Problem Set #5
ST441

Consider the following stochastic differential equation:

$$X_t = X_0 + \int_0^t b(s, X_s)\, ds + \int_0^t \sigma(s, X_s) dW_s \tag{0.1}$$

where W is a r-dimensional Brownian motion, b is a $d \times 1$ drift vector and σ is a $d \times r$ dispersion matrix.

1. Let X be a weak solution of (0.1) and define

$$a_{ij}(t, x) = \sum_{k=1}^{r} \sigma_{ik}(t, x)\sigma_{kj}t, x).$$

   Let L_s be the associated operator defined in lecture notes. Then, for every continuous function $f : \mathbb{R}_+ \times \mathbb{R}^d \mapsto \mathbb{R}$ belonging to $C^{1,2}((0, \infty) \times \mathbb{R}^d)$

$$M_t^f = f(t, X_t) - f(0, X_0) - \int_0^t \left\{ \frac{\partial f}{\partial s} + L_s f(X_s) \right\} ds$$

   is a local martingale. If g is another continuous function belonging to $\mathbb{C}^{1,2}((0, \infty) \times \mathbb{R}^d)$, then

$$\langle M^f, M^g \rangle = \sum_{i,j} \int_0^t a_{ij}(s, X_s) \frac{\partial f}{\partial x_i}(s, X_s) \frac{\partial g}{\partial x_j}(s, X_s)\, ds.$$

   Furthermore, if $f \in C_K^{1,2}((0, \infty) \times \mathbb{R}^d)$ and σ_{ij} are bounded on the support of f, then M^f is a martingale such that M_t^f is square integrable for each t.

2. Show that if P is a solution of the martingale problem, then it also solves the local martingale problem. Moreover, if σ_{ij} are locally bounded, then two problems have the same set of solutions.

3. Suppose that the coefficients b and σ are continuous and satisfy the linear growth condition

$$\| b(t,x) \|^2 + \| \sigma(t,x) \|^2 \leq K^2(1+ \| x \|^2)$$

for every $t \geq 0, x \in \mathbb{R}^d, y \in \mathbb{R}^d$, where K is a positive constant. If (X,W), $(\Omega,\mathcal{F},P)$, $(\mathcal{F}_t)$ is a weak solution to (0.1) with $E \| X_0 \|^{2m} < \infty$ for some $m \geq 1$, then for any $T > 0$, we have

$$E\left(\max_{s\leq t} \| X_s \|^{2m}\right) \leq C(1+E \| X_0 \|^{2m})e^C t; \qquad 0 \leq t \leq T,$$

$$E \| X_t - X_s \|^{2m} \leq C(1+E \| X_0 \|^{2m})(t-s)^m; \qquad 0 \leq t \leq T,$$

where C is a positive constant depending only on m, T, K and d.

4. Suppose that the linear growth condition is satisfied and $\sigma \equiv 1$. Show that there exists a weak solution. (Hint: Use Girsanov's theorem.)

5. Suppose that $d = 1$ and consider X^1 and X^2 satisfying

$$X_t^{(i)} = X_0 + \int_0^t b_i(s, X_s^{(i)})\, ds + \int_0^t \sigma(s, X_s) dW_s,$$

where W is a one-dimensional Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$. Assume further that 1) $b_1(t,x) \leq b_2(t,x)$, 2) σ is Lipschitz, i.e., $|\sigma(t,x) - \sigma(t,y)| \leq K|x-y|$ for some $K < \infty$, 3) $X_0^{(1)} \leq X_0^{(2)}, P$-a.s., and 4) either b_1 or b_2 is Lipschitz. Then,

$$P[X_t^{(1)} \leq X_t^{(2)}, \forall t \geq 0] = 1.$$

6. Suppose that b and σ do not depend on t and

$$\sum_{i,j} a_{ij}(x)\xi_i\xi_j \geq \lambda\|\xi\|;\ \forall x \in D, \xi \in \mathbb{R}^d \qquad \text{and some } \lambda > 0$$

for some open and *bounded* domain D. Let u be a solution of

$$Lu - ku = -g; \text{ in } D$$

with the boundary condition $u = f$ on the boundary of D, where k is a positive continuous function and, f and g are continuous functions. Further suppose that $E^x\tau_D < \infty$ where $\tau_d = \inf\{t \geq 0 : X_t \notin D\}$. Under the assumptions set out above show that for every $x \in D$

$$u(x) = E^x\left[f(X_{\tau_D})\exp\left(-\int_0^{\tau_D} k(X_s)ds\right) + \int_0^{\tau_D} g(X_s)\exp\left(-\int_0^t k(X_s)ds\right)dt\right].$$

(Hint: Consider the process

$$N_t = u(X_{t\wedge\tau_D})\exp\left(-\int_0^{t\wedge\tau_D} k(X_s)ds\right)+\int_0^{t\wedge\tau_D} g(X_s)\exp\left(-\int_0^s k(X_r)dr\right)ds,$$

and first show that it is a uniformly integrable martingale under P^x.)