Weak Dynamic Programming for Viscosity Solutions

B. Bouchard

Ceremade, Univ. Paris-Dauphine, and, Crest, Ensae

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Joint work with Nizar Touzi, CMAP, Ecole Polytechnique

Motivations

- Consider the control problem in standard form

$$V(t,x) \;:=\; \sup_{\nu \in \mathcal{U}} J(t,x;\nu)\;,\quad J(t,x;\nu) := \mathbb{E}\left[f(X_T^\nu) | X_t^\nu = x\right]$$

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b Continuity of the value function ?$(t,x) \in B_{r_i}(t_i,x_i) \mapsto \nu^\varepsilon(t_i,x_i)$ s.t.

$J(t,x;\nu^\varepsilon(t_i,x_i)) \geq J(t_i,x_i;\nu^\varepsilon(t_i,x_i)) - \varepsilon \geq V(t_i,x_i) - 2\varepsilon \geq V(t,x) - 3\varepsilon$

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- Our aim is to provide a <u>weak version</u>, much <u>easier to prove</u>.

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- Controlled process :

$$(\tau, \xi; \nu) \in \mathcal{S} \times \mathcal{U}_0 \longmapsto X^{\nu}_{\tau,\xi} \in \mathbb{H}^0_{\mathrm{rcll}}(\mathbb{R}^d)$$

with $[0, T] \times \mathbb{R}^d \subset \mathcal{S} \subset \left\{(\tau, \xi) : \tau \in \mathcal{T}_{[0,T]} \text{ and } \xi \in \mathbb{L}^0_\tau(\mathbb{R}^d)\right\}$.

Reward and Value functions

- Reward function

$$J(t, x; \nu) \quad := \quad \mathbb{E}\left[f\left(X_{t,x}^{\nu}(T)\right)\right]$$

defined for controls ν in

$$\mathcal{U} \quad := \quad \left\{\nu \in \mathcal{U}_0 : \ \mathbb{E}\left[|f(X_{t,x}^{\nu}(T))|\right] < \infty \ \forall \ (t, x) \in [0, T] \times \mathbb{R}^d\right\}.$$

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$$V(t,x) \; := \; \sup_{\nu \in \mathcal{U}_t} J(t,x;\nu) \quad \text{for} \quad (t,x) \in [0,T] \times \mathbb{R}^d.$$

Assumptions

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- **a.** For $\mathbb{P}$-a.e $\omega \in \Omega$, $\exists\ \tilde{\nu}_\omega \in \mathcal{U}_{\theta(\omega)}$ s.t.

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b. $\forall\ t \le s \le T$, $\theta \in \mathcal{T}^t_{[t,s]}$, $\tilde{\nu} \in \mathcal{U}_s$, and $\bar{\nu} := \nu \mathbf{1}_{[0,\theta]} + \tilde{\nu} \mathbf{1}_{(\theta,T]}$:

$$\mathbb{E}\left[f\left(X^{\bar{\nu}}_{t,x}(T)\right)|\mathcal{F}_\theta\right](\omega) = J(\theta(\omega), X^{\nu}_{t,x}(\theta)(\omega); \tilde{\nu}) \text{ for } \mathbb{P}-\text{a.e. } \omega \in \Omega.$$

The case where $J(\cdot;\nu)$ is l.s.c. and V is continuous

- Aim : Prove the DPP for $\tau \in \mathcal{T}^t_{[t,T]}$ (independent on $\mathcal{F}_t$)

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- More difficult one :

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Proof : Fix $(t_i, x_i)_{i\geq 1} := (\mathbb{Q}\times\mathbb{Q}^d)\cap([t,T]\times\mathbb{R}^d)$.

