Options hedging under liquidity costs

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Abstract

Following the framework of Çetin, Jarrow and Protter [4] we study the problem of super-replication in presence of liquidity costs under additional restrictions on the gamma of the hedging strategies in a generalized Black-Scholes economy. We find that the minimal super-replication price is different than the one suggested by the Black-Scholes formula and is the unique viscosity solution of the associated dynamic programming equation. This is in contrast with the results of [4] who find that the arbitrage free price of a contingent claim coincides with the Black-Scholes price. However, in [4] a larger class of admissible portfolio processes is used and the replication is achieved in the $L^2$ approximating sense.

Key words: Super-replication, liquidity cost, Gamma process, parabolic majorant, double stochastic integrals.


1 Introduction

Black-Scholes methodology for the pricing and hedging of options requires the market to be frictionless and competitive. In other words, traders can trade any quantity of the asset without changing its price and the trade is subject to no transaction costs and restrictions. There has been numerous works to relax these assumptions as it is now well known that the markets do not operate frictionless and perfectly competitive (see, e.g., [2], [3], [11], [12], [13], [14] and [18]).

Relaxation of both the frictionless and competitive market hypotheses introduces the notion of liquidity risk. Roughly speaking the liquidity risk is the additional risk due to the timing and size of a trade. Recently, several authors have proposed a number of methods to incorporate the liquidity risk into asset pricing theory (see [1], [4], [5], [6] and [21]).
common characteristic of all these works is that the liquidity risk appears as some nonlinear transaction cost which appears due to the imbalance between the supply and demand in the financial market which is relevant if an agent is attempting to trade large volumes in a short time. In this work we follow the approach introduced by Çetin, Jarrow and Protter [4] who introduced the so-called “supply curve” to model the asset price as a function of size and time. Starting with the given supply curve for, say, the stock, the authors show that the existence of the liquidity costs makes the trading strategies with infinite quadratic variation infeasible since they incur infinite liquidity costs. One important consequence of their modelling is that the continuous trading strategies of finite variation incur no liquidity costs; thus, the market is approximately complete (in an $L^2$-sense) if there exists a unique equivalent martingale measure for the ‘marginal price process’ (see [4] for details). In particular, they show that in a Black-Scholes type economy with liquidity costs the price of an option is given by the standard Black-Scholes formula and the approximate hedging strategy can be obtained by some appropriate averaging of the Black-Scholes hedge (see [5] for some further results and numerical and empirical studies).

The set of continuous and of finite variation processes is quite large and this is, indeed, the rationale behind the results of [4]. If any process in this set is admissible as a trading strategy, this means one can liquidate any position at any speed. However, in an illiquid market one is not always able to trade with infinite speed. A related work on such trading restriction can be found in Longstaff [20] who suggests a uniform bound on the time derivative of trading strategies to study the optimal portfolios in an illiquid market. The restrictions that we place on the trading strategies in this paper can be seen as a relaxation of the restrictions in [20]. First of all, we allow a trading strategy to have infinite variation. More precisely, the admissible trading strategies form a larger subset of semi-martingales (see (2.2)). As seen, the finite variation part of a trading strategy consists of a pure-jump component and an absolutely continuous component. The remaining infinite variation part is an integral with respect to the price process of the stock, which is a martingale since we work under the unique risk-neutral measure. The integrand in the absolutely continuous part of the trading strategy can be viewed as the rate of change of the trading strategy with respect to time while the integrand in the infinite variation part can be seen as the rate of change with respect to the changes in the stock price. As in [20] we assume these ‘derivatives’ are bounded (see Section 2 for the exact definitions). However, we do not impose uniform bounds over all admissible strategies. The price to pay for this relaxation is that we are no longer happy with the mere $L^2$-convergence but price contingent claims using super-replication arguments. In contrast with [4], we find that the super-replication price is different than the one suggested by the Black-Scholes formula. The restriction on the trading strategies is of Gamma-constraint type, which is studied in [7] in detail. Our main result, Theorem 3.1, gives the dynamic programming equation associated with the minimal super-replication price and states that the super-replication price is the unique viscosity solution of the dynamic programming equation. The techniques used are the ones developed in a series of papers by Soner and Touzi [22, 23, 24], by Cheridito, Soner and Touzi in [7, 8] and by Cheridito, Soner, Touzi and Victoir in [9].

The rationale for using the super-replication approach instead of other pricing techniques
such as utility based pricing is the following. Firstly, our approach gives a pricing interval, or equivalently a liquidity premium to be paid above the Black-Scholes price. No fair price produced by any approach can charge a premium larger that this premium and the premium depends on the risk profile of the seller. This premium can be calculated using the utility based pricing method of Hodges & Neuberger [17] (also see [14] for its use in the model with proportional transaction costs). Notice that the additional term liquidity premium will not be seen in such an approach, unless the utility of the seller exhibits a penalization for gamma of the process in addition to usual utility from the final wealth.

As we illustrate in Section 4 the optimal hedging strategy exhibits an asymmetry between the claims with convex and concave payoffs. For derivatives with convex payoff the hedging strategy is of dynamic Black-Scholes type. However, when the claim to be hedged has a concave payoff there are two options for the trader: either employ a buy-and-hold strategy at a higher cost of construction but no further liquidity costs, or employ a perfect Black-Scholes type replicating strategy but expect liquidity costs growing over time. Depending on the market conditions it might be cheaper to use the buy-and-hold type hedge than the replicating strategy when the liquidity cost associated with the replicating strategy is expected to be high. In Section 4 we show that this decision should be based on the level of concavity of the value function for the minimal super-replication price and give a precise level below which it is cheaper to use a buy-and-hold strategy.

The outline of the paper is as follows. Section 2 formulates the problem. Section 3 presents the main results. Section 4 describes the formal hedging strategy. Section 5 shows the viscosity property of the dynamic programming equation. Section 6 discusses the terminal condition, Section 7 finds the growth condition for the value function, while Section 8 shows the uniqueness of the solution.

2 Problem formulation

Throughout this paper, we fix a finite time horizon $T > 0$, and we consider a one-dimensional Brownian motion $W = \{W(t) , 0 \leq t \leq T \}$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We denote by $\mathbb{F} = \{\mathcal{F}(t) , 0 \leq t \leq T \}$ the smallest filtration that contains the filtration generated by $W$ and satisfies the usual conditions.

2.1 The financial market

The financial market consists of two assets, and the objective of the investor is to optimally allocate his wealth between these assets in order to hedge some contingent liability.

The first asset is non-risky. Without loss of generality, we normalize its price to unity, which means that this asset is taken as the numéraire.

The risky asset is subject to liquidity cost. Following Çetin, Jarrow and Protter [4], we account for the liquidity cost by modelling the price process of this asset as a function of the exchanged volume. We thus introduce a supply curve

$$S(t, S(t), \nu),$$
where \( \nu \in \mathbb{R} \) indicates the volume of the transaction, the process \( S(t) = S(t, S(t), 0) \) is the marginal price process defined by the stochastic differential equation

\[
\frac{dS(r)}{S(r)} = \mu(r, S(r)) \, dr + \sigma(r, S(r)) \, dW(r)
\] (2.1)

and some given initial condition \( S(0) \), and \( S : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R} \) is a smooth function representing the price per share for some given volume of transaction and the marginal price. In order to ensure that the stochastic differential equation (2.1) has a unique strong condition, we assume that the coefficient functions \( \mu, \sigma : [0, T] \times \mathbb{R}_+ \rightarrow \mathbb{R} \) satisfy the usual local Lipschitz and linear growth conditions.

In order to exclude arbitrage opportunities, we assume the existence of an equivalent martingale measure \( \mathbb{P}^0 \), i.e.

\[
\frac{dS(r)}{S(r)} = \sigma(r, S(r)) \, dW^0(r),
\]

where \( W^0 \) is a standard Brownian motion under \( \mathbb{P}^0 \), so that the process \( S \) is a martingale under \( \mathbb{P}^0 \).

We shall frequently move the time origin from zero to an arbitrary \( t \in [0, T] \), and we will denote by \( \{S_{t,s}(r), r \in [t, T]\} \) the process defined by (2.1) and the initial condition \( S_{t,s}(t) = s \).

### 2.2 Trading strategies

A trading strategy is defined by a pair \((X, Y)\) where \( X(t) \) denotes the wealth in the bank, and \( Y(t) \) is the number of shares held at each time \( t \) in portfolio. For reasons which will be clear later, we restrict the process \( Y \) to be of the form

\[
Y(r) = \sum_{n=0}^{N-1} y^n \mathbf{1}_{[r < \tau_{n+1}]} + \int_t^r \alpha(u) \, du + \int_t^r \Gamma(u) \, dS_{t,s}(u),
\] (2.2)

so that it has finite quadratic variation. Here, \( t = \tau_0 \leq \tau_1 \leq \ldots \) is an increasing sequence of \([t, T]\)-valued \( \mathbb{F}\)-stopping times, the random variable

\[
N := \inf\{n \in \mathbb{N} : \tau_n = T\}
\]

indicates the number of jumps, \( y^n \) is an \( \mathbb{R}^d\)-valued, \( \mathcal{F}(\tau_n)\)-measurable random variable satisfying \( y^n \mathbf{1}_{\{\tau_n = T\}} = 0 \); \( \alpha \) and \( \Gamma \) are two \( \mathbb{F}\)-progressively measurable real processes.

