AN INTRODUCTION TO MARKOV PROCESSES AND THEIR APPLICATIONS IN MATHEMATICAL ECONOMICS

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1. Markov Property

2. Brief review of martingale theory

3. Feller Processes

4. Infinitesimal generators

In the last sections we have seen how to construct a Markov process starting from a transition function. However, there are not many explicitly known transition functions and this would have limited the usefulness of the theory as we wouldn't be able to construct a large family of examples that can be used in applications. Fortunately, there is a way out. In physical or economic applications, the models are not always constructed using the probability distribution of the process but rather how the process moves from point to another. For Feller processes, this will be grasped by the so called *infinitesimal generator*s.

Definition 4.1. *Let (P_t) be Fellerian and define the operator $A : \mathbb{C} \mapsto \mathbb{C}$ by*

$$Af = \lim_{t\to\infty} \frac{1}{t}(P_t f - f).$$

The domain of A is denoted with $\mathscr{D}(A)$ and it contains the functions $f \in \mathbb{C}$ for which the above limit exists and belongs to $\mathbb{C}$. The operator A as defined is said to be the infinitesimal generator *of (P_t).*

By the very definition of the Markov property, if $f \in \mathbb{C}$

$$E[f(X_{t+h}) - f(X_t)|\mathcal{F}_t] = P_h f(X_t) - f(X_t).$$

Thus, if $f \in \mathscr{D}(A)$ we may write

$$E[f(X_{t+h}) - f(X_t)|\mathcal{F}_t] = hAf(X_t) + o(h).$$

Proposition 4.1. *Let (P_t) be Fellerian and A its generator.*

(1) *If $f \in \mathbb{C}$ then $\int_0^t P_s f\,ds \in \mathscr{D}(A)$ and*

$$P_t f - f = A\int_0^t P_s f\,ds.$$

(2) *If $f \in \mathscr{D}(A)$ and $t \geq 0$, then $P_t f \in \mathscr{D}(A)$ and*

$$\frac{d}{dt}P_t f = AP_t f = P_t Af.$$

(3) *If $f \in \mathscr{D}(A)$ and $t \geq 0$, then*

$$P_t f - f = \int_0^t AP_s f\,ds = \int_0^t P_s Af\,ds.$$

Proof. (1) Observe that

$$\begin{aligned} \frac{1}{h}(P_h - I)\int_0^t P_s f\,ds &= \frac{1}{h}\int_0^t (P_{s+h}f - P_s f)\,ds \\ &= \frac{1}{h}\left(\int_h^{t+h} P_s f\,ds - \int_0^t P_s f\,ds\right) \\ &= \frac{1}{h}\left(\int_t^{t+h} P_s f\,ds - \int_0^h P_s f\,ds\right) \end{aligned}$$

Since $s \mapsto P_s$ is continuous, we have that the above converges to $P_t f - f \in \mathbb{C}$ as $h \to 0$. This proves that $\int_0^t P_s f\,ds \in \mathscr{D}(A)$ and that $A\int_0^t P_s f\,ds = P_t f - f$.

(2) $P_t f \in \mathscr{D}(A)$ can be shown as above. In particular

$$AP_t f = \lim_{h\downarrow 0} \frac{P_{t+h}f - P_t f}{h} = \lim_{h\downarrow 0} P_t\left(\frac{P_h f - f}{h}\right) = P_t A f,$$

by Theorem 3.1. This shows that $t \mapsto P_t f$ has a right derivative which is equal to $P_t A f$. Moreover, the above also implies that $AP_t f = P_t A f$. In order to find the left derivative, consider

$$\begin{aligned} \lim_{h\downarrow 0} \frac{P_{t-h}f - P_t f}{-h} &= \lim_{h\downarrow 0} \frac{P_t f - P_{t-h}f}{h} \\ &= \lim_{h\downarrow 0} P_{t-h}\frac{P_h f - f}{h} = P_t A f \end{aligned}$$

by, again, Theorem 3.1.

(3) This is left as an exercise.

□

Corollary 4.1. *If A is the infinitesimal generator of a Feller semigroup (P_t), then $\mathscr{D}(A)$ is dense in $\mathbb{C}$ and A is a closed operator.*

Proof. Since

$$f = \lim_{t\downarrow 0} \frac{\int_0^t P_s f\,ds}{t},$$

and $\int_0^t P_s f\,ds \in \mathscr{D}(A)$ by the previous proposition, we have that $\mathscr{D}(A)$ is dense in $\mathbb{C}$. To show that A is closed let $(f_n) \subset \mathscr{D}(A)$ and $f_n \to f$, $Af_n \to g$ in $\mathbb{C}$. However, $P_t f_n - f_n = \int_0^t P_s A f_n\,ds$ implies that

$$P_t f - f = \int_0^t P_s g\,ds$$

by letting n tend to ∞. Dividing both sides of above by t and letting $t \downarrow 0$, we obtain $Af = g$. □

We will now connect A with the $(U^\alpha)_{\alpha>0}$, the resolvent of the semigroup (P_t) as defined in Chapter 2. We recall here that for any bounded linear operator A on $\mathbb{C}$, the resolvent set $\rho(A)$ of A consists of $\alpha \in \mathbb{R}$ such that the operator $\alpha - A$ is one-to-one, onto and $(\alpha - A)^{-1}$ is a bounded operator on $\mathbb{C}$.

