_i + y, \mathbb{R}^2)dy} & \text{if } S_{\tau_i} = s_m \\ \frac{F_{15}(\tau_i, B_{\tau_i}, s, C)}{1 - \int_0^{s-\tau_i} F_{15}(\tau_i, B_{\tau_i}, \tau_i + y, \mathbb{R}^2)dy} & \text{if } S_{\tau_i} = s_h. \end{cases}$$

Now, Theorem 3.2 and Theorem T8 in Chapter VIII of [7] yield that

$$M_t = \int_0^t \theta_2(s, P_s)dB_s + \sum_{i=1}^{N_t}\Delta P_{\tau_i} - \int_0^t\int_{\mathbb{R}^2} z_2 l_s(dz)ds$$

and

$$A_t = \int_0^t \theta_1(s, P_s)ds + \int_0^t\int_{\mathbb{R}^2} z_2 l_s(dz)ds,$$

where $z = (z_1, z_2) \in \mathbb{R}^2$. The result now follows since

$$\int_{\mathbb{R}^2} z_2 l_s(dz) = \theta_3(s, \tau_{N_s}, B_{\tau_{N_s}}).$$

□

4. Numerical studies

Numerical techniques to find conditional probabilities discussed in the previous section will be demonstrated via the example of the time of the next jump. Conditional probabilities for the type of the next jump and the size of the next jump can be computed applying similar numerical algorithms.
The function F_4 can be approximated by finding F_{16}, which solves

$$F_{16}(t_i, y_m, t_{n_1}) = e^{-\lambda_Z(t_{n_1}-t_i)}\Big(1 - D_1(t_{n_1}, t_i, y_m)\Big) + \Delta_1 \times \sum_{j=i+1}^{n_1} \lambda_Z e^{-\lambda_Z(t_j-t_i)} \times \Big[\Big(1 - D_1(t_j, t_i, y_m)\Big) +$$

(4.1)

$$+ \sum_{l=1}^{k_j} \mathbb{P}\Big(y_{l-1} - y_m < B_{t_j-t_i} \le y_l - y_m \mid B_s < h^h(t_i+s), \forall s \in [0, t_j - t_i]\Big) F_{16}(t_j, y_l, t_{n_1}) + \Phi_1(t_j, t_i, y_m)\Big],$$

with the boundary condition

$$F_{16}(t_{n_1}, y_m, t_{n_1}) = 0 \quad \text{for} \quad m = 0, 1, ..., k_{n_1},$$

$$k_j = \max\Big(0 \le l \le n_2 : y_l \le h^l(t_j)\Big), \quad j = 1, 2, ..., n_1.$$

In above, we take a mesh with uniform spacing given

$$t_i = t + i\Delta_1, i = 0, 1, ..., n_1, \quad \text{and} \quad y_m = C_1 + m\Delta_2, m = 0, 1, ..., n_2,$$

with

$$\Delta_1 = \frac{u-t}{n_1},\ n_1 \in \mathbb{N}, \quad \text{and} \quad \Delta_2 = \frac{C_2 - C_1}{n_2},\ n_2 \in \mathbb{N}.$$

Constants C_1 and C_2 are taken such that

$$\mathbb{P}(\min_{s\in[0,u-t]} B_s \le C_1) = \mathbb{P}(\max_{s\in[0,u-t]} B_s \ge -C_1) = 2\Phi\Big(\frac{C_1}{\sqrt{u-t}}\Big) = \epsilon \tag{4.2}$$

for some small $\epsilon > 0$ and

$$C_2 \ge \max_{s\in[0,u-t]} h^h(t+s).$$

The value F_{16} can be calculated applying backward induction to $i = 1, ..., n_1$. Also note that F_6 can be computed via procedure similar to the one applied for F_4, therefore, the details are omitted.

4.1. A numerical algorithm to calculate Brownian motion hitting probabilities and densities for two-sided curved boundaries.

Let

$$\tau = \inf\Big(t \ge 0 : B_t = f(t) \text{ or } B_t = g(t)\Big),$$

deterministic functions f and g are in the class $C^2\Big([0, u]\Big)$ and satisfy $f(t) < g(t)$, $\forall t \in [0, u]$, and a constant K is such that $f(u) \le K \le g(u)$.
According to [36],

$$\mathbb{P}\Big(\tau > u, B_u \le K\Big) = v_1(0, 0)$$

and

$$\mathbb{P}\Big(\tau < u, B_\tau = f(\tau)\Big) = v_2(0, 0),$$

where, for $0 < t < u$ and $f(t) < x < g(t)$, functions $v_1(t,x)$ and $v_2(t,x)$ solve the backward linear heat equation

$$\frac{\partial v_i}{\partial t} + \frac{1}{2}\frac{\partial^2 v_i}{\partial x^2} = 0, \quad i = 1, 2,$$

with corresponding boundary conditions

$$v_1(t, f(t)) = 0, \quad v_1(t, g(t)) = 0, \quad v_1(u, x) = \mathbb{I}\Big(x \leq K\Big)$$

and

$$v_2(t, f(t)) = 1, \quad v_2(t, g(t)) = 0, \quad v_2(u, x) = 0.$$

To find $v_1(0,0)$ and $v_2(0,0)$, one can use 3-sigma and rectangle rules approximating function h from (2.1) with

$$\frac{\gamma_1 x - w^D \times \Delta \times \sum_{i=1}^{n} \Phi\Big(\frac{x - K_i e^{-r(T-t)}}{\Sigma(t)}\Big) \frac{1}{\sqrt{2\pi\sigma_\kappa^2}} e^{-\frac{(K_i-\kappa)^2}{2\sigma_\kappa^2}} - \gamma_2}{\gamma_3} \tag{4.3}$$

where $K_i = (\kappa - 3\sigma_\kappa) + (i-1)\Delta + \frac{\Delta}{2}$ and $\Delta = \frac{(\kappa+3\sigma_\kappa)-(\kappa-3\sigma_\kappa)}{n} = \frac{6\sigma_\kappa}{n}$, $i = 0, 1, ..., n-1$, $n \in \mathbb{N}$, and then apply Crank-Nicolson finite difference method which is used for numerically solving the heat equation (see, e.g., [37] and [39]). Recall that rectangle rule approximates the value of a definite integral by finding the areas of rectangles with heights equal to corresponding values of the integrand.

