Multivariate utility maximization under proportional transaction costs

Luciano Campi, Mark Owen

U. Paris-Dauphine & Heriot-Watt U.

Contents

Introduction I : Formulation of the problem

A proper concave function $U : \mathbb{R}^d \to [-\infty, \infty)$ is called a utility function supported on $\mathbb{R}^d_+$ if

- $C_U := \mathrm{cl}(\mathrm{dom}(U)) = \mathrm{cl}\{x : U(x) > -\infty\} = \mathbb{R}^d_+$ and
- U is increasing with respect to $\mathbb{R}^d_+$-(partial) order.

Consider the following problem

$$V(x) := \sup\{\mathrm{E}\left[U(X)\right] : X \in \mathcal{A}^x_T\}$$

where $\mathcal{A}^x_T$ is the set of all attainable final gains from an initial portfolio x (to be defined later). **Main results :**

1. Existence of a unique solution under asympt. satiability of value function V
2. Multivariate duality à la Kramkov-Schachermayer (1999)
3. Including liquidation case, discussion of multivariate RAE

Introduction II : References

- Davis Norman (1990), Shreve Soner (1994) - BS-type model, intertemporal consumption, stochastic optimal control
- Cvitanić Karatzas (1996), Cvitanić Wang (2001) – BS-type model, liquidated terminal wealth, duality
- Kabanov (1999) – more general liquidated terminal wealth
- Deelstra Pham Touzi (2001) – Kabanov-Last framework, multivariate, non-smooth utility supported by solvency cone
- Kamizono (2001, 2004) – KL framework, direct utility of consumption

Introduction III : Our contributions

- Cover the case of discontinuous bid-ask processes, i.e. random and discontinuous prop. TC.
- Direct utility function (à la Kamizono), which separates investment and consumption assets in order to include liquidation (not in this talk).
- No restrictions on U such as $U(0) = 0$ or $\sup U(x) = \infty$. Can treat anything, including $U(0) = -\infty$.
- Prove existence of optimizer under the minimal condition of "Asymptotic Satiability" of the value function V, which is a weaker than RAE.

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TC : Kabanov's market model

Main features of the model : all expressed in physical units, d risky assets (e.g. foreign currencies), the terms of trading are given by a **bid-ask process** $\{\Pi_t(\omega), t \in [0, T]\}$: an adapted, càdlàg, $d \times d$ matrix-valued process s.t.

- $\Pi^{ij} > 0$, $1 \leq i, j \leq d$
- $\Pi^{ii} = 1$, $1 \leq i \leq d$
- $\Pi^{ij} \leq \Pi^{ik}\Pi^{kj}$, $1 \leq i, j, k \leq d$

Meaning : To buy 1 unit of currency j one has to pay $\Pi^{ij}_t(\omega)$ units of i (at time t when the state of world is ω)

TC : Solvency cones & price systems

- *solvency cone:* $K_t = \text{cone}\{e^i, \Pi_t^{ij} e^i - e^j : 1 \leq i, j \leq d\}$
- *cone of portfolios available at price 0:* $-K_t$
- *polar of* $-K_t$: $K_t^* = \{w \in \mathbb{R}^d : \langle v, w \rangle \geq 0, \forall v \in K_t\}$
- *Financial interpretation* : $w \in K_t^*$ iff $w \in \mathbb{R}_+^d$ **and** $\Pi_t^{ij} w^i \geq w^j \Rightarrow \Pi_t^{ij} \geq \frac{w^j}{w^i} \Rightarrow \Pi_t^{ij} = (1 + \lambda_t^{ij}) \frac{w^j}{w^i}$ for some $\lambda_t^{ij} \geq 0$
- Every $w \in K_t^*$ (resp. in its relative interior) is called *consistent (resp. strictly consistent) price system.*
- Frictionless case : If $\pi^{ij} \equiv 1 \quad \forall i, j$, then $K_t = \mathbb{R}_+^d$ and $K_t^* = \{x : x_1 = \cdots = x_d \geq 0\}$.
- Cones (K_t) induce the following order : Let Y_1, Y_2 be $\mathcal{F}_\tau$-meas. for some stopping time τ, then $Y_1 \succeq_\tau Y_2$ means $Y_1 - Y_2 \in K_\tau$.