In order to justify the continuous-time dynamics of our state variables, we report the following discrete-time argument from [4]. Let \( t = t_0 < \ldots < t_n = T \) be a partition of the interval \([0, T]\), and set \( \delta \psi(t_i) := \psi(t_i) - \psi(t_{i-1}) \) for any function \( \psi \). By the self-financing condition, it follows that

\[
\delta X(t_i) + \delta Y(t_i) S(t_i, \delta Y(t_i)) = 0, \quad 1 \leq i \leq n.
\]
Summing up these equalities, it follows from direct manipulations that
\[
X(T) + Y(T)S(T) = X(t) + Y(t)S(t) + \sum_{i=1}^{n} \delta Y(t_i)S(t_i, \delta Y(t_i)) - \sum_{i=1}^{n} \delta Y(t_i)S(t_i) - \sum_{i=1}^{n} \delta Y(t_i)S(t_i, \delta Y(t_i)) + \sum_{i=1}^{n} Y(t_{i-1})S(t_i) - \sum_{i=1}^{n} \delta Y(t_i)S(t_i) - S(t_i) .
\]

(2.3)

The continuous-time dynamics of the process
\[
Z := X + YS
\]
are obtained by taking limits in (2.3) as the time step of the partition max\{\(t_i - t_{i-1}\), 1 \(\leq i \leq n\}\) shrinks to zero. The last sum term in (2.3) is the term due to the liquidity cost. Under the smoothness assumption on \(\nu \mapsto S(t, s, \nu)\), it follows that
\[
\sum_{i=1}^{n} \delta Y(t_i)S(t_i, \delta Y(t_i)) - S(t_i) \quad \rightarrow \quad \int_{t}^{T} d[Y, Y]_r^{\ell} + \sum_{n=1}^{N} y^n [S(t_n, S(t_n), y^n) - S(t_n)]
\]
in probability, where
\[
\ell(t, s) := \left[4 \frac{\partial S}{\partial \nu}(t, s, 0)\right]^{-1}.
\]

(2.4)

In view of the form of the continuous-time process \(Y\) in (2.2), this provides
\[
Z(r) = Z(t) + \int_{t}^{r} Y(u) dS(u) - \int_{t}^{r} \frac{1}{4\ell(r, S(r))} \Gamma(r)^2 \sigma(r, S(r))^2 S(r)^2 dr - \sum_{n=0}^{N-1} y^n [S(t_n, S(t_n), y^n) - S(t_n)] \mathbf{1}_{\{r < r_{n+1}\}}.
\]

(2.5)

In the absence of jumps in the portfolio process, the process \(Z\) approaches the classical wealth process in frictionless markets for a large \(\ell\). Therefore, we will refer to \(\ell\) as the liquidity index of the market.

In the absence of liquidity cost, the process \(Z\) represents the total value of the portfolio of the investor. In the present setting, we assume that the investor is not subject to the liquidity cost at the final time \(T\). Then, although the process \(Z\) has no direct financial interpretation, its final value \(Z(T)\) is the total value of the investor portfolio at time \(T\).
2.3 Admissible trading strategies and the hedging problem

The purpose of the investor is to hedge without risk some given contingent claim
\[ G = g(S(T)) \]
for some function \( g : \mathbb{R}_+ \to \mathbb{R} \).

In order to formulate the super-hedging problem in the context of our financial market with liquidity cost, we need to restrict further the trading strategies as in [7].

Throughout this paper, we fix a parameter \( \beta \geq 0 \), and for an \( \mathcal{F} \)-progressively measurable process \( \{ H(r), t \leq r \leq T \} \) taking values in \( \mathbb{R} \), we define
\[
\| H \|_{t,s}^{\beta, \infty} := \left\| \sup_{t \leq r \leq T} \frac{|H(r)|}{1 + |S_{t,s}(r)|^\beta} \right\|_{L^\infty}.
\]

A trading strategy \( Y \) defined by (2.2) is said to be admissible if
\[
\| N \|_{\infty} < \infty, \quad \| Y \|_{t,s}^{\beta, \infty} + \| \alpha \|_{t,s}^{\beta, \infty} + \| \Gamma \|_{t,s}^{\beta, \infty} < \infty,
\]
and the process \( \Gamma \) is of the form
\[
\Gamma(r) = \sum_{n=0}^{N-1} \zeta^n 1_{\{ r < \tau_n + 1 \}} + \int_t^r \gamma^{(1)}(u)du + \int_t^r \gamma^{(2)}(u)dW(u),
\]
where \( \zeta^n \) is an \( \mathcal{F}(\tau_n) \)-measurable random variable satisfying \( \zeta^n 1_{\{ \tau_n = T \}} = 0 \), and \( \gamma^{(1)}, \gamma^{(2)} \) are two \( \mathcal{F} \)-progressively measurable processes valued in \( \mathbb{R} \) with
\[
\left\| \gamma^{(1)} \right\|_{t,s}^{\beta, \infty} + \left\| \gamma^{(2)} \right\|_{t,s}^{\beta, \infty} < \infty.
\]

We refer to [8] for a justification of such restrictions.

The collection of all admissible trading strategies \( Y = \{ Y(r), 0 \leq r \leq T \} \) is denoted by \( \mathcal{A}_{t,s} \). For every \( Y \in \mathcal{A}_{t,s} \), we denote by \( Z^Y_{t,s} \) the process defined by (2.5). The purpose of this paper is to solve the super-hedging problem
\[
V(t,s) := \inf \{ z \in \mathbb{R} : Z^Y_{t,s}(T) \geq g(S_{t,s}(T)) \text{ for some } Y \in \mathcal{A}_{t,s} \}.
\]

3 The main results

Since the function \( V \) is not known to have any regularity property, we shall characterize it by means of the corresponding dynamic programming equation in the viscosity sense. This requires to verify that \( V \) is locally bounded, which is obviously satisfied under the conditions
\[
g \text{ is bounded from below and } \sup_{s > 0} \frac{g(s)}{1 + s} < \infty.
\]

Indeed, the lower bound on \( g \) is immediately inherited by \( V \), and the affine growth condition guarantees the existence of a trivial buy-and hold strategy which super-hedges the contingent claim \( g(S(T)) \), thus producing a locally bounded upper bound for \( g \).
We also assume that

\[ \sigma \text{ is bounded and Lipschitz continuous,} \]
\[ \ell \text{ is locally Lipschitz continuous, and} \]
\[ l_{\delta} := \inf \{ \ell(t,s) : \delta \leq t \leq \delta^{-1}, t \in [0,T] \} > 0 \text{ for every } \delta > 0. \]

**Theorem 3.1** Assume (3.1), (3.2) and that the payoff function \( g \) is continuous. Then, \( V \) is the unique continuous viscosity solution of the dynamic programming equation

\[
\sup_{\beta \geq 0} \left( -V_t - \frac{1}{2} s^2 \sigma^2 (V_{ss} + \beta) - \frac{s^2 \sigma^2}{4\ell} (V_{ss} + \beta)^2 \right) = 0
\]

on \([0,T) \times (0,\infty)\), satisfying the terminal condition \( V(T,.) = g \) and the growth condition

\[-C \leq V(t,s) \leq C(1+s), (t,s) \in [0,T] \times \mathbb{R}_+, \text{ for some constant } C > 0.\]

The proof of this theorem is completed in the subsequent sections and summarized in Section 8.

Observe that the dynamic programming equation (3.3) is parabolic, i.e. non-increasing in \( V_{ss} \), as the differential operator appearing on the left-hand side is the parabolic envelope of the first guess operator

\[-V_t - \frac{1}{2} s^2 \sigma^2 V_{ss} = \frac{s^2 \sigma^2}{4\ell} V_{ss}^2. \]

We refer to [7] for more details on this issue.

Finally, by direct manipulation, we see that the maximizer in the dynamic programming equation (3.3) is given by

\[ \hat{\beta}(t,s) := (V_{ss}(t,s) + \ell(t,s))^-, \]

so that we can re-write the dynamic programming equation into

\[
-V_t - \frac{s^2 \sigma^2}{4\ell} \left[ -\ell^2 + ((V_{ss} + \ell)^+) \right] = 0.
\]

**4 Formal description of an optimal hedging strategy**

**4.1 A Black-Scholes structure condition on the liquidity index**

We now provide a formal description of an optimal hedging strategy for a payoff \( g(S_T) \) under liquidity costs, when the liquidity function satisfies the following structure condition.

**Assumption 4.1** The exists a smooth solution \( \phi \) to the PDE (3.3) with \( \phi_{ss} + \ell = 0. \)

The analysis of this section will be restricted to a formal discussion as we will ignore some admissibility restrictions and regularity conditions.
For concreteness, we examine the above assumption in the context of the classical Black-Scholes model, i.e. $\sigma(t, s) \equiv \sigma$, for some positive constant $\sigma$. Then, it is easily checked that with the supply function

$$S(s, \nu) := se^{\alpha s \nu / 4} \text{ so that } \ell(s) = \frac{1}{\alpha s^2},$$

Assumption 4.1 is satisfied by the function

$$\phi(t, s) := \frac{1}{4\alpha}(\sigma^2 t + 4 \ln s), \quad (t, s) \in [0, T] \times \mathbb{R}^+. \quad (4.1)$$

We also observe that Condition (3.2) is satisfied.

