Problem Set #2
ST441

1. Suppose that $(X_t, \mathcal{F}_t)$ is a Markov process and $\mathcal{G}_t \subset \mathcal{F}_t$ be a subfiltration. If X is adapted to $(\mathcal{G}_t)_{t \in \mathbf{T}}$, show that $(X_t, \mathcal{G}_t)$ is a Markov process, too.

2. Suppose that $(X_t, \mathcal{F}_t)$ is a Markov process with transition function P_t. Let $t_0 > 0$ and define $M_t := P_{t_0 - t} f(X_t)$ for $t \leq t_0$. Show that M is a P^x-martingale for every x.

3. Suppose that $(X_t, \mathcal{F}_t)$ is a right-continuous Markov process with values in $\mathbf{E}$ and let $x \in \mathbf{E}$ and $\sigma_x = \inf\{t \geq 0 : X_t \neq x\}$. Show that there exists a constant $a \geq 0$, depending on x, such that

$$P^x(\sigma_x > t) = e^{-at}.$$

(Remark: In above, if $a \in (0, \infty)$, note that σ_x becomes an exponential random variable. If this is the case x is called a *holding point*. What happens if $a = 0$? If $a = \infty$?)

4. Suppose that X is a right-continuous Markov process, e_p and e_q two independent exponential random variables with parameters p and q, independent of X. Prove that for a positive Borel function f

$$pU^p f(x) = E^x\left[f\left(X_{e_p}\right)\right], \qquad pqU^pU^q f(x) = E^x\left[f\left(X_{e_p + e_q}\right)\right]$$

and derive therefrom *the resolvent equation*, i.e. the equation given in Part 2 of Exerice 3.1 of the Lecture Notes on Markov and Feller process.

5. Show that Poisson process and Brownian motion are Feller processes.

6. Give an example of a Markov process that is not a strong Markov process. (Hint: Let the state space be $[0, \infty)$ and let X move deterministically at a constant speed to the right if it starts at $x > 0$; and if it starts at 0, X first waits for an exponential amount of time before it moves deterministically to the right at constant speed.)

7. A *convolution semi-group* is a family $(\mu_t,\, t \geq 0)$ of probability measures on $\mathbb{R}^d$ such that

   i) $\mu_t * \mu_s = \mu_{t+s}$ for any pair (s,t), where $*$ is the convolution operator;

   ii) $\mu_0 = \varepsilon_0$, where ε_0 is the point mass at 0, and μ_t converges weakly to ε_0 as $t \to 0$.

   (Can you think of any examples of such semigroups?) Show that if we set

   $$P_t(x,A) = \int_{\mathbb{R}^d} \mathbf{1}_A(x+y)\mu_t(dy),$$

   for any Borel set A, then (P_t) is a Feller transition function. Also show that if X is a Feller process with this transition function, then X has stationary independent increments. Conversely, if X is a Feller process with stationary independent increments, its transition function is given by a convolution semigroup. Such process are called *Lévy processes.*

8. Suppose that $(X_t, \mathcal{F}_t)$ is a Brownian motion and set $T_a := \inf\{t \geq 0 : X_t = a\}$. Then, the process $(T_a)_{a \geq 0}$ has stationary and independent increments.

9. Suppose that $(X_t, \mathcal{F}_t)$ is a strong Markov process with values in $\mathbf{E}$ and $A \in \mathscr{E}$. Let $T_A = \inf\{t \geq 0 : X_t \in A\}$. Show that $P^x(T_A = 0)$ is either 0 or 1 for each $x \in \mathbf{E}$.

10. Retaining the situation and notation of Problem 3 suppose further that X is strong Markov and $a \in (0,\infty)$. Show that $P^x(X_{\sigma_x} = x) = 0$, i.e. X can leave a holding point only by a jump.