Problem Set #3
ST441

This set of problems is devoted to the solution of *optimal stopping problem*. Recall the definitions of α-excessive functions and α-potential of a Feller transition function. We call f *excessive* if it is 0-excessive. 0-potential operator will be denoted with U. We assume that X is a càdlàg Feller process with a transition kernel $(P_t)_{t\geq 0}$ and assume that $(\mathcal{F}_t^0)$ is augmented with the null sets for each $t \geq 0$.

Definition 0.1 *Given a function g, the function G is an excessive majorant for g if G is excessive and $G \geq g$ pointwise. G is the least excessive majorant for g if i) G is an excessive majorant, and ii) if $\tilde{G}$ is any other excessive majorant, then $G \leq \tilde{G}$ pointwise.*

1. If f is excessive show that there exist functions $g_n \geq 0$ such that Ug_n increases to f. (Hint: Consider $g_n = n(f - P_{1/n}f)$.)

2. Suppose that $g \geq 0$, bounded and continuous. Let $g_0 = g$, $T_n = \{k2^{-n} : 0 \leq k \leq n2^n\}$, and define

$$g_n(x) = \max_{t \in T_n} P_t g_{n-1}(x), \qquad n = 1, 2, \ldots.$$

   Show that g_n is increasing. Let $H(x) := \lim_{n\to\infty} g_n(x)$ and show that H is lower semicontinuous (Hint: The limit of an increasing sequence of continuous functions is lower semicontinuous.).

3. Let H be as above and show that H is excessive. (Hint: First show that $H(x) \geq P_t g_n(x)$ for $t \in T_m$ for some m and large enough n. Then, take appropriate limits and use the fact that P_t is Fellerian. You will also need to make use of the following general fact on lower semicontinuous functions: If f is lower semicontinuous and $y \to x$, then $\liminf_{y\to x} f(y) \geq f(x)$.). Also show that $H = G$, the least excessive majorant of g.

4. Let g be as above and define

$$g^*(x) = \sup\{E^x g(X_T) : T \text{ a stopping time}\}.$$

Let

$$D = \{x : g(x) < G(x)\} \qquad \tau_D = \inf\{t : X_t \notin D\}.$$

Show that, if $P^x(\tau_D < \infty) = 1$, then $g^*(x) = G(x) = E^x g\left(X_{\tau_D}\right)$.

5. Suppose that there exists a Borel set A such that h is an excessive majorant of g, where $h(x) = E^x g\left(X_{\tau_A}\right)$ and $\tau_A = \inf\{t : X_t \in A\}$. Then, $g^*(x) = h(x)$.

6. Let B be a Brownian motion and let $X_t = B_{t \wedge T_{a,b}}$, where $T_{a,b}$ is the first exit from the interval $[a, b]$. Suppose g is a nonnegative function defined on $[a, b]$. Show that the least excessive majorant is the smallest concave functions G with $G \geq g$ pointwise.