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TC : Consistent price processes

An $\mathbb{R}^d_+ \setminus \{0\}$-valued, adapted process Z is a consistent price process if

- is a càdlàg martingale (*time consistency*)
- $Z_t \in K^*_t, \forall t \in [0,T]$
- If, moreover, $Z_\tau \in \mathrm{ri} K^*_\tau \ \forall \tau$ stopping time, and $Z_{\sigma-} \in \mathrm{ri} K^*_{\sigma-} \ \forall \sigma$ predictable s.t., Z is called a *strictly* consistent price process.

Relations with the usual concept of EMM: choose a numéraire Z^1, define $S_t = (1, Z^2_t/Z^1_t \dots Z^d_t/Z^1_t)$ and set $d\mathbb{Q}/d\mathbb{P} = Z^1_T/Z^1_0$, then S is a $\mathbb{Q}$-martingale.

Main Assumption

SCPS: there exists a strictly consistent price process Z^s.

TC : Admissible portfolios

Interpretation: $X_t = (X_t^1, \dots, X_t^d)$, X_t^i = number of units of asset i held in the portfolio V at time t.
A d-dim process X is an *admissible self-financing portfolio process* if

- is *predictable and finite variation* (not nec. càdlàg !)
- $X_\tau - X_\sigma \in -\mathcal{K}_{\sigma,\tau} = -\overline{conv}(\cup_{\sigma \le u < \tau} K_u, 0)$
- there exists a threshold $a > 0$ s.t. $X_T \succeq -a\mathbf{1}$ and $Z_\tau^s X_\tau \ge -a Z_\tau^s \mathbf{1}$ $\forall \tau$ stopping time and $\forall Z^s \in \mathcal{Z}^s$

We denote $\mathcal{A}^x$ the set of all admissible portfolio processes X s.t. $X_0 = x$, and $\mathcal{A}_T^x := \{X_T : X \in \mathcal{A}^x\}$.

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TC : Super-replication theorem

Let $Y \in L^0(\mathbb{R}^d, \mathcal{F}_T)$ a contingent claim such that $\exists a > 0$, $Y \succeq_T -a\mathbf{1}$ (i.e. $Y + a\mathbf{1} \in K_T$)

Theorem (C.-Schachermayer, 2006)

Under SCPS, the following sets are equal:

1. $\{x \in \mathbb{R}^d : \exists X \in \mathcal{A}^x, X_T \succeq Y\}$
2. $\{x \in \mathbb{R}^d : \langle Z_0, x\rangle \geq E[\langle Z_T, Y\rangle], \forall Z \in \mathcal{Z}^{(s)}\}$

where, we recall, $\mathcal{A}^x$ is the set of all admissible portfolio processes X s.t. $X_0 = x$

Let us come back to max U problem

Let U denote a utility function such that $C_U = \mathbb{R}^d_+$. Our objective is

$$V(x) := \sup\{\mathrm{E}\left[U(X)\right] : X \in \mathcal{A}^x_T\}.$$

For stating the main result we need multivariate Inada's conditions :

- *Essentially smoothness* (analogue of $U'(0) = \infty$)
- *Asymptotic satiability* (analogue of $U'(\infty) = 0$)

Essential smoothness: $U'(0) = \infty$

Definition

A utility function $U : \mathbb{R}^d \to [-\infty, \infty)$ is said to be *essentially smooth* if

1. U is differentiable in the interior of $\mathbb{R}^d_+$;
2. $\lim_{i\to\infty} |\nabla U(x_i)| = +\infty$ for any $x_i \in \mathbb{R}^d_+$ converging to a boundary point of $\mathbb{R}^d_+$.

Asymptotic satiability : $U'(\infty) = 0$

- Let U be a utility function, and let C_U be its support cone. We say that a utility function U is *asymptotically satiable* if given any $\epsilon > 0$ there exists an $x \in \mathrm{dom}(U)$ such that

$$\partial U(x) \cap [0, \epsilon)^d \neq \emptyset.$$

- Recall that the dual function of U is defined by

$$U^*(x^*) = \sup_{x \in \mathbb{R}^d} \{U(x) - \langle x, x^* \rangle\}$$

- One can prove that asympt. satiability of U is equivalent to $0 \in C_{U^*} := \mathrm{cl}(\mathrm{dom}(U^*))$.

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Main result

Assume that $V(x) < \infty$ for some $x \in \text{int}(\text{dom } V)$

Theorem

Suppose that $U : \mathbb{R}^d \to [-\infty, \infty)$ is a utility function supported on $\mathbb{R}^d_+$, essentially smooth, strictly concave on $\mathbb{R}^d_{++}$, and asymptotically satiable.