To compute

$$\mathbb{P}\Big(\tau > u, K_1 \leq B_u \leq K_2\Big), \quad u \in [0, T],$$

it can be used that

$$\mathbb{P}\Big(\tau > u, K_1 \leq B_u \leq K_2\Big) = \mathbb{P}\Big(\tau > u, B_u \leq K_2\Big) - \mathbb{P}\Big(\tau > u, B_u \leq K_1\Big). \tag{4.4}$$

Based on these results applied for $K = g(u)$ and Brownian motions B and $-B$, the values of D_m, $D_{m,1}$ and $D_{m,2}$ can be derived. Values of q^m, ϕ_m, $\phi_{m,1}$ and $\phi_{m,2}$ can be calculated according to rectangle rule: we can approximate the value of a density function $q(x)$ with $\frac{F(x+\eta)-F(x)}{\eta}$, where $F(x)$ denotes its corresponding cumulative distribution function and $\eta > 0$ is a small number. Recall that $D_{m,1}$, $D_{m,2}$, $\phi_{m,1}$ and $\phi_{m,2}$ are as defined in (A.5).

As an alternative to PDE approach, other numerical methods could be considered (e.g. approximation by piecewise linear boundaries (see, e.g., [27] and [32]) or Volterra integral equations approach (see, e.g., [11])).

4.2. **A numerical technique to calculate Brownian motion hitting probabilities and densities for one-sided curved boundaries.**

Let

$$\tau = \inf\Big(t \geq 0 : B_t = g(t)\Big),$$

deterministic function g is in the class $C^2\Big([0,u]\Big)$ and satisfies $g(0) > 0$, and constant K is such that $K \leq g(u)$.

Since

$$\begin{aligned}
&\mathbb{P}\Big(B_t < g(t), \forall t \in [0,u], \text{ and } B_u \le K\Big)\\
&= \mathbb{P}\Big(C < B_t < g(t), \forall t \in [0,u], \text{ and } B_u \le K\Big) + \mathbb{P}\Big(\min_{t\in[0,u]} B_t \le C, \quad B_t < g(t), \forall t \in [0,u], \text{ and } B_u \le K\Big)\\
&\le \mathbb{P}\Big(C < B_t < g(t), \forall t \in [0,u], \text{ and } B_u \le K\Big) + \mathbb{P}\Big(\min_{t\in[0,u]} B_t \le C\Big)\\
&= \mathbb{P}\Big(C < B_t < g(t), \forall t \in [0,u], \text{ and } B_u \le K\Big) + \mathbb{P}\Big(\max_{t\in[0,u]} B_t \ge -C\Big)
\end{aligned}$$

and

$$\mathbb{P}\Big(B_t < g(t), \forall t \in [0,u], \text{ and } B_u \le K\Big) \ge \mathbb{P}\Big(C < B_t < g(t), \forall t \in [0,u], \text{ and } B_u \le K\Big)$$

for all $C < 0$, $\mathbb{P}\Big(B_t < g(t), \forall t \in [0,u], \text{ and } B_u \le K\Big)$ can be approximated with

$$\mathbb{P}\Big(C_1 < B_t < g(t), \forall t \in [0,u], \text{ and } B_u \le K\Big), \tag{4.5}$$

where a constant C_1 is defined in (4.2). Probability (4.5) can be evaluated according to the PDE approach discussed in Section 4.1.

Based on these results applied for Brownian motions B and $-B$, probabilities Φ_1, Φ_2, D_1, D_2, D^l and D^u can be found. To calculate densities q_1, q_2, ϕ_1, ϕ_2, ϕ^l and ϕ^u, a rectangle rule can be used. As in the two-sided boundary case, as an alternative to PDE approach, other numerical methods can be applied as well (e.g. approximation by piecewise linear functions (see, e.g., [28] and [38]) or Volterra integral equations approach (see, e.g., [12] and [31])).

4.3. **Plots.** In view of above methods we can obtain the approximate conditional distribution for the time of the next jump. The set of parameters used to obtain the plots contained in Figures 3-5 is as follows: $r = 0.0001$, $t = 95$, $T = 100$, $\alpha_1 = 0.2$, $\sigma_\kappa = 1$, $\kappa = 100$, $\gamma_1 = 3$, $\gamma_2 = 40$, $\gamma_3 = 1$, $w^D = 20$, $\lambda_Z = 0.1$. Given these parameters the dynamics of low and high level boundaries, h^l and h^h, is illustrated in Figure 3.

Figure 4 plots the conditional probabilities for time to the next jump when the current state of the economy is at the low level while Figure 5 plots the corresponding probabilities for the middle level economy.