Proposition 4.2. *Let (P_t) be Fellerian with generator A. Then, $(0,\infty) \subset \rho(A)$ and for any $\alpha > 0$ and $f \in \mathbb{C}$:*

$$(\alpha - A)^{-1} f = U^\alpha f \left(= \int_0^\infty e^{-\alpha t} P_t f \, dt \right).$$

Proof. Let $\alpha > 0$. We have seen in the last chapter that U^α is a bounded operator with norm $\|U^\alpha\| = \frac{1}{\alpha}$. For any $f \in \mathbb{C}$

$$\begin{aligned}
\frac{1}{h}(P_h - I)U^\alpha f &= \frac{1}{h}\int_0^\infty e^{-\alpha t}(P_{t+h}f - P_t f)\, dt \\
&= \frac{1}{h}\left(e^{\alpha h}\int_h^\infty e^{-\alpha t} P_t f\, dt - \int_0^\infty e^{-\alpha t} P_t f\, dt \right) \\
&= \frac{e^{\alpha h} - 1}{h}\int_0^\infty e^{-\alpha t} P_t f\, dt - \frac{e^{\alpha h}}{h}\int_0^h e^{-\alpha t} P_t f\, dt.
\end{aligned}$$

Thus, by letting $h \downarrow 0$, we see that $U^\alpha f \in \mathscr{D}(A)$ and $AU^\alpha f = \alpha U^\alpha f - f$. That is,

$$(\alpha - A)U^\alpha f = f, \qquad f \in \mathbb{C}. \tag{4.1}$$

This shows in particular that $\alpha - A$ is onto. In addition, if $f \in \mathscr{D}(A)$, then we have

$$\begin{aligned}
U^\alpha A f &= \int_0^\infty e^{-\alpha t} P_t A f\, dt = \int_0^\infty A e^{-\alpha t} P_t f\, dt \\
&= A\int_0^\infty e^{-\alpha t} P_t f\, dt = AU^\alpha f.
\end{aligned}$$

So for $f \in \mathscr{D}(A)$

$$U^\alpha(\alpha - A)f = \alpha U^\alpha f - AU^\alpha f = \alpha U^\alpha f + f - \alpha U^\alpha f = f.$$

This shows that $\alpha - A$ is one-to-one. Thus, its inverse exists and is given by, via (4.1),

$$(\alpha - A)^{-1} = U^\alpha.$$

Since α was arbitrary, we have that $(0,\infty) \subset \rho(A)$. □

Proposition 4.3. *The infinitesimal generator of a Fellerian semigroup has the following positive maximum principle: if $f \in \mathscr{D}(A)$ and if x_0 is such that $0 \le f(x_0) = \sup\{f(x) : x \in \mathbf{E}\}$, then*

$$Af(x_0) \le 0.$$

Proof. We have $Af(x_0) = \lim_{t\downarrow 0} \frac{P_t f(x_0) - f(x_0)}{t}$. Since

$$P_t f(x_0) - f(x_0) \le f(x_0)(P_t(x_0, E) - 1) \le 0,$$

we have the result. □

A natural question at this point is the following: When can we say a linear operator A on $\mathbb{C}$ is the infinitesimal generator of a Fellerian semigroup? The celebrated *Hille-Yosida Theorem* gives the answer. The proof of this theorem can be found in Ethier and Kurtz[**?**].

Theorem 4.1. *A linear operator A on $\mathbb{C}$ is closable and its closure is the infinitesimal generator of a Fellerian semigroup if and only if*

a) *$\mathscr{D}(A)$ is dense in $\mathbb{C}$.*
b) *A satisfies the positive maximum principle.*

c) *For some $\alpha > 0$, $\alpha - A$ is dense in $\mathbb{C}$.*

The significance of the generators is also manifested in the following theorem where $(X_t, \mathcal{F}_t)$ is a Feller process with transition function (P_t).

Theorem 4.2. *If $f \in \mathscr{D}(A)$ then the process*

$$M_t^f = f(X_t) - f(X_0) - \int_0^t Af(X_s)\,ds$$

is a $(P^x, \mathcal{F}_t^0)$-martingale for any $x \in \mathbf{E}$.