### 4.2 Buy-and-hold versus dynamic hedging

Before turning to the description of a hedging strategy in the context of our financial market with liquidity costs, we would like to discuss the asymmetry between concavity and convexity for the point of view of superhedging. This will turn out to be the driving intuition for our hedging strategy.

The Black-Scholes hedging theory in a complete market says that the optimal superhedging strategy of some contingent claim is in fact a perfect replicating strategy, and consists in the dynamic strategy of holding at each time $r$, the number $\partial V / \partial s(r, S_r)$ of share of the underlying risky asset. In our context, this strategy is more expensive than in the frictionless Black-Scholes model since it induces a non-zero gamma process, implying a penalization on the wealth process.

A buy-and-hold strategy on some time interval $[t, \tau)$ is defined by $Y_r = Y_t$ for every $r \in [t, \tau)$. In particular, $\Gamma = 0$ on $[t, \tau)$, the wealth process is not subject to the liquidity cost penalty, and it is given by the same expression as in the classical frictionless framework:

$$Z_r = Z_t + Y_t (S_r - S_t) \quad \text{for } r \in [t, \tau).$$

For a concave payoff, a trivial static superhedging strategy is available. Indeed, performing the buy-and-hold strategy $Y_t = \partial g / \partial s(S_t)$ on $[t, \tau)$ (for a non-smooth $g$, let $Y_t$ be a measurable selection in the supergradient of $g$ at $S_t$), and starting from the capital $Z_t = g(S_t)$, it follows from the concavity of the payoff function $g$ that

$$Z_\tau = Z_t + Y_t (S_\tau - S_t) \geq g(S_\tau).$$

This discussion shows that, in our context of financial market with liquidity costs, when the super-replication value is concave, there is a trade-off between

- paying a higher cost for a buy-and-hold strategy, thus avoiding the liquidity costs,
- performing the perfect hedging strategy but paying the liquidity costs.

The hedging strategy which will be described in the subsequent subsection provides a precise definition of the level of concavity below which the liquidity cost induced by a perfect hedging strategy is so significant that it is cheaper to use a buy-and-hold strategy. In the next subsection we will describe a formal hedging strategy that resolves this issue. However, in our solution, we will split the price into two parts and replicate one part by
the classical Black-Scholes hedge and hedge the other part by a combination of a buy-and-hold strategy together with a classical hedge. The latter is achieved by optimally separating the space into regions in which one or the other strategy is optimal. So from a buy-and-hold strategy we understand this combination.

4.3 Hedging under liquidity costs

In order to discuss the hedging strategy, we introduce the following open set:

\[ C := \{(t, s) \in [0, T) \times (0, \infty) : V_{ss}(t, s) < -\ell(t, s)\} . \]

Observe that on \( C \), (3.5) reduces to

\[ -V_t + \frac{1}{4}s^2 \sigma^2(t, s)\ell(s) = 0 . \]  (4.2)

Under Assumption 4.1, we have that \( (V - \phi)_{ss} < 0 \) and \( (V - \phi)_t = 0 \) on \( C \). This implies that \( (V - \phi)_{tt} = (V - \phi)_{ts} = 0 \) on \( C \). Hence

\[ (t, s) \mapsto (V - \phi)(t, s) \text{ is concave and } (V - \phi)_t = 0 \text{ on } C . \]  (4.3)

Given an arbitrary initial position \( (t, s) \in C \), we define the exit time

\[ \theta := \inf\{u > t : (u, S_u) \notin C\} . \]

We now consider the initial capital \( V(t, s) \) at time \( t \), together with the hedging strategy \( \{Y_u, t \leq u < \theta\} \) defined by

\[ Y_t := V_s(t, s), \quad \Gamma_u := \phi_{ss}(u, S_u), \quad \text{and} \quad \alpha_u := \mathcal{L}\phi_u(u, S_u) . \]

In words, this hedging strategy consists in dynamically hedging the value function \( \phi \), and performing a buy-and-hold strategy in order to super-hedge the remaining value \( (V - \phi) \).

Then, we directly calculate for \( \tau \in (t, \theta) \) that:

\[
Z_{\tau} = V(t, s) + \int_t^\tau Y_u dS_u - \frac{1}{4} \int_t^\tau \ell^{-1}(u, S_u)\Gamma_u \sigma^2(u, S_u)S_u^2 du
\]

\[ = V(t, s) + V_s(t, s)(S_\tau - s) + \int_t^\tau \left( \int_t^u \mathcal{L}\phi_s(r, S_r) dr + \phi_{ss}(r, S_r) dS_r \right) dS_u
\]

\[ - \frac{1}{4} \int_t^\tau \ell(u, S_u)\sigma^2 S_u^2 du
\]

\[ = (V - \phi)(t, s) + (V - \phi)_s(t, s)[S_\tau - s] + \phi(\tau, S_\tau)
\]

\[ - \int_t^\tau \left( \mathcal{L}\phi(u, S_u) + \frac{1}{4}\ell(u, S_u)\sigma^2 S_u^2 \right) du , \]

where we applied Itô’s lemma twice to the process \( \phi(u, S_u) \). We next observe from Assumption 4.1 that

\[ \mathcal{L}\phi = \phi_t + \frac{1}{2}\sigma^2 s^2 \phi_{ss} = -\frac{1}{4}\ell(t, s)\sigma^2 s^2 \text{ on } C . \]
Together with (4.3), this implies that
\[
Z_{\tau} = (V - \phi)(t, s) + (V - \phi)_{\tau}(t, s)[S_{\tau} - s] + \phi(\tau, S_{\tau}) \\
\geq (V - \phi)(\tau, S_{\tau}) + \phi(\tau, S_{\tau}) = V(\tau, S_{\tau}).
\]

This shows that the above defined strategy is a super-hedging strategy in \( C \). Outside \( C \), one can show by Itô’s lemma that the hedging strategy consists in performing a perfect replicating Black-Scholes strategy on the total value function \( V \), i.e. \( Y_u = V_s(u, S_u) \).

In conclusion, the super-hedging strategy in our financial market with liquidity costs is formally described by applying successively a perfect dynamic replicating Black-Scholes strategy in \( C \), and the above mixed strategy consisting in dynamically hedging \( \phi \) and super-hedging the difference \( (V - \phi) \) by means of a buy-and-hold strategy.

5 Viscosity property

For an admissible trading strategy \( Y \in \mathcal{A}_{t,s} \) defined by \((\tau_n, y^n)_{n \geq 0}, \alpha, \Gamma)\), we define
\[
\|Y\|_{t,s} := \max \left\{ \|N\|_{L^\infty}, \|Y\|_{t_s}^{\beta,\infty}, \|\alpha\|_{t_s}^{\beta,\infty}, \|\Gamma\|_{t_s}^{\beta,\infty}, \|\gamma(1)\|_{t_s}^{\beta,\infty}, \|\gamma(2)\|_{t_s}^{\beta,\infty} \right\}.
\]

By definition of \( \mathcal{A}_{t,s} \), we have
\[
\mathcal{A}_{t,s} = \bigcup_{M>0} \mathcal{A}_{t,s}^M \quad \text{where} \quad \mathcal{A}_{t,s}^M := \{ Y \in \mathcal{A}_{t,s} : \|Y\|_{t,s} \leq M \}.
\]

Throughout this section, we shall use the following classification notation in the theory of viscosity solutions. For a locally bounded function \( u \) from some domain \( D \subset \mathbb{R}^k \to \mathbb{R} \), we introduce the lower and the upper semi-continuous envelopes
\[
u_s(x) := \liminf_{D \ni x' \to x} u(x') \quad \text{and} \quad \nu^*(x) := \limsup_{D \ni x' \to x} u(x').
\]

5.1 Viscosity supersolution property

We follow [7] by passing to a weak formulation of the problem (2.7). For a Brownian motion \( \tilde{W} \) on a probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})\), satisfying the usual conditions, we define the process \( \tilde{S}_{t,s} \) and \( \tilde{A}_{t,s}^M \) on \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})\) exactly as \( S_{t,s} \) and \( A_{t,s}^M \) are defined on \((\Omega, \mathcal{F}, \mathbb{P})\).

For \( M > 0 \), we introduce a weak formulation of the super-hedging problem
\[
tilde{v}^M(t, s) := \inf \left\{ z \in \mathbb{R} : \tilde{Z}_{t,s}(T) \geq g(\tilde{S}_{t,s}(T)) \quad \text{for some} \quad (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{W}) \quad \text{and} \quad \tilde{Y} \in \tilde{A}_{t,s}^M \right\},
\]

(5.1)

together with the lower semi-continuous function
\[
tilde{v}(t, s) := \left( \inf_{M>0} \tilde{v}^M \right)_{S_{t,s}}(t, s) = \liminf_{(t', s') \to (t, s)} \inf_{M>0} \tilde{v}^M(t', s'), \quad (t, s) \in [0, T] \times \mathbb{R}_+.
\]

(5.2)

The purpose of this section is to derive the following viscosity property for \( \tilde{v} \).