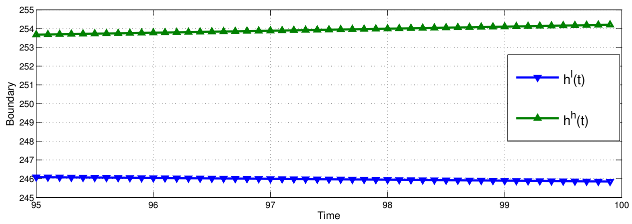

FIGURE 3. Low and high level boundaries calculated for some set of parameters: $r = 0.0001$, $t = 95$, $T = 100$, $\alpha_1 = 0.2$, $\sigma_\kappa = 1$, $\kappa = 100$, $\gamma_1 = 3$, $\gamma_2 = 40$, $\gamma_3 = 1$, $w^D = 20$.

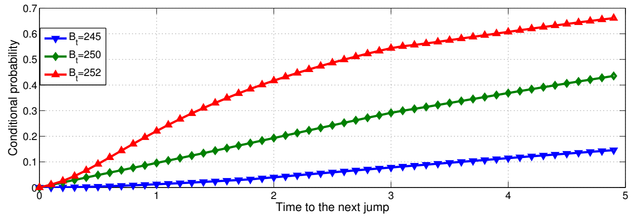

FIGURE 4. Conditional probability for the time of the next jump given $S_t = s_l$: $r = 0.0001$, $t = 95$, $T = 100$, $\alpha_1 = 0.2$, $\sigma_\kappa = 1$, $\kappa = 100$, $\gamma_1 = 3$, $\gamma_2 = 40$, $\gamma_3 = 1$, $w^D = 20$, $\lambda_Z = 0.1$.

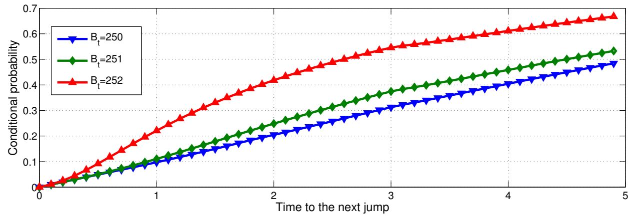

FIGURE 5. Conditional probability for the time of the next jump given $S_t = s_m$: $r = 0.0001$, $t = 95$, $T = 100$, $\alpha_1 = 0.2$, $\sigma_\kappa = 1$, $\kappa = 100$, $\gamma_1 = 3$, $\gamma_2 = 40$, $\gamma_3 = 1$, $w^D = 20$, $\lambda_Z = 0.1$.

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Appendix A. Proofs not contained in the main text

Proof of Theorem 2.1. This theorem will be proved in several steps.

Step 1 First, it will be shown that there exist some $\delta_1 \in (0,T)$ and $\Delta_1 > 0$ such that

$$h^h(t) - h^l(t) \geq \Delta_1, \ \forall t \in (T-\delta_1, T).$$

According to (2.5) and (2.6),

$$A_1 = \lim_{t \uparrow T} p_1(t) = \kappa - \sqrt{-2\sigma_\kappa^2 \ln\Big(\frac{\gamma_1}{w^D}\sqrt{2\pi\sigma_\kappa^2}\Big)}$$

and

$$A_2 = \lim_{t \uparrow T} p_2(t) = \kappa + \sqrt{-2\sigma_\kappa^2 \ln\Big(\frac{\gamma_1}{w^D}\sqrt{2\pi\sigma_\kappa^2}\Big)},$$

which means that $A_1 < A_2$.
Then

$$\begin{aligned}
&\lim_{t\uparrow T}\int_{-\infty}^{\infty}\Phi\Big(\frac{Ke^{-r(T-t)}-p_1(t)}{\Sigma(t)}\Big)\frac{1}{\sqrt{2\pi\sigma_\kappa^2}}e^{-\frac{(K-\kappa)^2}{2\sigma_\kappa^2}}\,dK\\
&=\int_{-\infty}^{\infty}\Phi\Big(\lim_{t\uparrow T}\frac{Ke^{-r(T-t)}-p_1(t)}{\Sigma(t)}\Big)\frac{1}{\sqrt{2\pi\sigma_\kappa^2}}e^{-\frac{(K-\kappa)^2}{2\sigma_\kappa^2}}\,dK\\
&=\int_{A_1}^{\infty}\frac{1}{\sqrt{2\pi\sigma_\kappa^2}}e^{-\frac{(K-\kappa)^2}{2\sigma_\kappa^2}}\,dK
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{t\uparrow T}\int_{-\infty}^{\infty}\Phi\Big(\frac{Ke^{-r(T-t)}-p_2(t)}{\Sigma(t)}\Big)\frac{1}{\sqrt{2\pi\sigma_\kappa^2}}e^{-\frac{(K-\kappa)^2}{2\sigma_\kappa^2}}\,dK\\
&=\int_{-\infty}^{\infty}\Phi\Big(\lim_{t\uparrow T}\frac{Ke^{-r(T-t)}-p_2(t)}{\Sigma(t)}\Big)\frac{1}{\sqrt{2\pi\sigma_\kappa^2}}e^{-\frac{(K-\kappa)^2}{2\sigma_\kappa^2}}\,dK\\
&=\int_{A_2}^{\infty}\frac{1}{\sqrt{2\pi\sigma_\kappa^2}}e^{-\frac{(K-\kappa)^2}{2\sigma_\kappa^2}}\,dK.
\end{aligned}$$