Suppose in addition ***one of the following conditions*** *:*

1. *V is asymptotically satiable*
2. *U^* satisfies the growth condition*

$$U^*(\epsilon x^*) \leq \zeta(\epsilon)(U^*(x^*)^+ + 1)$$

for all $x^* \in \mathbb{R}^d_{++}, \epsilon \in (0, 1]$ and for some positive function ζ (stronger that 1)

Then the optimal investment problem has a unique solution $\widehat{X}_x$.

Growth condition & asymptotic elasticity I

- One-dim. RAE : $\limsup_{x\to\infty} \frac{xU'(x)}{U(x)} < 1$ (e.g. $U(x) = \ln x$)
- d-dim "natural" analogue (as in, e.g., DPT) :

$$AE(U) := \limsup_{|x|\to\infty} \frac{\langle x, \nabla U(x)\rangle}{U(x)} < 1$$

- But $U(x_1, x_2) = \ln x_1 + \ln x_2$ does not satisfy 2-RAE, while $\ln x$ satisfies 1-RAE (that's why DPT assume $U(0) = 0$)
- In other terms, d-RAE is not very robust wrt adding $d \geq 2$ one-dim utility functions
- Nonetheless $U(x_1, x_2) = \ln x_1 + \ln x_2$ does satisfy growth condition, so our existence result can be applied to it.

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Growth condition & asymptotic elasticity II

We need two more definitions : let U be a utility function supported on $\mathbb{R}^d_+$

- U, essentially smooth, satisfies Multivariate RAE if it is bounded below and

$$\sup_{c \in \mathbb{R}} \liminf_{\substack{x \in \text{int}(\mathbb{R}^d_+) \\ |x| \to \infty}} \frac{U(x) + c}{\langle x, \nabla U(x) \rangle} > 1. \tag{3.1}$$

  where $|x| := \max\{|x_1|, \ldots, |x_d|\}$.

- U is multivariate risk-averse (MVRA) if $\forall x \in \text{dom}(U)$, $x' \in \mathbb{R}^d$ s.t. $x' \succeq_{\mathbb{R}^d_+} x$, and all $z \in \mathbb{R}^d_+$ we have

$$U(x+z) - U(x) \geq U(x'+z) - U(x').$$

- If $U(x) = \sum_i U_i(x_i)$ additive, then concavity of each U_i is enough to get MVRA, but $U(x) = \sqrt{x_1 x_2}$

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$$U(x+z) - U(x) \geq U(x'+z) - U(x').$$

- If $U(x) = \sum_i U_i(x_i)$ additive, then concavity of each U_i is enough to get MVRA, but $U(x) = \sqrt{x_1 x_2}$

Growth condition & asymptotic elasticity II

We need two more definitions : let U be a utility function supported on $\mathbb{R}^d_+$

- U, essentially smooth, satisfies Multivariate RAE if it is bounded below and

$$\sup_{c\in\mathbb{R}} \liminf_{\substack{x\in\text{int}(\mathbb{R}^d_+)\\ |x|\to\infty}} \frac{U(x)+c}{\langle x, \nabla U(x)\rangle} > 1. \tag{3.1}$$

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Growth condition & asymptotic elasticity III

Lemma

Let U be a utility function with $C_U = \mathbb{R}^d_+$, essentially smooth, strictly concave on $\mathbb{R}^d_{++}$, multivariate risk averse and asympt. satiable.
Suppose that U is bounded below and satisfies multiv. RAE. Then U^* satisfies the growth condition, i.e. there exists a function $\zeta : (0,1] \to [0,\infty)$ such that for all $\epsilon \in (0,1]$ and all $x^* \in \mathbb{R}^d_{++}$

$$U^*(\epsilon x^*) \leq \zeta(\epsilon)(U^*(x^*)^+ + 1).$$

Duality I

- Define $\mathcal{C} = \mathcal{A}_T^0 \cap L^\infty(\mathbb{R}^d)$ and $\mathbb{U}_x : L^\infty(\mathbb{R}^d) \to [-\infty, \infty)$ by
$$\mathbb{U}_x(X) = \mathrm{E}\left[U(x+X)\right].$$
Then $\sup_{X \in \mathcal{C}} \mathbb{U}_x(X) \leq V(x)$.
- Define the dual cone of $\mathcal{C}$ by
$$\mathcal{D} := \{m \in \mathrm{ba}(\mathbb{R}^d) : m(X) \leq 0 \quad \forall X \in \mathcal{C}\}.$$
- Then
$$\begin{aligned}\sup_{X \in \mathcal{C}} \mathbb{U}_x(X) &\leq \sup_{X \in L^\infty} \inf_{m \in \mathcal{D}} \{\mathbb{U}_x(X) - m(X)\} \\ &\leq \inf_{m \in \mathcal{D}} \sup_{X \in L^\infty} \{\mathbb{U}_x(X) - m(X)\} =: \inf_{m \in \mathcal{D}} \mathbb{U}_x^*(m).\end{aligned}$$

