Problem Set #4
ST441

1. Let $\mathbb{C}^2$ be the subspace of twice continuously differentiable functions in $\mathbb{C}$ whose first and second derivatives also belong to $\mathbb{C}$. Let X be the linear Brownian motion. Show that if $f \in \mathbb{C}$, then for any $\alpha > 0$, $U^\alpha f \in \mathbb{C}^2$ and $\alpha U^\alpha f - f = \frac{1}{2}(U^\alpha f)''$.

2. Let B be a scalar Brownian motion and $X = |B|$. Show that X is a Feller process. Find its transition function, infinitesimal generator and determine the domain of its infinitesimal generator.

3. Determine the infinitesimal generator of a Poisson process with parameter λ.

4. Let X be a Feller process with t.f. (P_t) and its generator A, and c a positive Borel function.

   (a) Prove that one can define a homogeneous t.f. Q_t by setting

   $$Q_t(x, A) = E^x \left[\mathbf{1}_A(X_t) \exp\left(-\int_0^t c(X_s)\, ds \right) \right].$$

   This t.f. corresponds to the killing of the trajectories of X at the rate $c(X)$.

   (b) If $f \in \mathscr{D}(A)$ and c is continuous and bounded, prove that

   $$\lim_{t \downarrow 0} \frac{Q_t f - f}{t} = Af - cf,$$

   i.e. the generator of (Q_t) is given by $A - c$.

5. Let A be an operator on $\mathbb{C}(\mathbb{R})$ defined by

$$Af(x) = \frac{1}{2} a(x) f''(x) + b(x) f'(x)$$

for some continuous functions a and b such that $a \geq 0$. Show that there exists a Feller process with this generator. (Hint: Apply Hille-Yosida Theorem.)