Hence,

$$\begin{aligned}
\lim_{t\uparrow T}\Big(h^h(t)-h^l(t)\Big) &= \frac{1}{\gamma_3}\Big(w^D\int_{A_1}^{A_2}\frac{1}{\sqrt{2\pi\sigma_\kappa^2}}e^{-\frac{(K-\kappa)^2}{2\sigma_\kappa^2}}\,dK - 2\gamma_1\sqrt{-2\sigma_\kappa^2\ln\Big(\frac{\gamma_1}{w^D}\sqrt{2\pi\sigma_\kappa^2}\Big)}\Big)\\
&= \frac{2}{\gamma_3}\Big(\gamma_1\sqrt{2\pi\sigma_\kappa^2}e^{\frac{z^2}{2}}\int_0^z\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}\,dy-\gamma_1\sigma_\kappa z\Big)\\
&=: f(z),
\end{aligned}$$

where

$$z=\sqrt{-2\ln\Big(\frac{\gamma_1}{w^D}\sqrt{2\pi\sigma_\kappa^2}\Big)}>0.$$

Since $f(0)=0$ and $f'(z)=\frac{2\gamma_1\sqrt{2\pi\sigma_\kappa^2}ze^{\frac{z^2}{2}}\int_0^z\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}\,dy}{\gamma_3}$ is positive for $z>0$ and 0 for $z=0$, we obtain that

$$\lim_{t\uparrow T}\Big(h^h(t)-h^l(t)\Big)>0.$$

Finally, one can take, e.g., $\Delta_1=\frac{1}{2}\lim_{t\uparrow T}\Big(h^h(t)-h^l(t)\Big)$ and use the definition of the limit.

Step 2 Second, it will be proved that there exists some $\Delta_2>0$ such that

$$h^h(t)-h^l(t)\ge\Delta_2,\ \forall t\in[0,T-\delta_1].$$

Assume that $t\in[0,T-\delta_1]$. Then (2.5) and (2.6) imply that

$$\begin{aligned}
p_2(t)-p_1(t) &= 2\sqrt{-2(\sigma_\kappa^2e^{-r(T-t)}+\Sigma^2(t))\ln\Big(\frac{\gamma_1}{w^D}\sqrt{2\pi(\sigma_\kappa^2e^{-2r(T-t)}+\Sigma^2(t))}\Big)}\\
&\ge 2\sqrt{-2(\sigma_\kappa^2e^{-rT}+\alpha_1^2\frac{1-e^{-2r\delta_1}}{2r})\ln\Big(\frac{\gamma_1}{w^D}\sqrt{2\pi(\frac{\alpha_1^2}{2r}+(\sigma_\kappa^2-\frac{\alpha_1^2}{2r})e^{-2rT})}\Big)}\\
&=: \delta_2>0,
\end{aligned}$$

which means that, for all $y\in[-\frac{\delta_2}{2},\frac{\delta_2}{2}]$,

$$p_1(t)\le\kappa e^{-r(T-t)}+y\le p_2(t)$$

and, hence,

$$h^h(t) \geq h(t, \kappa e^{-r(T-t)} + y) \geq h^l(t). \tag{A.1}$$

Furthermore,

$$\begin{aligned} h_x(t, \kappa e^{-r(T-t)} + y) &= \frac{1}{\gamma_3}\Bigg(\gamma_1 - \frac{w^D}{\sqrt{2\pi\Big(\sigma_\kappa^2 e^{-2r(T-t)} + \Sigma^2(t)\Big)}} e^{-\frac{y^2}{2(\sigma_\kappa^2 e^{-r(T-t)} + \Sigma^2(t))}}\Bigg) \\ &\leq \frac{1}{\gamma_3}\Bigg(\gamma_1 - \frac{w^D}{\sqrt{2\pi\Big(\sigma_\kappa^2 e^{-2rT} + \Sigma^2(0)\Big)}} e^{-\frac{y^2}{2(\sigma_\kappa^2 e^{-rT} + \alpha_1^2 \frac{1-e^{-2r\delta_1}}{2r})}}\Bigg) \end{aligned}$$

Assumption 1 guarantees that there exists some positive $\delta_3 \leq \frac{\delta_2}{2}$ such that

$$\begin{aligned} h_x(t, \kappa e^{-r(T-t)} - \delta_3) &= h_x(t, \kappa e^{-r(T-t)} + \delta_3) \\ &\leq \frac{1}{\gamma_3}\Bigg(\gamma_1 - \frac{w^D}{\sqrt{2\pi\Big(\sigma_\kappa^2 e^{-2rT} + \Sigma^2(0)\Big)}} e^{-\frac{\delta_3^2}{2(\sigma_\kappa^2 e^{-rT} + \Sigma^2(T-\delta_1))}}\Bigg) \\ &=: -\delta_4 < 0. \end{aligned}$$

Moreover,

$$h_{xx}(t,x) = \frac{w^D(x - \kappa e^{-r(T-t)})}{\gamma_3\sqrt{2\pi(\sigma_\kappa^2 e^{-2r(T-t)} + \Sigma^2(t))}(\sigma_\kappa^2 e^{-r(T-t)} + \Sigma^2(t))} e^{-\frac{(\kappa e^{-r(T-t)} - x)^2}{2(\sigma_\kappa^2 e^{-r(T-t)} + \Sigma^2(t))}},$$

that is, function $h_x(t,x)$ is a decreasing function of x for $x \leq \kappa e^{-r(T-t)}$ and an increasing function of x for $x \geq \kappa e^{-r(T-t)}$. This means that, for $x \in [\kappa e^{-r(T-t)} - \delta_3, \kappa e^{-r(T-t)} + \delta_3]$,

$$h_x(t,x) \leq \max\Big(h_x(t, \kappa e^{-r(T-t)} - \delta_3), h_x(t, \kappa e^{-r(T-t)} + \delta_3)\Big) \leq -\delta_4.$$

Thus, by the mean value theorem and in view of (A.1),

$$h^h(t) - h^l(t) \geq h(t, \kappa e^{-r(T-t)} - \delta_3) - h(t, \kappa e^{-r(T-t)} + \delta_3) \geq 2\delta_3\delta_4 > 0.$$

Step 3 Finally, it will be shown that there exists some $\Delta > 0$ such that

$$h^h(t) - h^l(t) \geq \Delta, \ \forall t \in [0, T).$$

Indeed, one can take $\Delta = \min(\Delta_1, \Delta_2)$, and the result follows.

