Problem Set #1
ST441

1. Suppose that $(X_t, \mathcal{F}_t)$ is a Brownian motion and set $S_t := \sup_{s \le t} X_s$. Show that $((X_t, S_t), \mathcal{F}_t)$ is a Markov process and determine its transition probabilities.

2. Suppose that $(X_t, \mathcal{F}_t)$ is a Brownian motion, f a non-negative, bounded, Borel measurable function, and $A_t = \int_0^t f(X_s)\, ds$. Show that $((X_t, A_t), \mathcal{F}_t)$ is a Markov process.

3. Suppose that $(N_t, \mathcal{F}_t)$ is a Poisson process with parameter λ. Let $x \in \mathbb{R}$ and define $X_t = x + N_t$. Show that $(X_t, \mathcal{F}_t)$ is a Markov process and determine its transition probabilities.

4. Suppose m is a measure on the Borel subsets $\mathcal{E}$ of a metric space E. Suppose for each $t > 0$, there exist jointly measurable non-negative functions $p_t : E \times E \mapsto \mathbb{R}$ such that $\int p_t(x, y) m(dy) = 1$ for each x and t and define

$$P_t(x, A) = \int_A p_t(x, y) m(dy), \qquad A \in \mathcal{E}.$$

Show that the kernels P_t satisfy the Chapman-Kolmogorov equations if and only if

$$\int p_s(x, y) p_t(y, z) m(dy) = p_{s+t}(x, z).$$

for every $s, t \ge 0$, every $x \in E$, and m-almost surely.

5. The Ornstein-Uhlenbeck process X started at x is a continuous Gaussian process with $E^x X_t = e^{-t/2} x$ and covariance

$$\mathrm{Cov}(X_s, X_t) = e^{-(s+t)/2}(e^{s \wedge t} - 1).$$

Show that X is a Markov process in its own filtration and determine its transition probabilities.