Problem Set #7
ST441

1. The infinitesimal generator, A, of a δ-dimensional squared Bessel process for $\delta > 0$ is given by

$$Af = 2x\frac{d^2}{dx^2}f + \delta\frac{d}{dx}f.$$

   Let T_y denote the first hitting time of y. Find $E^x\left[e^{-\lambda T_y}\right]$ for all $x, y \in [0, \infty)$ and $\lambda > 0$.

2. Extend the Kushner-Stratonvich equations in the lecture notes to the setting when $d[W, B]_t = \rho(X_t, Y_t)dt$ for some bounded measurable ρ.

3. Extend the Kushner-Stratonovich equations to a multidimensional setting.

4. Let X be the unobserved signal given by

$$X_t = X_0 + \int_0^t \sigma(s)dW_s, \qquad \forall t \in [0, 1],$$

   where X_0 is a normal random variable with mean zero and variance $s(0)$, and σ is a continuous and deterministic function such that

$$\int_0^1 \sigma^2(s)ds < \infty.$$

   There also exists an observation process Y given by

$$Y_t = B_t + \int_0^t \frac{X_s - Y_s}{f(s)}ds,$$

   where f is a deterministic continuous function. Suppose B and W are independent Brownian motions.

a) Find the equation for $\hat{X}$ given the observation process Y.

b) Let $v(t) := \mathbb{E}[(X_t - \widehat{X}_t)^2 | \mathcal{F}_t^Y]$ where $\mathcal{F}^Y$ is the filtration generated by Y and $\widehat{X}_t = \mathbb{E}[X_t | \mathcal{F}_t^Y]$. Show that v solves the differential equation

$$f^2(t)v'(t) + v^2(t) = \sigma^2(t)f^2(t).$$

(Hint: If $Z \sim N(\mu, \sigma^2)$ then $\mathbb{E}Z^3 = \mu(\mu^2 + 3\sigma^2)$.)

c) Let $s(t) := s(0) + \int_0^t \sigma^2(s)ds$. Suppose $f(t) = s(t) - t$. Using the differential equation above show that, given $\mathcal{F}_t^Y$, X_t is a normal random variable with mean $\widehat{X}_t$ and variance $s(t) - t$. Also show that $\widehat{X}$ is a Brownian motion.