□

Proof of Theorem 3.3. The proof of this theorem will be done in several steps. Denote by τ the remaining time to the first arrival after t in the sunspot process Z. Recall that τ is independent of $\mathcal{F}_t^P$ and Z is a Poisson process with intensity λ_Z. Hence, τ has an exponential distribution with parameter λ_Z. Let

$$\mathcal{F}_t^{P,\tau} = \sigma\{(P_s, 0 \leq s \leq t), \tau\}.$$

Step 1 Calculation of the conditional probability on the set $[S_t = s_l]$: By the law of iterated expectations,

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[\tau_{N_t+1} < u, S_{\tau_{N_t+1}} \in C_1, J_{N_t+1} \in C_2\Big] \mid \mathcal{F}_t^P\Big) \\
&= \mathbb{E}^{\mathbb{P}}\Big(\mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[\tau_{N_t+1} < u, S_{\tau_{N_t+1}} \in C_1, J_{N_t+1} \in C_2\Big] \mid \mathcal{F}_t^{P,\tau}\Big) \mid \mathcal{F}_t^P\Big) \\
&= \mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[\tau \geq u-t\Big]\mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[\tau_{N_t+1} < u, S_{\tau_{N_t+1}} \in C_1, J_{N_t+1} \in C_2\Big] \mid \mathcal{F}_t^{P,\tau}\Big) \mid \mathcal{F}_t^P\Big) \\
&+ \mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[\tau < u-t\Big]\mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[\tau_{N_t+1} < t+\tau, S_{\tau_{N_t+1}} \in C_1, J_{N_t+1} \in C_2\Big] \mid \mathcal{F}_t^{P,\tau}\Big) \mid \mathcal{F}_t^P\Big) \\
&+ \mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[\tau < u-t\Big]\mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[B_{t+\tau} \leq h_2(t+\tau), t+\tau \leq \tau_{N_t+1} < u, S_{\tau_{N_t+1}} \in C_1, J_{N_t+1} \in C_2\Big] \mid \mathcal{F}_t^{P,\tau}\Big) \mid \mathcal{F}_t^P\Big) \\
&+ \mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[\tau < u-t\Big]\mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[B_{t+\tau} > h_2(t+\tau), t+\tau \leq \tau_{N_t+1} < u, S_{\tau_{N_t+1}} \in C_1, J_{N_t+1} \in C_2\Big] \mid \mathcal{F}_t^{P,\tau}\Big) \mid \mathcal{F}_t^P\Big).
\end{aligned}
$$

The first term in this decomposition corresponds to the scenario that there are no shock arrivals on $[t, u)$ at all and, hence, Brownian motion hits the boundary h^h on (t, u). The new state of the state process is equal to s_h and the jump size is $J^u(\tau_{N_t+1})$.

The second term corresponds to the scenario that the first shock arrival time is $t + \tau < u$ and Brownian motion hits the boundary h^h on $(t, t+\tau)$. As in the first scenario, the process switches to s_h, the jump size is equal to $J^h(\tau_{N_t+1})$.

According to the third scenario, the first shock arrival time is $t + \tau < u$, the Brownian motion value stays smaller than the value of the boundary h^h on $(t, t + \tau)$ and at the time of the shock $B_{t+\tau} \leq h^l(t + \tau)$. As a consequence, there is no jump at time $t + \tau$.

The fourth scenario is the same as the third one with the only difference that $B_{t+\tau} > h^l(t + \tau)$. Therefore, the price jumps at time $t + \tau$. With probability p_{lh}, the new state of the state process is s_h and the jump size is $J^{lh}(t + \tau, B_{t+\tau})$. With probability $1 - p_{lh}$, the new state of the state process is s_m and the jump size is $J^{lm}(t + \tau, B_{t+\tau})$.

In view of the independence of τ and $\mathcal{F}_t^P$, the first and second terms are equal to

$$e^{-\lambda_Z(u-t)} \int_t^u \mathbb{I}(s_h \in C_1, J^h(y) \in C_2)\phi_1(y, t, B_t)dy$$

and

$$\int_0^{u-t} \lambda_Z e^{-\lambda_Z r}\Big[\int_t^{t+r} \mathbb{I}(s_h \in C_1, J^h(y) \in C_2)\phi_1(y, t, B_t)dy\Big]dr,$$

where

$$\phi_1(u, t, y) = -\frac{\partial D_1(u, t, y)}{\partial u}, \quad D_1(u, t, y) = \mathbb{P}\Big(B_s < h^h(t + s) - y, \forall s \in [0, u - t]\Big). \tag{A.2}$$