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Proof : Fix $(t_i, x_i)_{i\geq 1} := (\mathbb{Q}\times\mathbb{Q}^d)\cap([t,T]\times\mathbb{R}^d)$.
For $i \geq 1$, fix $r_i > 0$ and $\nu^i \in \mathcal{U}_{t_i}$ such that

$$J(t,x;\nu^i)+\varepsilon \geq J(t_i,x_i;\nu^i) \geq V(t_i,x_i)-\varepsilon \geq V(t,x)-2\varepsilon,$$

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Given $\nu \in \mathcal{U}_t$, define

$$\nu^\varepsilon \quad := \quad \mathbf{1}_{[t,\tau]}\nu + \mathbf{1}_{(\tau,T]} \sum_{i\geq 1} \mathbf{1}_{A_i}(\tau, X^\nu_{t,x}(\tau))\nu^i \ .$$

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Proof : Then,

$$\mathbb{E}\left[f\left(X_{t,x}^{\nu^\varepsilon}(T)\right)|\mathcal{F}_\tau\right] \quad = \quad \sum_{i\geq 1} J(\tau, X_{t,x}^{\nu}(\tau); \nu^i)\mathbf{1}_{A_i}\left(\tau, X_{t,x}^{\nu}(\tau)\right)$$

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&\geq \sum_{i\geq 1}\left(V(\tau, X_{t,x}^{\nu}(\tau)) - 3\varepsilon\right)\mathbf{1}_{A_i}\left(\tau, X_{t,x}^{\nu}(\tau)\right)
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- The lower-semicontinuity of $J(\cdot;\nu)$ is very important in this proof : It is in general not difficult to obtain.
- The upper-semicontinuity of V is also very important : It is much more difficult to obtain, especially when controls are not uniformly bounded (singular control).

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To derive the PDE in the viscosity sense, try to obtain :

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φ being smooth it should be much easier to prove!!

The weak DPP

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$$V(t,x) \ \ge \ \sup_{\nu \in \mathcal{U}_t^\varphi} \mathbb{E}\left[\varphi(\theta^\nu, X^\nu_{t,x}(\theta^\nu))\right],$$

where $\mathcal{U}_t^\varphi =$
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Remark : The minimum can be taken to be local if $\{(\theta^\nu, X^\nu_{t,x}(\theta^\nu)),\ \nu \in \mathcal{U}_t\}$ is bounded in $\mathbb{L}^\infty$. In practice φ is taken to be $C^{1,2}$.

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$$J(t,x;\nu^i)+\varepsilon \geq J(t_i,x_i;\nu^i) \geq V(t_i,x_i)-\varepsilon \geq \varphi(t_i,x_i)-\varepsilon \geq \varphi(t,x)-2\varepsilon,$$

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$$\nu^\varepsilon \quad := \quad \mathbf{1}_{[t,\theta^\nu]}\nu + \mathbf{1}_{(\theta^\nu,T]} \sum_{i\geq 1} \mathbf{1}_{A_i}(\theta^\nu, X^\nu_{t,x}(\theta^\nu))\nu^i \ .$$

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Then,

$$\mathbb{E}\left[f\left(X^{\nu^\varepsilon}_{t,x}(T)\right) | \mathcal{F}^\nu_\theta\right] = \sum_{i \geq 1} J(\theta^\nu, X^\nu_{t,x}(\theta^\nu); \nu^i)\mathbf{1}_{A_i}\left(\theta^\nu, X^\nu_{t,x}(\theta^\nu)\right)$$

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&\geq \quad \sum_{i \geq 1} \left(\varphi(\theta^\nu, X^\nu_{t,x}(\theta^\nu)) - 3\varepsilon\right) \mathbf{1}_{A_i}\left(\theta^\nu, X^\nu_{t,x}(\theta^\nu)\right)
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Proof. For $i \geq 1$, fix $r_i > 0$ and $\nu^i \in \mathcal{U}_{t_i}$ such that on $A_i := (t_i - r_i, t_i] \times B_i$, disjoint sets that cover $[t, T] \times \mathbb{R}^d$. Given $\nu \in \mathcal{U}_t$, define

$$\nu^\varepsilon \quad := \quad \mathbf{1}_{[t,\theta^\nu]}\nu + \mathbf{1}_{(\theta^\nu,T]} \sum_{i \geq 1} \mathbf{1}_{A_i}(\theta^\nu, X^\nu_{t,x}(\theta^\nu))\nu^i \ .$$

and

$$\begin{aligned} V(t,x) \quad &\geq \quad J(t,x;\nu^\varepsilon) \\ &= \quad \mathbb{E}\left[\mathbb{E}\left[f\left(X^{\nu^\varepsilon}_{t,x}(T)\right) | \mathcal{F}^\nu_\theta\right]\right] \\ &\geq \quad \mathbb{E}\left[\varphi\left(\theta^\nu, X^\nu_{t,x}(\theta^\nu)\right)\right] - 3\varepsilon \ . \end{aligned}$$