**Proposition 5.1** The function \( \tilde{v} \) is a lower semicontinuous viscosity super-solution of the PDE (3.3).
Observe that \( \tilde{v} \leq V \) by definition. In the subsequent Section 5.2, we will prove that a convenient majorant of \( V \) is a subsolution of the PDE (3.3). Then, the comparison result of Section 8, will provide the equality \( \tilde{v} = V \).

The proof of Proposition 5.1 follows the same line of argument than [7]. The first step is the following partial dynamic programming principle, whose proof is the same as Lemma 5.1 in [7].

**Lemma 5.1** Let \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{W})\) be a probability space supporting a Brownian motion. For \((t, s) \in [0, T) \times \mathbb{R}_+\), let \( z \in \mathbb{R} \), \( M > 0 \), and \( \tilde{Y} \in \tilde{\mathcal{A}}^M \) be such that

\[
\tilde{Z}^\tilde{Y}_{t, s, z}(T) \geq g \left( \tilde{S}_{t, s}(T) \right).
\]

Then, for every stopping time \( \theta \) with values in \([t, T]\), we have

\[
\tilde{Z}^\tilde{Y}_{t, s, z}(\theta) \geq \tilde{v}^M \left( \theta, \tilde{S}_{t, s}(\theta) \right).
\]

The following property of \( \tilde{v}^M \) can be proved by the same arguments as Lemma 5.2 in [7].

**Lemma 5.2** For a sufficiently large \( M \), the function \( \tilde{v}^M \) is finite and lower semicontinuous. Moreover existence holds for the problem \( \tilde{v}^M \).

**Proof of Proposition 5.1** By the stability of viscosity solution, it is sufficient to prove that \( \tilde{v}^M \) is a viscosity super-solution of the PDE (3.3) for a sufficiently large \( M > 0 \), see Corollary 5.5 in [7]. Since \( \tilde{v}^M \) is lower semicontinuous by Lemma 5.2, we fix some \((t_0, s_0) \in Q := [0, T) \times \mathbb{R}_+\) and \( \varphi \in C^\infty(Q) \) with

\[
0 = (\tilde{v}^M - \varphi)(t_0, s_0) = \min_{Q} (\tilde{v}^M - \varphi), \tag{5.3}
\]

and we prove that, for some \( \beta \geq 0 \):

\[
-\varphi_t(t_0, s_0) - \frac{1}{2} \sigma^2(t_0, s_0) [\varphi_{ss}(t_0, s_0) + \beta] - \ell(t_0, s_0) \sigma^2(t_0, s_0) [\varphi_{ss}(t_0, s_0) + \beta]^2 \geq 0 \tag{5.4}
\]

1. With \( z_0 := \tilde{v}^M(t_0, s_0) = \varphi(t_0, s_0) \), it follows from Lemma 5.2 that, for some probability space equipped with a Brownian motion \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{W})\), there exists an admissible trading strategy \( \tilde{Y} \in \tilde{\mathcal{A}}^M \) such that

\[
\tilde{Z}^\tilde{Y}_{t_0, s_0, z_0}(T) \geq g \left( \tilde{S}_{t_0, s_0}(T) \right). \tag{5.5}
\]

For ease of notation, we set \((\tilde{S}, \tilde{Z}) = (\tilde{S}_{t_0, s_0}, \tilde{Z}^\tilde{Y}_{t_0, s_0, z_0})\). Let \( \tilde{\tau}_1 \) be the first jump time after \( t_0 \) appearing in the trading strategy \( \tilde{Y} \) and define the family of stopping times

\[
\theta^n := \tilde{\tau}_1 \wedge (t_0 + \eta) \wedge \inf \left\{ r > t_0 : |\ln \tilde{S}(r) - \ln s_0| \geq 1 \right\} \quad \text{for} \quad \eta > 0.
\]

Notice that \( \theta^n > t_0 \) \( \tilde{P} \)-almost surely for every \( \eta > 0 \). By the partial dynamic programming principle of Lemma 5.1,

\[
\tilde{Z}(\theta^n) \geq \tilde{v}^M \left( \theta^n, \tilde{S}(\theta^n) \right). \tag{5.6}
\]
Since \( \tilde{v}^M \geq \varphi \), it follows from (2.5) that
\[
\int_0^{\theta_0^*} Y(r) d\tilde{S}(r) - \int_0^{\theta_0^*} \ell \left( r, \tilde{S}(r) \right) \tilde{\sigma} \left( r, \tilde{S}(r) \right) \Gamma(r) dr - \varphi \left( \theta_0^*, \tilde{S}(\theta_0^*) \right) \geq 0.
\]

By twice applying Itô’s lemma, this provides
\[
\int_{t_0}^\theta f(r) dr + \int_{t_0}^{\theta} \left( c + \int_{t_0}^r a(u) du + \int_{t_0}^r b(u) d\tilde{S}(u) \right) d\tilde{S}(r) \geq 0, \tag{5.7}
\]
where
\[
c := \tilde{y}^0 - D_\varphi(t_0, s_0), \quad f(r) := -\mathcal{L}_\varphi \left( r \land \theta_0^*, \tilde{S}(r \land \theta_0^*) \right) - (\ell \tilde{\sigma})^2 \left( r \land \theta_0^*, \tilde{S}(r \land \theta_0^*) \right) \tilde{\Gamma} \left( r \land \theta_0^* \right)^2, \]
\[
a(r) := \tilde{a}(r \land \theta_0^*) - \mathcal{L}_D (D_\varphi) \left( r \land \theta_0^*, \tilde{S}(r \land \theta_0^*) \right), \quad b(r) := \tilde{\Gamma}(r \land \theta_0^*) - D^2 \varphi \left( r \land \theta_0^*, \tilde{S}(r \land \theta_0^*) \right),
\]
and for a smooth function \( \phi \),
\[
\mathcal{L}_\phi := \phi_t + \frac{1}{2} \sigma^2 \phi_{ss}.
\]

2. Note that, by our choice of the stopping time \( \theta_0^* \), the processes \( f, a \) and \( b \) are bounded. Hence, there exists a constant \( C_1 > 0 \) such that for all \( \eta > 0 \),
\[
\left| \int_{t_0}^\theta f(r) dr \right| \leq C_1 \eta. \tag{5.8}
\]

Moreover, the process \( m(r) := 1_{\{r < \theta_0^*\}} \tilde{\sigma}(r, \tilde{S}(r)) \), \( r \in [t_0, T] \), satisfies the continuity assumption (A.3) of Proposition A.3 of [7] at \( t_0 \) for \( \varepsilon = 0 \). Therefore, it follows from Proposition A.3 that for every constant \( \varepsilon > 0 \), almost surely,
\[
\lim_{\eta \downarrow 0} \eta^{-3/2+\varepsilon} \int_{t_0}^\theta \left( \int_{t_0}^r a(u) du \right) d\tilde{S}(r) = 0. \tag{5.9}
\]

It can easily be checked that \( \tilde{S} \) is almost surely Hölder-continuous of order 1/3. Hence, the process \( m \) satisfies the continuity assumption A.1 of Theorem A.1 of [7] for \( \varepsilon = 2/3 \), and it follows from Theorem A.1 that there exists a constant \( C_2 > 0 \) such that
\[
\limsup_{\eta \downarrow 0} \frac{1}{\eta \log \log \frac{1}{\eta}} \left| \int_{t_0}^\theta \left( \int_{t_0}^r b(u) d\tilde{S}(u) \right) d\tilde{S}(r) \right| = \limsup_{\eta \downarrow 0} \frac{1}{\eta \log \log \frac{1}{\eta}} \left| \int_{t_0}^{r_0+\eta} \left( \int_{t_0}^r b(u)m(u) d\tilde{W}(u) \right) m(r) d\tilde{W}(r) \right| \leq C_2. \tag{5.10}
\]

Hence, it follows from (5.7), (5.8), (5.9), (5.10) and the law of the iterated logarithm for Brownian motion (see e.g. [19]) that
\[
c = \tilde{y}^0 - D_\varphi(t_0, s_0) = 0. \tag{5.11}
\]
3. By (5.11), we can rewrite (5.7) as
\[ \int_{t_0}^{\theta_0} f(r)dr + \int_{t_0}^{\theta_0} \left( \int_{t_0}^{r} a(u)du + \int_{t_0}^{r} b(u)d\tilde{S}(u) \right) d\tilde{S}(r) \geq 0. \quad (5.12) \]

It follows from (5.12), (5.8) and (5.9) that
\[ \liminf_{\eta \searrow 0} \frac{1}{\eta \log \log \frac{1}{\eta}} \int_{t_0}^{\theta_0} \left( \int_{t_0}^{r} b(u)d\tilde{S}(u) \right)^T d\tilde{S}(r) \geq 0. \quad (5.13) \]

Since \( b \) is right-continuous, it follows from (5.13) and Theorem A.1 that
\[ \beta := b(t_0) = \tilde{\Gamma}(t_0) - D^2\varphi(t_0, s_0) \geq 0. \quad (5.14) \]

By the boundedness and continuity of the process \( f \), we obtain from (5.12) and (5.9) that
\[ f(t_0) \geq \limsup_{\eta \searrow 0} \frac{1}{\eta} \int_{t_0}^{\theta_0} \int_{t_0}^{r} -b(u)d\tilde{S}(u)d\tilde{S}(r). \quad (5.15) \]

Since \( b \) is of the form (2.6), it satisfies the continuity assumption A.2 of Theorem A.2 of [7]. Hence, we get from (5.14) and Theorem A.2 that
\[ \limsup_{\eta \searrow 0} \frac{1}{\eta} \int_{t_0}^{\theta_0} \int_{t_0}^{r} -b(u)d\tilde{S}(u)d\tilde{S}(r) = \frac{1}{2} m(t_0)^2 \beta = \frac{1}{2} \tilde{\sigma}(t_0, s_0)^2 \beta. \]

Together with (5.15), this shows that
\[ f(t_0) - \frac{1}{2} \tilde{\sigma}(t_0, s_0)^2 \beta \geq 0, \]
which completes the proof, by the definition of \( f \).