The third term is equal to

$$\mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[\tau < u-t\Big]\mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[B_{t+\tau} \le h^l(t+\tau), t+\tau \le \tau_{N_t+1} < u, S_{\tau_{N_t+1}} \in C_1, J_{N_t+1} \in C_2\Big] \mid \mathcal{F}_t^{P,\tau}\Big) \mid \mathcal{F}_t^P\Big)$$
$$= \mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[\tau < u-t\Big]\mathbb{E}^{\mathbb{P}}\Big(\mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[B_{t+\tau} \le h^l(t+\tau), (B_s < h^h(s), \forall s \in [t,t+\tau))\Big]$$
$$\mathbb{I}\Big(\tau_{N_t+1} < u, S_{\tau_{N_t+1}} \in C_1, J_{N_t+1} \in C_2\Big) \mid \mathcal{F}_{t+\tau}^P\Big) \mid \mathcal{F}_t^{P,\tau}\Big) \mid \mathcal{F}_t^P\Big)$$
$$= \mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[\tau < u-t\Big]\mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[B_{t+\tau} \le h^l(t+\tau), (B_s < h^h(s), \forall s \in [t,t+\tau))\Big]$$
$$F_1(t+\tau, B_{t+\tau}, u, C_1, C_2) \mid \mathcal{F}_t^{P,\tau}\Big) \mid \mathcal{F}_t^P\Big)$$
$$= \int_0^{u-t} \lambda_Z e^{-\lambda_Z r}\Big[\int_{-\infty}^{h^l(t+r)} q_1(x;r,t,B_t)F_1(t+r,x,u,C_1,C_2)dx\Big]dr,$$

where $q_1(x;r,t,y)$ is the density of B_r on the set $\Big[B_s < h^h(t+s)-y, \forall s \in [0,r]\Big]$, and the fourth term is equal to

$$\mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[\tau < u-t\Big]\mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[B_{t+\tau} > h^l(t+\tau), t+\tau \le \tau_{N_t+1} < u, S_{\tau_{N_t+1}} \in C_1, J_{N_t+1} \in C_2\Big] \mid \mathcal{F}_t^{P,\tau}\Big) \mid \mathcal{F}_t^P\Big)$$
$$= \mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[\tau < u-t\Big]\mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[B_{t+\tau} > h^l(t+\tau), (B_s < h^h(s), \forall s \in [t,t+\tau))\Big]$$
$$\mathbb{I}\Big(S_{\tau_{N_t+1}} \in C_1, J_{N_t+1} \in C_2\Big) \mid \mathcal{F}_t^{P,\tau}\Big) \mid \mathcal{F}_t^P\Big)$$
$$= \int_0^{u-t} \lambda_Z e^{-\lambda_Z r}\Big[\int_{h^l(t+r)}^{h^h(t+r)} q_1(x;r,t,B_t)\Big(p_{lh}\mathbb{I}(s_h \in C_1, J^{lh}(t+r,x) \in C_2)$$
$$+ p_{lm}\mathbb{I}(s_m \in C_1, J^{lm}(t+r,x) \in C_2)\Big)dx\Big]dr.$$

Combining all the terms, we get that

$$\begin{aligned} F_1(t,B_t,u,C_1,C_2) &= e^{-\lambda_Z(u-t)}\int_t^u \mathbb{I}(s_h \in C_1, J^h(y) \in C_2)\phi_1(y,t,B_t)dy \\ &+ \int_0^{u-t}\lambda_Z e^{-\lambda_Z r}\Big[\int_t^{t+r} \mathbb{I}(s_h \in C_1, J^h(y) \in C_2)\phi_1(y,t,B_t)dy \\ &+ \int_{-\infty}^{h^l(t+r)} q_1(x;r,t,B_t)F_1(t+r,x,u,C_1,C_2)dx \\ &+ \int_{h^l(t+r)}^{h^h(t+r)} q_1(x;r,t,B_t)\Big(p_{lh}\mathbb{I}(s_h \in C_1, J^{lh}(t+r,x) \in C_2) \\ &+ p_{lm}\mathbb{I}(s_m \in C_1, J^{lm}(t+r,x) \in C_2)\Big)dx\Big]dr. \end{aligned} \tag{A.3}$$

Step 2 Calculation of conditional probability on the set $[S_t = s_m]$: According to the first scenario, there are no shock arrivals on $[t,u)$ at all and, hence, Brownian motion hits one of the two boundaries h^h or h^l on (t,u). If it hits h^h earlier than h^l, then the new state of the state process is s_h and the jump size is equal to $J^h(t+\tau_{N_t+1})$. If it hits h^l earlier than h^h, then the new state of the state process is s_l and the jump size is equal to $J^l(t+\tau_{N_t+1})$. According to the second scenario, the first shock arrival time is $t+\tau < u$ and Brownian motion hits one of the two boundaries h^h or h^l on $(t,t+\tau)$, then the new state of the state process and the jump size are determined by the same mechanism as in the first scenario. Finally, according to the third scenario, the first shock arrival time is $t+\tau < u$ and Brownian motion stays between both boundaries h^h and h^l on $[t,t+\tau]$. With probability p_{mh}, the new state of the state process is s_h and the jump size is $J^{mh}(t+\tau, B_{t+\tau})$. With probability $1-p_{mh}$, the new state of the state process is s_l and the jump size is $J^{ml}(t+\tau, B_{t+\tau})$. Taking this decomposition, we obtain the formula for F_2:

$$
\begin{aligned}
&F_2(t, B_t, u, C_1, C_2) \\
&= e^{-\lambda_Z(u-t)} \int_t^u \Big[\mathbb{I}(s_h \in C_1, J^h(y) \in C_2)\phi_{m,1}(y,t,B_t) + \mathbb{I}(s_l \in C_1, J^l(y) \in C_2)\phi_{m,2}(y,t,B_t)\Big] dy \\
&+ \int_0^{u-t} \lambda_Z e^{-\lambda_Z r} \Big[\int_t^{t+r} \Big[\mathbb{I}(s_h \in C_1, J^h(y) \in C_2)\phi_{m,1}(y,t,B_t) + \mathbb{I}(s_l \in C_1, J^l(y) \in C_2)\phi_{m,2}(y,t,B_t)\Big] dy \\
&\qquad + \int_{h^l(t+r)}^{h^h(t+r)} q^m(x; r, t, B_t)\Big(p_{mh}\mathbb{I}(s_h \in C_1, J^{mh}(t+r, x) \in C_2) \\
&\qquad\qquad + p_{ml}\mathbb{I}(s_l \in C_1, J^{ml}(t+r, x) \in C_2)\Big) dx\Big] dr,
\end{aligned} \tag{A.4}
$$

where $q^m(x; r, t, y)$ is the density of B_r on the set $\Big[h^l(t+s) - y < B_s < h^h(t+s) - y, \forall s \in [0, r]\Big]$ and

$$
\begin{aligned}
\phi_{m,1}(u,t,y) &= \frac{\partial D_{m,1}(u,t,y)}{\partial u}, \quad D_{m,1}(u,t,y) = \mathbb{P}\Big(\tau(t,y) \le u-t, B_{\tau(t,y)} = h^h(t+\tau(t,y)) - y\Big), \\
\phi_{m,2}(u,t,y) &= \frac{\partial D_{m,2}(u,t,y)}{\partial u}, \quad D_{m,2}(u,t,y) = \mathbb{P}\Big(\tau(t,y) \le u-t, B_{\tau(t,y)} = h^l(t+\tau(t,y)) - y\Big),
\end{aligned}
$$

$$
\tau(t,y) = \inf\{s \ge 0 : B_s = h^l(t+s) - y \quad \text{or} \quad B_s = h^h(t+s) - y\}. \tag{A.5}
$$

Step 3 Calculation of conditional probability on the set $[S_t = s_h]$: The conditional probability on the set $[S_t = s_h]$ satisfies

$$
\begin{aligned}
F_3(t, B_t, u, C_1, C_2) &= e^{-\lambda_Z(u-t)} \int_t^u \mathbb{I}(s_l \in C_1, J^l(y) \in C_2)\phi_2(y,t,B_t) dy \\
&+ \int_0^{u-t} \lambda_Z e^{-\lambda_Z r} \Big[\int_t^{t+r} \mathbb{I}(s_l \in C_1, J^l(y) \in C_2)\phi_2(y,t,B_t) dy \\
&+ \int_{h^h(t+r)}^{\infty} q_2(x; r, t, B_t) F_3(t+r, x, u, C_1, C_2) dx \\
&+ \int_{h^l(t+r)}^{h^h(t+r)} q_2(x; r, t, B_t)\Big(p_{hl}\mathbb{I}(s_l \in C_1, J^{hl}(t+r, x) \in C_2) \\
&\qquad + p_{hm}\mathbb{I}(s_m \in C_1, J^{hm}(t+r, x) \in C_2)\Big) dx\Big] dr,
\end{aligned} \tag{A.6}
$$

where $q_2(x; r, t, y)$ is the density of B_r on the set $\Big[B_s > h^l(t+s) - y, \forall s \in [0, r]\Big]$ and

$$
\phi_2(u,t,y) = -\frac{\partial D_2(u,t,y)}{\partial u}, \quad D_2(u,t,y) = \mathbb{P}\Big(B_s > h^l(t+s) - y, \forall s \in [0, u-t]\Big). \tag{A.7}
$$

The calculation procedure is patterned after Step 2.

□

Proof of Lemma 3.1. Calculations pattern after Theorem 3.3 and yield the following:

$$
\begin{aligned}
F_7(u,t,B_t,C) = e^{-\lambda_Z(u-t)} &\int_t^u \mathbb{I}(p^h(y,h^h(y)) \in C_1, J^h(y) \in C_2)\phi_1(y,t,B_t)dy \\
&+ \int_0^{u-t} \lambda_Z e^{-\lambda_Z r}\Big[\int_t^{t+r} \mathbb{I}(p^h(y,h^h(y)) \in C_1, J^h(y) \in C_2)\phi_1(y,t,B_t)dy \\
&+ \int_{-\infty}^{h^l(t+r)} q_1(x;r,t,B_t)F_7(u,t+r,x,C)dx \\
&+ \int_{h^l(t+r)}^{h^h(t+r)} q_1(x;r,t,B_t)\Big(p_{lh}\mathbb{I}(p^h(t+r,x) \in C_1, J^{lh}(t+r,x) \in C_2) \\
&\quad + p_{lm}\mathbb{I}(p^m(t+r,x) \in C_1, J^{lm}(t+r,x) \in C_2)\Big)dx\Big]dr,
\end{aligned}
$$