The weak DPP

Using test functions makes the proof straightforward :

$$\sup_{\nu \in \mathcal{U}_t^{\varphi}} \mathbb{E}\left[\varphi(\theta^\nu, X_{t,x}^\nu(\theta^\nu))\right] \leq V(t,x)$$

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Remark : If $\{X_{t,x}^\nu(\theta^\nu),\ \nu\in\mathcal{U}_t\}$ is bounded in $\mathbb{L}^\infty$, one can approximate V_* from below by smooth functions and obtain :

$$\sup_{\nu\in\mathcal{U}_t} \mathbb{E}\left[V_*(\theta^\nu, X_{t,x}^\nu(\theta^\nu))\right] \leq V(t,x) \leq \sup_{\nu\in\mathcal{U}_t} \mathbb{E}\left[V^*(\theta^\nu, X_{t,x}^\nu(\theta^\nu))\right]$$

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- Controlled process

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- $\mathcal{U}$ = square integrable progressively measurable processes with values in $U \subset \mathbb{R}^d$
- f is l.s.c with f^- with linear growth, μ and σ Lipschitz continuous.

Example : Verification of the assumptions

For all $(t, x) \in [0, T] \times \mathbb{R}^d$ and $\nu \in \mathcal{U}_t$:

L.s.c. $(t', x') \to (t, x) \Rightarrow X^{\nu}_{t',x'}(T) \to X^{\nu}_{t,x}(T)$ in $\mathbb{L}^2$

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A2 (Causality) $\forall\ \tilde{\nu} \in \mathcal{U}_t : \nu = \tilde{\nu}$ on $A \subset \mathcal{F} \Rightarrow X^{\nu}_{t,x} = X^{\tilde{\nu}}_{t,x}$ on A.

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A3 (Stability under concatenation) $\forall\ \tilde{\nu} \in \mathcal{U}_t,\ \theta \in \mathcal{T}^t_{[t,T]}$:
$\nu \mathbf{1}_{[0,\theta]} + \tilde{\nu}\mathbf{1}_{(\theta,T]} \in \mathcal{U}_t$.

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a. For $\mathbb{P}$-a.e $\omega \in \Omega$, $\exists\ \tilde{\nu}_\omega \in \mathcal{U}_{\theta(\omega)}$ s.t.

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Proof. Canonical space : $W(\omega) = \omega$. Set $\mathbf{T}_s(\omega) := (\omega_r - \omega_s)_{r \geq s}$ and $\omega^s := (\omega_{r \wedge s})_{r \geq 0}$.

$$\begin{aligned}
\mathbb{E}\left[f\left(X^{\nu}_{t,x}(T)\right)|\mathcal{F}_\theta\right](\omega) &= \int f\left(X^{\nu(\omega^{\theta(\omega)} + \mathbf{T}_{\theta(\omega)}(\omega))}_{\theta(\omega), X^{\nu}_{t,x}(\theta)(\omega)}(T)(\mathbf{T}_{\theta(\omega)}(\omega))\right) d\mathbb{P}(\mathbf{T}_{\theta(\omega)}(\omega)) \\
&= \int f\left(X^{\nu(\omega^{\theta(\omega)} + \mathbf{T}_{\theta(\omega)}(\tilde{\omega}))}_{\theta(\omega), X^{\nu}_{t,x}(\theta)(\omega)}(T)(\mathbf{T}_{\theta(\omega)}(\tilde{\omega}))\right) d\mathbb{P}(\tilde{\omega}) \\
&= J(\theta(\omega), X^{\nu}_{t,x}(\theta)(\omega); \tilde{\nu}_\omega)
\end{aligned}$$

where, $\tilde{\nu}_\omega(\tilde{\omega}) := \nu(\omega^{\theta(\omega)} + \mathbf{T}_{\theta(\omega)}(\tilde{\omega})) \in \mathcal{U}_{\theta(\omega)}$.