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Duality II

- Recall that $U^*(x^*) = \sup_{x \in \mathbb{R}^d} \{U(x) - \langle x, x^* \rangle\}$
- For any $X \in \mathcal{A}_T^x$ and $m \in \mathcal{D}$

$$U(X) \leq U^*\left(\frac{dm^c}{d\mathbb{P}}\right) + \left\langle X, \frac{dm^c}{d\mathbb{P}} \right\rangle$$

- Taking expectation, one has

$$\begin{aligned} \mathrm{E}\left[U(X)\right] &\leq \mathrm{E}\left[U^*\left(\frac{dm^c}{d\mathbb{P}}\right) + \left\langle X, \frac{dm^c}{d\mathbb{P}} \right\rangle\right] \\ &\leq \mathrm{E}\left[U^*\left(\frac{dm^c}{d\mathbb{P}}\right)\right] + m(x) \end{aligned}$$

- One can prove that $\mathbb{U}_x^*(m) = \mathrm{E}\left[U^*(\frac{dm^c}{d\mathbb{P}})\right] + m(x)$ for $m \in \mathrm{ba}(\mathbb{R}_+^d)$, so that $V(x) \leq \inf_{m \in \mathcal{D}} \mathbb{U}_x^*(m)$.

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Duality III

Proposition (Lagrange Duality Theorem)

1. *If* $x \in \text{int}(C_V)$ *then* $\sup_{X \in \mathcal{C}} \mathbb{U}_x(X) = V(x) = \min_{m \in \mathcal{D}} \mathbb{U}^*_x(m) \in \mathbb{R}$.
2. *If* $x \notin C_V$ *then* $\sup_{X \in \mathcal{C}} \mathbb{U}_x(X) = V(x) = \inf_{m \in \mathcal{D}} \mathbb{U}^*_x(m) = -\infty$.

In the first case we let $\widehat{m} \in \mathcal{D}$ *denote the minimizer. Then*

$$\widehat{X} := -\nabla U^* \left(\frac{d\widehat{m}^c}{d\mathbb{P}} \right)$$

is the optimizer for the primal problem.

Duality IV : Sketch of the proof

Any candidate optimizer $\widehat{X}$ must satisfy

1. $U(\widehat{X}) = U^*\left(\frac{\mathrm{d}\widehat{m}^c}{\mathrm{d}\mathbb{P}}\right) + \left\langle \widehat{X}, \frac{\mathrm{d}\widehat{m}^c}{\mathrm{d}\mathbb{P}} \right\rangle$;
2. $\widehat{X} \in \mathcal{A}_T^x$; and
3. $\mathrm{E}\left[\left\langle \widehat{X}, \frac{\mathrm{d}\widehat{m}^c}{\mathrm{d}\mathbb{P}} \right\rangle\right] = \widehat{m}(x)$.

These are equivalent to

1. $\widehat{X} = \left(-\nabla U^*\left(\frac{\mathrm{d}\widehat{m}^c}{\mathrm{d}\mathbb{P}}\right), \underline{0}\right)$; and
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Take 1 as definition of $\widehat{X}$. We prove 2 by variational analysis, here the asymptotic satiability of V turns out to be crucial.

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The liquidation case : consumption vs investment assets

- Consider $U(x) = \tilde{U}(x_1)$ where $\tilde{U}$ is a u.s.c. utility function on $\mathbb{R}_+$, which corresponds to liquidation to the first asset.
- Define the liquidating utility function $\bar{U}$ as

$$\bar{U}(x) := \sup\{\tilde{U}(\xi) : (\xi, \underline{0}) \in L^0_+(x - K_T)\}, \quad x \in \mathbb{R}^d$$

- Notice that $\bar{U}(x) = \tilde{U}(I(x))$ where $I(\cdot)$ is the liquidation function expressed in *physical units*, i.e.

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- One can prove that

$$\sup_{X \in \mathcal{A}^x_T} \mathrm{E}\left[U(X)\right] = \sup_{X \in \mathcal{A}^x_T} \mathrm{E}\left[\tilde{U}(I(X_{T-}))\right].$$

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