(A.8)

$$
\begin{aligned}
F_8(u,t,B_t,C) = e^{-\lambda_Z(u-t)} &\int_t^u \Big[\mathbb{I}(p^h(y,h^h(y)) \in C_1, J^h(y) \in C_2)\phi_{m,1}(y,t,B_t) \\
&\quad + \mathbb{I}(p^l(y,h^l(y)) \in C_1, J^l(y) \in C_2)\phi_{m,2}(y,t,B_t)\Big]dy \\
&+ \int_0^{u-t} \lambda_Z e^{-\lambda_Z r}\Big[\int_t^{t+r}\Big[\mathbb{I}(p^h(y,h^h(y)) \in C_1, J^h(y) \in C_2)\phi_{m,1}(y,t,B_t) \\
&\quad + \mathbb{I}(p^l(y,h^l(y)) \in C_1, J^l(y) \in C_2)\phi_{m,2}(y,t,B_t)\Big]dy \\
&\quad + \int_{h^l(t+r)}^{h^h(t+r)}\Big(p_{mh}\mathbb{I}(p^h(t+r,x) \in C_1, J^{mh}(t+r,x) \in C_2) \\
&\quad + p_{ml}\mathbb{I}(p^l(t+r,x) \in C_1, J^{ml}(t+r,x) \in C_2)\Big)q^m(x;r,t,B_t)dx\Big]dr,
\end{aligned}
$$

$$
\begin{aligned}
F_9(u,t,B_t,C) = e^{-\lambda_Z(u-t)} &\int_t^u \mathbb{I}(p^l(y,h^l(y)) \in C_1, J^l(y) \in C_2)\phi_2(y,t,B_t)dy \\
&+ \int_0^{u-t} \lambda_Z e^{-\lambda_Z r}\Big[\int_t^{t+r} \mathbb{I}(p^l(y,h^l(y)) \in C_1, J^l(y) \in C_2)\phi_2(y,t,B_t)dy \\
&+ \int_{h^h(t+r)}^{\infty} q_2(x;t,B_t,r)F_9(u,t+r,x,C)dx \\
&+ \int_{h^l(t+r)}^{h^h(t+r)} q_2(x;t,B_t,r)\Big(p_{hl}\mathbb{I}(p^l(t+r,x) \in C_1, J^{hl}(t+r,x) \in C_2) \\
&\quad + p_{hm}\mathbb{I}(p^m(t+r,x) \in C_1, J^{hm}(t+r,x) \in C_2)\Big)dx\Big]dr.
\end{aligned}
$$

In above representation, J^i and J^{ij} are as defined in (3.1) and $\phi_{m,1}$ and $\phi_{m,2}$ are as defined in (A.5). Recall that $q^m(x;r,t,y)$ is the density of B_r on the set $\Big[h^l(t+s) - y < B_s < h^h(t+s) - y, \forall s \in [0,r]\Big]$. □

Proof of Lemma 3.2. Applying Leibniz's rule for differentiating integrals to F_7, F_8 and F_9, we obtain $F_{10}(u,t,B_t,C)$ satisfies

$$\begin{aligned} F_{10}(u,t,B_t,C) &= e^{-\lambda_Z(u-t)}\mathbb{I}(p^h(u,h^h(u))\in C_1, J^h(u)\in C_2)\phi_1(u,t,B_t) \\ &\quad + \lambda_Z e^{-\lambda_Z(u-t)}\Big[\int_{h^l(u)}^{h^h(u)} q_1(x;u-t,t,B_t)\Big(p_{lh}\mathbb{I}(p^h(u,x)\in C_1, J^{lh}(u,x)\in C_2) \\ &\qquad + p_{lm}\mathbb{I}(p^m(u,x)\in C_1, J^{lm}(u,x)\in C_2)\Big)dx\Big] \\ &\quad + \int_0^{u-t}\lambda_Z e^{-\lambda_Z r}\Big[\int_{-\infty}^{h^l(t+r)} q_1(x;r,t,B_t)F_{10}(u,t+r,x,C)dx\Big]dr, \end{aligned} \tag{A.9}$$

$$\begin{aligned} F_{11}(u,t,B_t,C) &= e^{-\lambda_Z(u-t)}\Big[\mathbb{I}(p^h(u,h^h(u))\in C_1, J^h(u)\in C_2)\phi_{m,1}(u,t,B_t) \\ &\quad + \mathbb{I}(p^l(u,h^l(u))\in C_1), J^l(u)\in C_2)\phi_{m,2}(u,t,B_t)\Big] \\ &\quad + \lambda_Z e^{-\lambda_Z(u-t)}\Big[\int_{h^l(u)}^{h^h(u)} q^m(x;u-t,t,B_t)\Big(p_{mh}\mathbb{I}(p^h(u,x)\in C_1, J^{mh}(u,x)\in C_2) \\ &\qquad + p_{ml}\mathbb{I}(p^l(u,x)\in C_1, J^{ml}(u,x)\in C_2)\Big)dx\Big] \end{aligned} \tag{A.10}$$

and $F_{12}(u,t,B_t,C)$ satisfies

$$\begin{aligned} F_{12}(u,t,B_t,C) &= e^{-\lambda_Z(u-t)}\mathbb{I}(p^l(u,h^l(u))\in C_1, J^l(u)\in C_2)\phi_2(u,t,B_t) \\ &\quad + \lambda_Z e^{-\lambda_Z(u-t)}\Big[\int_{h^l(u)}^{h^h(u)} q_2(x;t,B_t,u-t)\Big(p_{hl}\mathbb{I}(p^l(u,x)\in C_1, J^{hl}(u,x)\in C_2) \\ &\qquad + p_{hm}\mathbb{I}(p^m(u,x)\in C_1, J^{hm}(u,x)\in C_2)\Big)dx\Big] \\ &\quad + \int_0^{u-t}\lambda_Z e^{-\lambda_Z r}\Big[\int_{h^h(t+r)}^{\infty} q_2(x;t,B_t,r)F_{12}(u,t+r,x,C)dx\Big]dr. \end{aligned} \tag{A.11}$$

In particular, for $C=\mathbb{R}^2$, indicator functions in (A.9), (A.10), and (A.11) are equal to 1, and the result for $g^{(i+1)}(u,\mathbb{R}^2)$ follows. □

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