Proof. Since f and Af are bounded, M^f is integrable. Moreover,

$$\begin{aligned} E^x\left[M_t^f \middle| \mathcal{F}_s^0\right] &= M_s^f + E^x\left[f(X_t) - f(X_s) - \int_s^t Af(X_r)\,dr \middle| \mathcal{F}_s^0\right] \\ &= M_s^f + E^{X_s}\left[f(X_{t-s}) - f(X_0) - \int_0^{t-s} Af(X_r)\,dr\right] \\ &= M_s^f + P_{t-s}f(X_s) - f(X_s) - \int_0^{t-s} P_r Af(X_s)\,dr \\ &= M_s^f \end{aligned}$$

by Proposition 4.1. □

Conversely, we have the

Theorem 4.3. *If $f \in \mathbb{C}$ and there exists a function $g \in \mathbb{C}$ such that*

$$f(X_t) - f(X_0) - \int_0^t g(X_s)\,ds$$

is a $(P^x, \mathcal{F}_t^0)$-martingale for every $x \in \mathbf{E}$, then $f \in \mathscr{D}(A)$ and $Af = g$.

Proof. By taking expectation with respect to P^x we have

$$P_t f(x) - f(x) - \int_0^t P_s g(x)\,ds = 0.$$

Thus,

$$\left\|\frac{P_t f - f}{t} - g\right\| = \left\|\frac{1}{t}\int_0^t (P_s g - g)\,ds\right\| \le \frac{1}{t}\int_0^t \|P_s g - g\|\,ds,$$

which goes to 0 as $t \to 0$. □

Given a t.f. there are actually a few cases when A and $\mathscr{D}(A)$ are completely known. In general, one has to be content with the subspaces of $\mathscr{D}(A)$.

Exercise 4.1. *Let $\mathbb{C}^2$ be the subspace of twice continuously differentiable functions in $\mathbb{C}$ whose first and second derivatives also belong to $\mathbb{C}$. Let X be the linear Brownian motion. Show that if $f \in \mathbb{C}$, then for any $\alpha > 0$, $U^\alpha f \in \mathbb{C}^2$ and $\alpha U^\alpha f - f = \frac{1}{2}(U^\alpha f)''$.*

Proposition 4.4. *Let X be the linear Brownian motion. Then, $\mathscr{D}(A) = \mathbb{C}^2$ and $Af = \frac{1}{2}f''$.*

Proof. From Proposition 4.2 we know that $\mathscr{D}(A) = U^\alpha(\mathbb{C})$ for any $\alpha > 0$, and that $AU^\alpha f = \alpha U^\alpha f - f$. In view of the exercise above, we have that if $f \in \mathbb{C}$, $U^\alpha f \in \mathbb{C}^2$ and $AU^\alpha f = \frac{1}{2}(U^\alpha f)''$. This shows that $\mathscr{D}(A) \subset \mathbb{C}^2$.

Conversely, if $g \in \mathbb{C}^2$, define a function f by

$$f = \alpha g - \frac{1}{2} g''.$$

Observe that $g - U^\alpha f \in \mathbb{C}$ and satisfies the ODE $y'' - 2\alpha y = 0$, whose only solution in $\mathbb{C}$ is 0. Thus, $g = U^\alpha f$, i.e. $g \in \mathscr{D}(A)$. This completes the proof of that $\mathscr{D}(A) = \mathbb{C}^2$ and $Ag = \frac{1}{2}g''$ for any $g \in \mathscr{D}(A)$ since one can always find an $f \in \mathbb{C}$ such that $g = U^\alpha f$. □

Remark 1. *For Brownian motion in higher dimensions, it can still be shown that* $Af = \frac{1}{2}\Delta f$ *where* Δ *is the Laplacian and* $f \in \mathbb{C}^2$*. However, we do not have that* $\mathscr{D}(A) = \mathbb{C}^2$*.*

Recall that in view of Proposition 4.1 we have that

$$\frac{d}{dt} P_t f = \frac{1}{2}(P_t f)'',$$

when P is the t.f. for the linear Brownian motion. The above implies for any $f \in \mathbb{C}$

$$g(t,x) = \int_{\mathbb{R}} p(t,x,y) f(y)\, dy,$$

where p is the transition density of the linear Brownian motion, solves the heat equation

$$u_t = \frac{1}{2} u_{xx}.$$

However, this is just a restatement of the fact that $p(t,x,y)$ is the fundamental solution of the heat eqaution.