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$$\mathbb{E}\left[f\left(X^{\bar{\nu}}_{t,x}(T)\right)|\mathcal{F}_\theta\right](\omega) = J(\theta(\omega), X^{\nu}_{t,x}(\theta)(\omega); \tilde{\nu}) \text{ for } \mathbb{P}-\text{a.e. } \omega \in \Omega.$$

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Proof.

$$\mathbb{E}\left[f\left(X^{\bar{\nu}}_{t,x}(T)\right)|\mathcal{F}_\theta\right](\omega) = \int f\left(X^{\tilde{\nu}(\omega^{\theta(\omega)}+\mathbf{T}_{\theta(\omega)}(\tilde{\omega}))}_{\theta(\omega), X^{\nu}_{\mathbf{t},\mathbf{x}}(\theta)(\omega)}(T)(\mathbf{T}_{\theta(\omega)}(\tilde{\omega}))\right) d\mathbb{P}(\tilde{\omega}),$$

and therefore

$$\begin{aligned}\mathbb{E}\left[f\left(X^{\bar{\nu}}_{t,x}(T)\right)|\mathcal{F}_\theta\right](\omega) &= \int f\left(X^{\tilde{\nu}(\mathbf{T}_{\mathbf{s}}(\tilde{\omega}))}_{\theta(\omega), X^{\nu}_{\mathbf{t},\mathbf{x}}(\theta)(\omega)}(T)(\mathbf{T}_{\theta(\omega)}(\tilde{\omega}))\right) d\mathbb{P}(\tilde{\omega}) \\ &= J(\theta(\omega), X^{\nu}_{t,x}(\theta)(\omega); \tilde{\nu}) .\end{aligned}$$

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- Set $\theta_n := \inf \left\{ s \geq t_n \ : \ (s, X^u_{t_n, x_n}(s)) \notin B_r \right\}$ so that

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for some $\iota_n \longrightarrow 0$

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$$
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& \leq \ \mathbb{E}\left[\varphi(\theta_n, X^u_{t_n, x_n}(\theta_n))\right] - r^4
\end{aligned}
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Example : Super-solution property

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- Assume that $-\mathcal{L}^u \varphi(t_0, x_0) < 0$, for some $u \in U$.
- Set $\tilde{\varphi}(t, x) := \varphi(t, x) - |t - t_0|^4 - |x - x_0|^4$.
- Then, $-\mathcal{L}^u \tilde{\varphi}(t, x) \ \leq \ 0$ on $B_r := \{|t - t_0| \leq r, \ |x - x_0| \leq r\}$.
- Fix $(t_n, x_n, V(t_n, x_n)) \to (t_0, x_0, V_*(t_0, x_0))$.
- Set $\theta_n := \inf \left\{ s \geq t_n \ : \ (s, X^u_{t_n, x_n}(s)) \notin B_r \right\}$ so that

$$
\begin{aligned}
V(t_n, x_n) + \iota_n = \tilde{\varphi}(t_n, x_n) \quad & \leq \quad \mathbb{E}\left[\tilde{\varphi}(\theta_n, X^u_{t_n, x_n}(\theta_n))\right] \\
& \leq \quad \mathbb{E}\left[\varphi(\theta_n, X^u_{t_n, x_n}(\theta_n))\right] - r^4
\end{aligned}
$$

for some $\iota_n \to 0$, i.e. $V(t_n, x_n) \leq \mathbb{E}\left[\varphi(\theta_n, X^u_{t_n, x_n}(\theta_n))\right] - r^4/2$.

Example : Super-solution property

- $(t_0, x_0) \in [0, T) \times \mathbb{R}^d$ s.t. $0 = (V_* - \varphi)(t_0, x_0) = \min(V_* - \varphi)$
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for some $\iota_n \to 0$, i.e. $V(t_n, x_n) \leq \mathbb{E}\left[\varphi(\theta_n, X^u_{t_n,x_n}(\theta_n))\right] - r^4/2$.

- While $V(t_n, x_n) \geq \mathbb{E}\left[\varphi(\theta_n, X^u_{t_n,x_n}(\theta_n))\right]$ by the weak DPP.

Extensions

- Optimal control with running gain

Extensions

- Optimal control with running gain
- Optimal stopping

Extensions

- Optimal control with running gain
- Optimal stopping
- Mixed optimal control/stopping, impulse control, ...