5.2 Viscosity subsolution property

We now define the function
\[ v^M(t, s) := \inf \left\{ z \in \mathbb{R} : Z^Y_{t, s, z}(T) \geq g(S_{t, s}(T)) \mbox{ for some } Y \in \mathcal{A}_{t, s}^M \right\}. \quad (5.16) \]

Recalling that \( \mathcal{A}_{t, s}^M \nearrow \mathcal{A}_{t, s} \) as \( M \to 0 \), we see that
\[ v^M(t, s) \searrow V(t, s) \mbox{ as } M \to 0. \quad (5.17) \]

Since \( (v^M)^* \geq v^M \), the upper semicontinuous function
\[ v(t, s) := \inf_{M > 0} (v^M)^*(t, s), \quad (t, s) \in [0, T] \times [0, \infty)^d \]
\[ (5.18) \]

is a majorant of \( v \), and therefore,
\[ v \geq V^* \quad (5.19) \]

The purpose of this section is the following viscosity property for \( v \).
Proposition 5.2 Assume that $g$ satisfies (3.1). Then the function $v$ is an upper semicontinuous viscosity subsolution of the PDE (3.3).

Since $v \geq V^* \geq V_s \geq \hat{v}$, and $\hat{v}$ (resp. $v$) is a viscosity supersolution (resp. subsolution) of (3.3), it will follow from the comparison result of Section 8 that $v = V^* = V_s = \hat{v}$. In particular, equality holds in (5.19).

The proof of Proposition 5.2 again follows the same line of argument than [7]. We first start by the following partial dynamic programming principle for the family $(v^M)_{M>0}$, whose proof is the same as in [7].

Lemma 5.3 Let $t \in [0, T)$, $s \in (0, \infty)^d$, $z \in \mathbb{R}$ and $\theta$ a $[t, T]$-valued stopping time. Let $M_1, M_2 > 0$ and $Y \in A_1^{M_1}$, such that

$$Z(t,s,z)(\theta) > v^{M_2}(\theta, S(t,s)(\theta)).$$

Then there exists a control $Y \in A_1^{M_1+M_2}$ such that

$$Z(t,s,z)(T) \geq g(S(t,s)(T)).$$

Proof of Proposition 5.2 Let $(t_0, s_0) \in [0, T) \times \mathbb{R}_+$ and $\varphi \in C^\infty([0, T) \times \mathbb{R}_+)$ be such that

$$0 = (v - \varphi)(t_0, s_0) > (v - \varphi)(t, s) \text{ for all } (t, s) \neq (t_0, s_0).$$

Assume that for some $\beta \geq 0$, 

$$h(t_0, s_0) := \left(-\varphi_t - \frac{1}{2}\sigma^2(\varphi_{ss} + \beta) - \ell\ell^2(\varphi_{ss} + \beta)^2\right)(t_0, s_0) > 0,$$

and let us work towards a contradiction.

1. Since the function $h$ is continuous, the set

$$\mathcal{N} := \{(t, s) \in [0, T) \times \mathbb{R}_+ \cap B_1(t_0, s_0) : h(t, s) > 0\}$$

defines an open neighborhood of $(t_0, s_0)$. Here, $B_1(t_0, s_0)$ is the open unit ball in $\mathbb{R}^2$ centered at $(t_0, s_0)$. Choose a constant $M_1 \geq 2$ such that for each fixed pair $(\hat{t}, \hat{s}) \in \mathcal{N}$, all the functions

$$\varphi_s(t, s) + \beta(s - \hat{s}), \varphi_s(t, s), \varphi_{ss}(t, s) + \beta,$$

are bounded by $M_1$ on $\mathcal{N}$. Now recall that $v = \inf_{M>0}(v^M)^*$ and $(t_0, s_0)$ is a strict maximizer of $v - \varphi$. Then, since $\partial\mathcal{N}$ is compact and $(v^M)^* - \varphi$ is upper semicontinuous for all $M$, we deduce that there exists an $\eta > 0$ and an $M_2 > 0$ such that

$$(v^{M_2})^*(t, s) \leq \varphi(t, s) - 4\eta \text{ for all } (t, s) \in \partial\mathcal{N}.$$ 

2. Let $M_3 := M_1 + M_2$. There exists a $(\hat{t}, \hat{s}) \in \mathcal{N}$ such that

$$v^{M_3}(\hat{t}, \hat{s}) \geq (v^{M_3})^*(t_0, s_0) - \eta \geq v(t_0, s_0) - \eta = \varphi(t_0, s_0) - \eta \geq \varphi(\hat{t}, \hat{s}) - 2\eta. \text{ (5.20)}$$

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Let $\hat{S} := S_{\hat{t}, \hat{s}}$, and consider the stopping time

$$\theta := \inf \left\{ t \geq \hat{t} : (t, \hat{S}(t)) \notin N \right\}.$$ 

Then, $\theta > \hat{t}$ and $(\theta, \hat{S}(\theta)) \in \partial N$ because the process $\hat{S}$ is almost surely continuous. Therefore,

$$(v^{M_2})(\theta, \hat{S}(\theta)) \leq \varphi(\theta, \hat{S}(\theta)) - 4\eta. \quad (5.21)$$

We finally consider the initial position

$$\hat{z} := v^{M_3}(\hat{t}, \hat{s}) - \eta, \quad (5.22)$$

together with the trading strategy

$$\hat{Y}(r) := 1_{[\hat{t}, \theta]}(r) \left( \varphi_s(r, \hat{S}_r) + \beta(\hat{S}_r - \hat{s}) \right), \quad r \geq \hat{t}.$$ 

By our choice of $M_1$, the control $\hat{Y}$ is in $A_{\hat{t}, \hat{s}}^{M_1}$.

3. Denote $\hat{Z} := Z_{\hat{t}, \hat{s}, \hat{z}}^{\hat{Y}}$. Then,

$$\hat{Z}(\theta) - v^{M_2}(\theta, \hat{S}(\theta)) = v^{M_3}(\hat{t}, \hat{s}) - v^{M_2}(\theta, \hat{S}(\theta)) - \eta$$

$$+ \int_{\hat{t}}^{\theta} \hat{Y}(r)d\hat{S}(r) - \int_{\hat{t}}^{\theta} (\ell \bar{\sigma}^2(\varphi_{ss} + \beta)^2) (r, \hat{S}(r))dr$$

$$\geq \varphi(\hat{t}, \hat{s}) - v^{M_2}(\theta, \hat{S}(\theta)) - 3\eta$$

$$+ \int_{\hat{t}}^{\theta} \hat{Y}(r)d\hat{S}(r) - \int_{\hat{t}}^{\theta} (\ell \bar{\sigma}^2(\varphi_{ss} + \beta)^2) (r, \hat{S}(r))dr$$

by (5.20). Using (5.21) and Itô’s lemma, we see that

$$\hat{Z}(\theta) - v^{M_2}(\theta, \hat{S}(\theta)) \geq \varphi(\hat{t}, \hat{s}) - \varphi(\theta, \hat{S}(\theta)) + 4\eta - 3\eta$$

$$+ \int_{\hat{t}}^{\theta} \hat{Y}(r)d\hat{S}(r) - \int_{\hat{t}}^{\theta} (\ell \bar{\sigma}^2(\varphi_{ss} + \beta)^2) (r, \hat{S}(r))dr$$

$$= \int_{\hat{t}}^{\theta} h(r, \hat{S}(r))dr + \frac{1}{2} \beta(\hat{S}(\theta) - \hat{s})^2 + \eta.$$ 

Since $\beta \geq 0$, this provides

$$\hat{Z}(\theta) - v^{M_2}(\theta, \hat{S}(\theta)) \geq \int_{\hat{t}}^{\theta} h(r, \hat{S}(r))dr + \eta \geq \eta,$$

by definition of $\theta$ as the first exit time from $\mathcal{N}$. Hence, it follows from Lemma 5.3 that $v^{M_3}(\hat{t}, \hat{s}) \leq \hat{z}$, contradicting (5.22).
6 Terminal condition

We now derive the behavior of the functions $\tilde{v}$ and $v$ on the boundary $\{T\} \times \mathbb{R}_+$.  

**Proposition 6.1** Assume that the payoff function $g$ satisfies \( (3.1) \), and is lower semicontinuous. Then $\tilde{v}(T,.) \geq g(.)$ on $\mathbb{R}_+$. 