Exercise 4.2. *Let* X *be a Feller process with t.f.* (P_t) *and its generator* A*, and* c *a positive Borel function.*

1. *Prove that one can define a homogeneous t.f.* Q_t *by setting*

$$Q_t(x, A) = E^x\left[\mathbf{1}_A(X_t) \exp\left(-\int_0^t c(X_S)\, ds\right)\right].$$

   This t.f. corresponds to the killing of the trajectories of X *at the rate* $c(X)$*.*

2. *If* $f \in \mathscr{D}(A)$ *and* c *is continuous, prove that*

$$\lim_{t \downarrow 0} \frac{Q_t f - f}{t} = Af - cf,$$

   i.e. the generator of (Q_t) *is given by* $A - c$*.*

Exercise 4.3. *Let* A *be an operator on* $\mathbb{C}(\mathbb{R})$ *defined by*

$$Af(x) = \frac{1}{2} a(x) f''(x) + b(x) f'(x)$$

for some continuous functions a *and* b *such that* $a \geq 0$*. Show that there exists a Feller process with this generator.*

Recall that we have seen in Theorem 4 that if $f \in \mathcal{D}(A)$ then $f(X_t) - \int_0^t Af(X_s)\, ds$ is a martingale. However, this property is valid for a more general class of functions that are not necessarily continuous, or more enerally not in the domain of A. The following definition is relevant in this respect:

Definition 4.2. *If X is a Markov process, then a Borel measurable function f is said to belong to the domain $\mathbb{D}_A$ of the extended infinitesimal generator if there exists a Borel measurable function g such that a.s. $\int_0^t |g(X_s)|\,ds < \infty$, for every t, and*

$$\left(f(X_t) - f(x) - \int_0^t g(X_s)\,ds\right)_{t\geq 0}$$

is a $(P^x, \mathcal{F}_t^0)$-martingale for any $x \in \mathbf{E}$.

5. Diffusion processes

In this section we will restrict our attention to a special class of real valued Markov processes, which we will call diffusions, and see their first connection to the solutions of stochastic differential equations. We assume that $\mathbf{E} = \mathbb{R}^d$, a is matrix field on $\mathbb{R}^d$ and b is a vector field on $\mathbb{R}^d$ such that

i) the maps $x \mapsto a(x)$ and $x \mapsto b(x)$ are Borel measurable and locally bounded,
ii) for each x the matrix $a(x)$ is symmetric and nonnegative, i.e. for every $\lambda \in \mathbb{R}^d$

$$\sum_{i,j} a_{ij}\lambda_i\lambda_j \geq 0.$$

With such a pair (a, b) we associate the linear operator

$$L = \frac{1}{2}\sum_{i,j=1}^d a_{ij}(\cdot)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(\cdot)\frac{\partial}{\partial x_i}.$$

Definition 5.1. *A Markov process X is said to be a diffusion process with generator L if*

i) *it has continuous paths,*
ii) *for any $x \in \mathbb{R}^d$ and any $f \in C_K^\infty(\mathbb{R}^d)$,*

$$E^x[f(X_t)] = f(x) + E^x\left[\int_0^t Lf(X_s)\,ds\right].$$

In above $C_K^\infty(\mathbb{R}^d)$ is the class of infinitely differentiable functions on $\mathbb{R}^d$ with a compact support. In the sequel we will shortly write C_K^∞. In this case we further say that X has the diffusion coefficient a and drift b.

We emphasize here that the assumption of continuity of paths in particular implies that the process has infinite lifetime, i.e. $\zeta = \inf\{t > 0 : X_t = \Delta\} = \infty$.

Remark 2. *Note that we could also define time-inhomogeneous diffusions by letting*

$$L_s = \frac{1}{2}\sum_{i,j=1}^d a_{ij}(s,\cdot)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(s,\cdot)\frac{\partial}{\partial x_i},$$

for time dependent a and b. In this case the we should expect the diffusion process X to satisfy

$$E_s^x[f(X_u)] = f(x) + E_s^x\left[\int_s^u L_t f(X_t)\,dt\right],$$

for any s and $f \in C_K^\infty$ where E_s^x is the expectation with respect to the probability measure P_s^x defined by the inhomogeneous transition function, $P_{s,t}$ of X. However, in the sequel we will mostly restrict our attention to homogeneous diffusions.

For any $f \in C_K^2$ we define:

$$M_t^f := f(X_t) - f(X_0) - \int_0^t Lf(X_s)\, ds.$$

The process M^f is continuous and locally bounded.

Proposition 5.1. *The condition ii) in Definition 5.1 is equivalent to each of the following:*

iii) *For any $f \in C_K^\infty$ M^f is a martingale for any P^x;*
iv) *For any $f \in C^2$, M^f is a local martingale for any P^x.*

Proof. This is left as an exercise. □

In view of the above proposition we see that any function in C_K^2 is in the domain of the extended generator of X. Moroever, if X is Feller, $C_K^2 \subset \mathcal{D}(A)$.

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