Problem Set #6
ST441

1. Using the methods employed in the proof of Theorem 6.1 show that for $l < x < y < r$
$$s(x) = s(y)P^x(T_y < T_l).$$
In view of the above show that s is strictly increasing and continuous.

2. Suppose that X is a diffusion with the infinitesimal generator
$$L = \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx},$$
where σ and b are locally bounded functions on $\mathbf{E}$ and $\sigma > 0$. Prove that the scale function is given by
$$s(x) = \int_c^x \exp\left(-\int_c^y 2b(z)\sigma^{-2}(z)dz\right) dy$$
where c is an arbitrary point in the interior of $\mathbf{E}$.

3. A 3-dimensional Bessel process is a Markov process on $(0, \infty)$ satisfying
$$X_t = x + \int_0^t \frac{1}{X_s} ds + B_t, \qquad x > 0.$$
Compute its scale function.

4. If X is a one-dimensional diffusion satisfying the hypothesis imposed in this chapter and ϕ is a homeomorphism from $\mathbf{E}$ to $\tilde{\mathbf{E}}$, then $\tilde{X} = \phi(X)$ also satisfies these hypothesis on $\tilde{\mathbf{E}}$. Prove that $\tilde{s} = s \circ \phi^{-1}$ and that $\tilde{m}$ is the image of m under ϕ, i.e $\tilde{m}(\tilde{I}) = m(\phi^{-1}(\tilde{I}))$ for any $\tilde{I} \subset \mathrm{int}(\tilde{\mathbf{E}})$.

5. Let X be as in Problem 2. Show that $m(dx) = \frac{2}{s'(x)\sigma^2(x)}dx$. (Hint: First find the speed measure for the process $\tilde{X} = s(X)$ using the relationship between the speed measure and the infinitesimal generator by observing that $\tilde{s}(x) = x$. Then, show that that $m(dx) = \tilde{m}(s(x))s'(x)$ using the previous exercise.)

6. Let X be as in Problem 2, $I =]l, r[$ and suppose $X_0 = x \in \mathrm{int}(I)$. We distinguish four cases:

   (a) $s(l+) = -\infty, s(r-) = \infty$. Then,

   $$P^x(\sigma_I = \infty) = P^x\left(\sup_{0 \le t < \infty} X_t = r\right) = P^x\left(\inf_{0 \le t < \infty} X_t = l\right) = 1.$$

   In particular, the process X is recurrent, i.e. for any $y \in I$, we have

   $$P^x(X_t = y; \text{ for some} t \in [0, \infty)) = 1.$$

   (b) $s(l+) > -\infty, s(r-) = \infty$. Then,

   $$P^x\left(\lim_{t \to \sigma_I} X_t = l\right) = P^x\left(\sup_{t < \sigma_I} X_t < r\right) = 1.$$

   (c) $s(l+) = -\infty, s(r-) < \infty$. Then,

   $$P^x\left(\lim_{t \to \sigma_I} X_t = r\right) = P^x\left(\inf_{t < \sigma_I} X_t > l\right) = 1.$$

   (d) $s(l+) > -\infty, s(r-) < \infty$. Then,

   $$P^x\left(\lim_{t \to \sigma_I} X_t = l\right) = 1 - P^x\left(\lim_{t \to \sigma_I} X_t = r\right) = \frac{s(r-) - s(x)}{s(r-) - s(l+).}$$