**Proof.** For $z > \tilde{v}(T, s)$, there is a sequence $(M_n, t_n, s_n) \rightarrow (\infty, T, s)$ such that $z > \tilde{v}^{M_n}(t_n, s_n)$, and the definition of the relaxed problem \( (5.1) \), we have 

$$Z_{t_n,s_n,z}^\tilde{v}(T) \geq g(S_{t_n,s_n}(T)) \quad \text{for some} \quad (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{W}) \quad \text{and} \quad \tilde{Y} \in \tilde{A}^{M_n}. $$

This implies that $z + \int_t^T \tilde{Y}(u) d S_{t_n,s_n}(u) \geq g(S_{t_n,s_n}(T))$. Let $\mathbb{P}^0$ be the equivalent martingale measure for $S$, i.e., the probability measure under which $S$ is a martingale. Then $z \geq \mathbb{E}^{\mathbb{P}^0}[g(S_{t_n,s_n}(T))]$. Since $g$ is bounded from below and lower semicontinuous, it follows from Itô’s lemma that $z \geq g(s)$, and the required result follows from the arbitrariness of $z > \tilde{v}(T, s)$. 

Since $\tilde{v} \leq V$, by the definition of $\tilde{v}$, the previous proposition shows in fact that $\tilde{v}(T, \cdot) = g(.)$ on $\mathbb{R}_+$. Similarly, in order to show that $v(T, \cdot) = g$, it is sufficient to prove that $v(T, \cdot) \leq g$.

**Proposition 6.2** Assume that the payoff function $g$ is continuous and satisfies \( (3.1) \). Then $v(T, \cdot) \leq g(.)$ on $\mathbb{R}_+$. 

**Proof.** First we claim that for every $\epsilon > 0$ and $s_0$ there exist a smooth approximating function $h^\epsilon$ and a point $s_\epsilon$ satisfying 

$$g^\epsilon \geq g, \quad g^\epsilon(s_0) \leq g(s_0) + \epsilon, \quad g^\epsilon_{ss}(s) = 0, \quad \forall s \geq s_\epsilon. \quad (6.1)$$

Indeed for $\alpha > 0$ set 

$$h^\alpha(s) := g(s_0) + \epsilon/2 + \begin{cases} 
0, & |s - s_0| \leq 1/\alpha, \\
\alpha(s - s_0 - 1/\alpha)^2, & s - s_0 \geq 1/\alpha, \\
\alpha(s - s_0 + 1/\alpha)^2, & s - s_0 \leq -1/\alpha.
\end{cases}$$

In view of \( (3.1) \), for sufficiently large $\alpha$, $h^\alpha \geq g + \epsilon/4$. Fix $\alpha$ so that this inequality satisfied and for sufficiently large $S$ set 

$$h^{\alpha,S}(s) := \begin{cases} 
h^\alpha(s), & s \geq S, \\
h^\alpha(S) + (h^\alpha)'(S)(s - S), & s \geq S.
\end{cases}$$

Again in view of \( (3.1) \), for sufficiently large $S$, $h^{\alpha,S} \geq g + \epsilon/4$. Fix $S$ and $\alpha$ satisfying this inequality. Then $h^{\alpha,S}$ satisfies all the conditions stated in \( (6.1) \) except that it may not be smooth. Also $h^{\alpha,S}(s_0) = g(s_0) + \epsilon/2$. Now, choose a symmetric bump function $\eta$ satisfying $\eta \geq 0$, $\int \eta dx = 1$ and support of $\eta$ is in the unit interval. For $\delta > 0$ set $\eta^{\delta}(x) := \eta(x/\delta)/\delta$ and $h^{S,\alpha,\delta} = h^{S,\alpha} * \eta^{\delta}$ (here we extend $h^{S,\alpha}$ to negative numbers by continuity, and * is the
convolution). Notice that \( h^{S,\alpha} \) is Lipschitz, and therefore \( h^{S,\alpha,\delta} \) converges uniformly to \( h^{S,\alpha} \). Since \( h^{S,\alpha} \geq g + \epsilon/4 \), this implies that for all sufficiently small \( \delta \), \( h^{S,\alpha,\delta} \geq h^{S,\alpha} - \epsilon/4 \geq g \). Hence, for sufficiently small \( \delta \) the function \( g^\epsilon = h^{S,\alpha,\delta} \) satisfies (6.1).

For \( \lambda > 0 \) set \( w^\lambda(t, s) := g^\epsilon(s) + \lambda(T - t) \). Then,

\[
I[w^\lambda](t, s) := -w^\lambda_t - \frac{1}{2}s^2\sigma^2(t, s)w^\lambda_{ss} - \ell(t, s)s^2\sigma^2(t, s)[w^\lambda_{ss}]^2 \geq \lambda - \frac{1}{2}s^2\sigma^2(t, s)g^\epsilon_{ss} - \ell(t, s)s^2\sigma^2(t, s)[g^\epsilon_{ss}]^2 \geq 0,
\]

provided that

\[
\lambda \geq c_\epsilon := \sup_{0 < s \leq s'} \left\{ \frac{1}{2}s^2\sigma^2(t, s)g^\epsilon_{ss} + \ell(t, s)s^2\sigma^2(t, s)[g^\epsilon_{ss}]^2 \right\}.
\]

Fix \( (t, s) \in [0, T] \times \mathbb{R}^+ \). With initial wealth \( Z(t) = w^\lambda(t, s) \) use the portfolio process \( Y(u) := w^\lambda_s(u, S_{t,s}(u)) \) for \( u \in [t, T] \). Since \( w^\lambda \) is smooth, \( Y \) is admissible and \( \Gamma(u) = w^\lambda_s(u, S_{t,s}(u)) \) for \( u \in [t, T] \). Moreover, by the above inequality and the \( Z \)-dynamics (2.5),

\[
Z(T) - g^\epsilon(S_{t,T}(T)) = Z(T) - w^\lambda(T, S_{t,T}(T)) = Z(t) - w^\lambda(t, S_{t,s}(t)) + \int_t^T dZ(u) - d[w^\lambda(u, S_{t,s}(u))]
\]

\[
= \int_t^T I[w^\lambda](u, S_{t,s}(u)) \, du \geq 0.
\]

Hence, \( Z(T) = g^\epsilon(S_{t,T}(T)) \geq g(S_{t,T}(T)) \). Therefore, this portfolio is super-replicating. Moreover since \( w^\lambda_{ss}(s) = g^\epsilon_{ss}(s) = 0 \) for all large \( s \), \( Y \in A^M_{t,s} \) for all sufficiently large \( M \). Hence, by the definition of the minimal super-replication price \( v^M \), we have

\[
v^M(t, s) \leq w^\lambda(t, s), \quad \forall (t, s), \quad \Rightarrow \quad v^M(t, s) \leq g^\epsilon(s) + \lambda(T - t).
\]

Since \( g^\epsilon \) is continuous, \( v(T, s) \leq (v^M)^*(T, s) \leq g^\epsilon(s) \), for all \( \epsilon > 0 \) and \( s \). Using (6.1), we conclude that \( v(T, s_0) \leq g(s_0) \).

\[\square\]

### 7 Growth condition

In this section, we prove that the growth condition (3.1) placed on the pay-off \( g \) implies a similar growth condition on the minimal super-replication prices, \( \tilde{v} \) and \( v \).

**Proposition 7.1** Assume (3.1). Then, there is a constant \( C \) so that both \( \tilde{v} \) and \( v \) are bounded from below and satisfy the following growth condition.

\[
-C \leq v(t, s) \leq C[1 + s], \quad -C \leq \tilde{v}(t, s) \leq C[1 + s], \quad \forall \ t \in [0, T], \ s \geq 0.
\]
**Proof.** Let $-C$ be a lower bound for $g$. Fix any initial point $(t,s)$, and let $Y$ be a super-replicating portfolio. Since the corresponding wealth process $Z^Y$ is a supermartingale, we have the following inequalities

$$Z(t) \geq \mathbb{E}[Z(T) \mid \mathcal{F}_t] \geq \mathbb{E}[g(S_{t,s}(T)) \mid \mathcal{F}_t] \geq -C.$$ 

Hence we have the lower bound.

To prove the upper bound, we proceed as in Proposition 6.2. First observe that $w(t,s) := C[1 + s]$ satisfies

$$I[w](t,s) := -w_t(t,s) - \frac{1}{2} s^2 \sigma^2(t,s) w_{ss}(t,s) - \ell(t,s) s^2 \sigma^2(t,s) w_{ss}^2 = 0.$$ 

Let $Y(u) = C$ and $Z$ be the wealth process with initial wealth $Z(t) = w(t,s)$. Then, as in Proposition 6.2, we conclude that $Z^Y$ is super-replicating. Hence, the functions $\tilde{v}$ and $v$ are both less than $w$.

8 Uniqueness

In the previous sections we proved that $\tilde{v}$ is a super-solution of (3.3) and the terminal data, $v$ is a subsolution of (3.3) and the terminal data, and both $\tilde{v}$ and $v$ satisfy the linear growth condition (7.1). To complete the proof of the main Theorem 3.1, we need to prove a comparison result for (3.3). Due to the quadratic nonlinearity in (3.3), standard results do not directly apply to this equation. Moreover, due to the lack of homogenity in the $s$-variable, the techniques used in [2] do not apply either. However, we use the special structure of the equation coming from the fact that it is one dimensional, and consider the following equivalent equation.

$$-A(t,s) v_t + F(t,s, v_{ss}) = 0,$$  \hspace{1cm} (8.1)

where

$$A(t,s) := \frac{4 \ell(t,s)}{s^2 \sigma^2(t,s)}, \quad F(t,s, v_{ss}) := \left[ \ell^2(t,s) - (v_{ss} + \ell(t,s))^+ \right].$$  \hspace{1cm} (8.2)

**Proposition 8.1** Let Condition (3.2) hold, and let $\tilde{w}$ be a lower semi-continuous supersolution of (8.1) and $w$ be an upper semi-continuous subsolution of (8.1). Further assume that $\tilde{w}$ and $w$ satisfy the growth condition (7.1) and the boundary conditions

$$w(T,s) \leq \tilde{w}(T,s), \quad \forall \ s \geq 0,$$  \hspace{1cm} (8.3)

$$w(t,0) \leq \tilde{w}(t,0), \quad \forall \ t \in [0,T].$$  \hspace{1cm} (8.4)

Then, $w \leq \tilde{w}$ on $[0,T] \times \mathbb{R}^+$.

**Proof.**

1. Set

$$\psi(t,s) := w(t,s) - \tilde{w}(t,s).$$
The goal is to show that $\psi \leq 0$ on $[0, T] \times \mathbb{R}^+$. We suppose to the contrary and assume that there exists $(t_0, s_0) \in [0, T] \times \mathbb{R}^+$ such that $\psi(t_0, s_0) > 0$. Since $\psi \leq 0$ on the parabolic boundary $([T] \times \mathbb{R}^+) \cup ([0, T] \times \{0\})$ and $\psi$ is upper semi-continuous, it is clear that $t_0 < T$ and $s_0 > 0$. Moreover, again by the upper semi-continuity, there exists $\delta > 0$ so that for any compact subset $K \subset [0, T] \times \mathbb{R}^+$ containing $(t_0, s_0)$ we have

$$\sup_K \psi = \sup_{\mathcal{N} \cap K} \psi \quad \text{where} \quad \mathcal{N} := [0, T - 2\delta] \times [2\delta, \delta^{-1}].$$

2. Following the usual trick in the theory of viscosity solutions [10], we construct a strict super-solution to (8.1). In view of the previous step, we only need this property on the domain $\mathcal{N}$.

For $\gamma \geq 1$, we set

$$\eta(t, s) := \left[ s \ln(s) + \gamma \right] (T - t + 1).$$

so that, for $(t, s) \in \mathcal{N}$

$$I[\eta](t, s) := -A(s)\eta(t, s) + F(s, \eta_{ss}(t, s))$$

$$= A(s)[s \ln(s) + \gamma] - \frac{(T - t + 1)^2}{s^2} - \frac{2\ell(s)(T - t + 1)}{s}$$

$$= \frac{1}{s^2} \left( \frac{4\ell(s)}{\sigma^2(t, s)} [s \ln(s) + \gamma] - (T - t + 1)^2 - 2s\ell(s)(T - t + 1) \right)$$

$$= \frac{1}{s^2} \left( \frac{4\ell(s)}{\sigma^2(t, s)} [s \ln(s) + \gamma] - c - cs\ell(s) \right)$$

$$\geq \frac{\ell(s)}{s^2 \sigma^2(t, s)} \left( 2\gamma - \frac{\sigma^2(t, s)}{\ell(s)} c + [4s \ln(s) + 2\gamma - cs^2(t, s)] \right).$$

By our assumption (3.2),

$$\sup_{\mathcal{N}} \left\{ \frac{\sigma^2(t, s)}{\ell(s)} + \sigma^2(t, s) \right\} < \infty.$$ 

Hence, there is $\gamma \geq 1$ so that $c[\eta] := \inf_{\mathcal{N}} I[\eta] > 0$.

Let $C$ be the constant in (7.1). We can choose $\gamma \geq 1$ so that in addition to above inequality, we also have

$$\eta(t, s) \geq C[1 + s] \geq \max\{ w(t, s) ; \tilde{w}(t, s) \}.$$ 

(8.5)

3. For $\mu \in [0, 1]$ set

$$w^\mu := (1 - \mu) \tilde{w} + \mu \eta.$$ 

Let $I[\cdot]$ be defined as in the previous step. Then, by the concavity of $F$, on $\mathcal{N}$,

$$I[w^\mu] \geq (1 - \mu) I[\tilde{w}] + \mu I[\eta] \geq \mu c[\eta].$$

Hence, $w^\mu$ is a strict super-solution of (8.1) on $\mathcal{N}$.

4. Set

$$\psi^\mu(t, s) := w(t, s) - w^\mu(t, s).$$
In step 1, we assumed that \( \psi(t_0, s_0) > 0 \). Hence for \( \mu \) sufficiently small we also have \( \psi^\mu(t_0, s_0) > 0 \). In view of (8.4), (8.3) and (8.5), \( \psi^\mu \leq 0 \) on the parabolic boundary \( \{(T) \times \mathbb{R}^+ \} \cup \{0, T \} \times \{0\} \). Also for all \( \mu > 0 \), the growth of \( \eta \) is faster than linear and by (7.1) and step 1, we conclude that \( \psi^\mu \) attains its maximum at some point \( (t^\mu, s^\mu) \in [0, T - 2\delta] \times [2\delta, (2\delta)^{-1}] \subset \mathcal{N} \):

\[
\psi^\mu(t^\mu, s^\mu) = \sup_{\mathcal{N}} \psi^\mu = \sup_{[0, T] \times [0, \infty)} \psi^\mu.
\]

Fix \( \mu > 0 \) satisfying above.

5. Let \( \mu \) be as above and for \( \alpha > 0 \), consider the auxiliary function

\[
\Phi^{\alpha, \mu}(t, s; \bar{t}, \bar{s}) := w(t, s) - w^\mu(\bar{t}, \bar{s}) - \frac{\alpha}{2} \|t - \bar{t}\|^2 + |s - \bar{s}|^2.
\]

In view of the previous step, for all small \( \mu > 0 \) and sufficiently large \( \alpha \geq 1 \), there is a maximizer \( (t^{\alpha, \mu}, s^{\alpha, \mu}, \bar{t}^{\alpha, \mu}, \bar{s}^{\alpha, \mu}) \) of \( \Phi^{\alpha, \mu} \). Moreover, as \( \alpha \) tends to infinity, \( (t^{\alpha, \mu}, s^{\alpha, \mu}, \bar{t}^{\alpha, \mu}, \bar{s}^{\alpha, \mu}) \) approaches to \( (t^\mu, s^\mu, \bar{t}^\mu, \bar{s}^\mu) \). Since \( (t^\mu, s^\mu) \in [0, T - 2\delta] \times [2\delta, (2\delta)^{-1}] \subset \mathcal{N} \), for all large \( \alpha \), \( (t^{\alpha, \mu}, s^{\alpha, \mu}, \bar{t}^{\alpha, \mu}, \bar{s}^{\alpha, \mu}) \subset \mathcal{N} \times \mathcal{N} \).

Also,

\[
\lim_{\alpha \to \infty} \alpha \|t^{\alpha, \mu} - \bar{t}^{\alpha, \mu}\|^2 + |s^{\alpha, \mu} - \bar{s}^{\alpha, \mu}|^2 = 0, \quad c_\mu := \sup_{\alpha > 1} \|s^{\alpha, \mu}\| + |\bar{s}^{\alpha, \mu}| < \infty. \tag{8.6}
\]

6. By the Crandall-Ishii Lemma (see [10] or Section V.6 in [15]), there are \( a^{\alpha, \mu} \leq b^{\alpha, \mu} \) such that

\[
(q^\alpha, p^\alpha, a^{\alpha, \mu}) \in \mathcal{D}^{(1,2)}w(t^{\alpha, \mu}, s^{\alpha, \mu}), \quad (q^\alpha, p^\alpha, b^{\alpha, \mu}) \in \mathcal{D}^{-(1,2)}w(\bar{t}^{\alpha, \mu}, \bar{s}^{\alpha, \mu}),
\]

\[
q^\alpha := \alpha \|t^{\alpha, \mu} - \bar{t}^{\alpha, \mu}\|, \quad p^\alpha := \alpha \|s^{\alpha, \mu} - \bar{s}^{\alpha, \mu}\|,
\]

and the sets \( \mathcal{D}^{(1,2)}, \mathcal{D}^{-(1,2)} \) are defined in [10, 15]. Formally, \( q^\alpha \) is the generalized time derivative, \( p^\alpha \) is the generalized space derivative and \( a^{\alpha, \mu}, b^{\alpha, \mu} \) are the generalized second derivatives. We now use the viscosity property of \( w \) and \( w^\mu \) to obtain,

\[
-A(t^{\alpha, \mu}, s^{\alpha, \mu})q^\alpha + F(t^{\alpha, \mu}, s^{\alpha, \mu}, a^{\alpha, \mu}) \leq 0, \tag{8.7}
\]

\[
-A(\bar{t}^{\alpha, \mu}, \bar{s}^{\alpha, \mu})q^\alpha + F(\bar{t}^{\alpha, \mu}, \bar{s}^{\alpha, \mu}, b^{\alpha, \mu}) \geq \mu c[\eta]. \tag{8.8}
\]

Moreover, as in page 217 in [15], we can show that \( a^{\alpha, \mu}, b^{\alpha, \mu} \) satisfy \( |a^{\alpha, \mu}| + |b^{\alpha, \mu}| \leq \alpha \) and

\[
\begin{bmatrix}
  a^{\alpha, \mu} & 0 \\
  0 & -b^{\alpha, \mu}
\end{bmatrix}
\leq 3\alpha \begin{bmatrix}
  1 & -1 \\
  -1 & 1
\end{bmatrix}.
\]

Using (8.6), (8.9) and the local Lipschitz property of the coefficients (3.2), we will show in Lemma 8.1 below that there is a constant \( C_\mu \) such that

\[
|A(t^{\alpha, \mu}, s^{\alpha, \mu}) - A(\bar{t}^{\alpha, \mu}, \bar{s}^{\alpha, \mu})| |q^\alpha| \leq C_\mu \alpha \left( |t^{\alpha, \mu} - \bar{t}^{\alpha, \mu}|^2 + |s^{\alpha, \mu} - \bar{s}^{\alpha, \mu}|^2 \right),
\]

\[
-F(t^{\alpha, \mu}, s^{\alpha, \mu}, a^{\alpha, \mu}) + F(\bar{t}^{\alpha, \mu}, \bar{s}^{\alpha, \mu}, b^{\alpha, \mu}) \leq C_\mu \alpha |t^{\alpha, \mu} - \bar{t}^{\alpha, \mu}|^2 + \alpha |s^{\alpha, \mu} - \bar{s}^{\alpha, \mu}|^2 + |t^{\alpha, \mu} - \bar{t}^{\alpha, \mu}| + |s^{\alpha, \mu} - \bar{s}^{\alpha, \mu}|.
\]
7. Subtract (8.7) from (8.8) and then use (8.10). The result is,

\[ \mu c[\eta] \leq C_\mu \alpha \left[ |t^{\alpha,\mu} - \bar{t}^{\alpha,\mu}|^2 + |s^{\alpha,\mu} - s^{\alpha,\mu}|^2 \right]. \]

We let \( \alpha \) tend to infinity and use (8.6). This implies that \( \mu c[\eta] \leq 0 \). However, this contradicts the fact that \( \mu \) and \( c[\eta] \) are strictly positive. Hence, there is no \((t_0, s_0)\) as in step 1. Therefore, \( \psi \leq 0 \) on \([0, T] \times \mathbb{R}^+\). \( \square \)

We complete the above proof by proving the technical estimate (8.10).

**Lemma 8.1** Assume (3.2), then (8.10) holds for all \( \alpha \geq 1 \).

**Proof.**

1. In view of (3.2), the coefficient \( A \) defined by (8.2) is locally Lipschitz on \( \mathcal{N} \). Since by 8.6 \( s^{\alpha,\mu} \) and \( \bar{s}^{\alpha,\mu} \) are uniformly bounded in \( \alpha \), there exists a constant, \( C_\mu \) possibly depending on \( \mu \) so that

\[
|A(t^{\alpha,\mu}, s^{\alpha,\mu}) - A(t^{\bar{\alpha},\mu}, s^{\bar{\alpha},\mu})| |q^\alpha| \leq C_\mu \left[ |t^{\alpha,\mu} - t^{\bar{\alpha},\mu}| + |s^{\alpha,\mu} - s^{\bar{\alpha},\mu}| \right] |q^\alpha| \\
= C_\mu \alpha \left[ |t^{\alpha,\mu} - t^{\bar{\alpha},\mu}|^2 + |s^{\alpha,\mu} - s^{\bar{\alpha},\mu}| \right] |t^{\alpha,\mu} - t^{\bar{\alpha},\mu}| \\
\leq C_\mu \alpha \left[ |t^{\alpha,\mu} - t^{\bar{\alpha},\mu}|^2 + |s^{\alpha,\mu} - s^{\bar{\alpha},\mu}|^2 \right].
\]

2. We continue by proving the second inequality in (8.10). To simplify the presentation, we suppress the superscripts in our notation, i.e. \( s = s^{\alpha,\mu}, a = a^{\alpha,\mu} \) etc. By the definition of \( F \), (8.2),

\[
-F(t^{\alpha,\mu}, s^{\alpha,\mu}, a^{\alpha,\mu}) + F(\bar{t}^{\alpha,\mu}, \bar{s}^{\alpha,\mu}, b^{\alpha,\mu}) \\
= -F(t, s, a) + F(\bar{t}, \bar{s}, b) \\
= \ell^2(\bar{t}, \bar{s}) - \ell^2(t, s) + (a + \ell(t, s))^2 - (b + \ell(\bar{t}, \bar{s}))^2 \\
= C_\mu \left[ (|t^{\alpha,\mu} - t^{\bar{\alpha},\mu}|^2 s^{\alpha,\mu} - s^{\bar{\alpha},\mu}) + (a + \ell(t, s))^2 - (b + \ell(\bar{t}, \bar{s}))^2. \right.
\]

If \( (a + \ell(t, s))^2 - (b + \ell(\bar{t}, \bar{s}))^2 \leq 0 \), then the proof of the required estimate is complete. We then continue to prove the estimate in the case

\[ (a + \ell(t, s))^2 - (b + \ell(\bar{t}, \bar{s}))^2 > 0. \] (8.12)

Since the matrix inequality (8.9) implies that \( a \leq b \), it follows from the increase of the function \( z \mapsto z^+ \) that

\[
(a + \ell(t, s))^2 - (b + \ell(\bar{t}, \bar{s}))^2 \\
= \left( (a + \ell(t, s))^+ - (b + \ell(\bar{t}, \bar{s}))^+ \right) \left( (a + \ell(t, s))^+ + (b + \ell(\bar{t}, \bar{s}))^+ \right) \\
\leq \left( (a + \ell(t, s))^+ - (b + \ell(\bar{t}, \bar{s}))^+ \right) \left( (b + \ell(t, s))^+ + (b + \ell(\bar{t}, \bar{s}))^+ \right) \\
\leq ((a - b) + C_\mu (|s - \bar{s}| + |t - \bar{t}|)) (|b| + C_\mu).
\]

3. We will now use again the restriction (8.9) to estimate the right hand side of the final inequality in step 2. We already know that (8.9) implies that \( a \leq b \), but it is stronger than
that. Indeed, multiply (8.9) by a general two vector \((x, y)\) both from right and left. The result is,

\[ ax^2 - by^2 \leq 3\alpha(x - y)^2, \quad \forall \ x, y \in \mathbb{R}^1. \]

By choosing \(x = y\), we obtain \(a \leq b\). But this choice may not be optimal and by calculus we conclude that

\[ a - b \leq -\frac{b^2}{3\alpha + b}. \]

4. Observe that the above estimates implies that \(\frac{b^2}{3\alpha + b} \geq C_\mu(|t - \bar{\ell}| + |s - \bar{s}|)\) contradicts (8.12). Hence \(\frac{b^2}{3\alpha + b} \leq C_\mu(|t - \bar{\ell}| + |s - \bar{s}|)\). Since \(|b| \leq \alpha\), this implies that

\[ b^2 \leq C_\mu \alpha \left(|t - \bar{\ell}| + |s - \bar{s}|\right). \]

with a possibly different constant denoted by \(C_\mu\) again. We substitute this into the estimate of step 2. The result is

\[
(a + \ell(t, s))^{\pm 2} - (b + \ell(\bar{\ell}, \bar{s}))^{\pm 2} \\
\leq C_\mu(|t - \bar{\ell}| + |s - \bar{s}|)(|b| + C_\mu) \\
\leq C_\mu(|t - \bar{\ell}| + |s - \bar{s}|) + C_\mu b \left(|t - \bar{\ell}| + |s - \bar{s}|\right) \\
\leq C_\mu(|t - \bar{\ell}| + |s - \bar{s}|) + C_\mu \left(|b|^2 + 1\right) \left(|t - \bar{\ell}| + |s - \bar{s}|\right) \\
\leq C_\mu(|t - \bar{\ell}| + |s - \bar{s}|) + C_\mu \left(C_\mu \alpha \left(|t - \bar{\ell}| + |s - \bar{s}|\right) + 1\right) \left(|t - \bar{\ell}| + |s - \bar{s}|\right) \\
\leq C_\mu \left(|t - \bar{\ell}| + |s - \bar{s}| + \alpha |s - \bar{s}|^2 + \alpha |t - \bar{\ell}|^2\right).
\]

\[ \Box \]

**Proof of Theorem 3.1** We showed in section 5.1 that \(\tilde{v}\) is a supersolution of (3.3) and in section 5.2 that \(v\) is a subsolution of (3.3). Therefore, due to the equivalent version (3.5), \(\tilde{v}\) is a supersolution and \(v\) is a subsolution of (8.1). Further in section 6 we proved the terminal condition (8.3) and in section 7 that both satisfy the growth condition (7.1). To prove the boundary inequality (8.4), we observe that starting from any \((t, 0)\) only the trivial portfolio \(Y \equiv 0\), can be admissible. Using this and the techniques used in section 6, we can prove (8.4). Then, the comparison result Proposition 8.1 implies that \(v \leq \tilde{v}\). However, by construction we have \(\tilde{v} \leq V \leq V^* \leq v\). Therefore, all are equal and in particular, \(V\) is continuous and is the unique viscosity solution of (8.1) and (3.3). \[ \Box \]

**References**


