Polynomial processes and applications to option pricing

Christa Cuchiero
(joint work with M. Keller-Ressel and J. Teichmann)

Vienna University of Technology
Institute for Mathematical Methods in Economics

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  - Lévy driven SDEs,
  - Jacobi processes, etc.

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  3. The semimartingale characteristics of S_T are known: Monte Carlo simulation methods.
- The class of processes which we describe ranges between 2 and 3 since European option prices can be calculated explicitly (up to matrix exponentials) for a dense set of claims. $\Rightarrow$Variance reduction techniques.

Setting and notation

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$$P_t f(x) := \mathbb{E}[f(X_t^x)] = \int_S f(\xi) p_t(x, d\xi), \quad x \in S,$$

defined on functions $f : S \to \mathbb{R}$ where $\mathbb{E}[f(X_t^x)] < \infty$.

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defined on functions $f : S \to \mathbb{R}$ where $\mathbb{E}[f(X_t^x)] < \infty$.
- $\mathcal{A}$: infinitesimal generator,

$$\mathcal{A}f = \lim_{t\to 0} \frac{P_t f - f}{t}.$$

Definition of polynomial processes

- $\mathrm{Pol}_{\leq m}(S)$: finite dimensional vector space of polynomials up to degree $m \geq 0$ on S, that is the restriction of polynomials on $\mathbb{R}^n$ to S.
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Definition

We call an S-valued time-homogeneous Markov process *m*-polynomial if,

$$P_t f(x) \in \mathrm{Pol}_{\leq m}(S)$$

for all $f \in \mathrm{Pol}_{\leq m}(S)$ and $t \geq 0$. If X is m-polynomial for all $m \geq 0$, then it is called polynomial.

Characterization of polynomial processes

Theorem (1)

Let X be a time-homogeneous Markov process with state space S and semigroup (P_t), pointwise continuous at $t = 0$. Then, the following assertions are equivalent:

(a) *X is m-polynomial for some $m \geq 0$.*

(b) *There exists a linear map A on $\mathrm{Pol}_{\leq m}(S)$, such that (P_t) restricted to $\mathrm{Pol}_{\leq m}(S)$ can be written as*

$$P_t|_{\mathrm{Pol}_{\leq m}(S)} = e^{tA}$$

for all $t \geq 0$.

Characterization of polynomial processes

Theorem (Continuation)

(c) *The infinitesimal generator $\mathcal{A}$ is well defined on $\mathrm{Pol}_{\leq m}(S)$ and maps $\mathrm{Pol}_{\leq m}(S)$ to itself.*

(d) *The Kolmogorov backward equation for an initial value $f(\cdot, 0) \in \mathrm{Pol}_{\leq m}(S)$*

$$\partial_t f(x,t) = \mathcal{A} f(x,t)$$

has a real analytic solution for all times $t \in \mathbb{R}$. In particular, $f(\cdot, t) \in \mathrm{Pol}_{\leq m}(S)$.

Sketch of the proof

(a)⇒(b)

- The semigroup

$$P_{(\cdot)} : \mathbb{R}_+ \to \mathcal{L}(\mathrm{Pol}_{\leq m}(S)) \tag{1}$$

satisfies for all $t, s \geq 0$ the Cauchy functional equation

$$\begin{cases} P_{t+s} = P_t P_s, \\ P_0 = Id \end{cases} .$$

- Finite dimensionality of $\mathrm{Pol}_{\leq m}(S)$ and continuity of (1) at $t = 0$ imply $P_t = e^{tA}$.

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(b)⇒(c)

- For $f \in \mathrm{Pol}_{\leq m}(S)$ the generator is given by

$$\mathcal{A}f = \lim_{t \to 0} \frac{P_t f - f}{t} = \lim_{t \to 0} \frac{e^{tA} f - f}{t} = Af,$$

which is obviously well defined with respect to $\| \cdot \|_{\mathrm{Pol}_{\leq m}}$ and in $\mathrm{Pol}_{\leq m}(S)$.

Sketch of the proof

(c)⇒(d) - Since $\mathcal{A}$ maps $\mathrm{Pol}_{\leq m}(S)$ to itself, the Kolmogorov backward equation can be understood as a linear ODE in the classical sense whose solution is $e^{tA}f(\cdot, 0)$.

Sketch of the proof

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(d)⇒(a)

- For any initial value f in an appropriate Banach space, $P_t f$ is the unique solution of the Kolmogorov backward equation.
- On $\mathrm{Pol}_{\leq m}(S)$ it must therefore be equal to $e^{tA}f$.
  $\Rightarrow P_t f \in \mathrm{Pol}_{\leq m}(S) \Rightarrow X$ is m-polynomial.

Corollary

Corollary

Let X be an m-polynomial process with semigroup (P_t), continuous at $t = 0$ and let $f \in \mathrm{Pol}_{\leq m}(S)$ be fixed. Then there exists a unique function $Q : \mathbb{R} \times S \to \mathbb{R}$, being real analytic in time and $Q(t, \cdot) \in \mathrm{Pol}_{\leq m}(S)$ for all $t \in \mathbb{R}$, such that

(a) $Q(0, x) = f(x)$ *and*

(b) $Q(t - s, X_s)$ *is a martingale for* $s \geq 0$.

Moreover, $Q(-s, X_s)$ is a time space harmonic function for the m-polynomial process X.

Feller semimartingales

Aim: Find sufficient conditions for m-polynomial processes in terms of the infinitesimal generator of Feller processes.

- Conservative Feller semigroup (P_t) with $C_c^\infty(S) \subset D(\mathcal{A})$.
- There exist functions a_{kl}, b_k and a kernel $K(x, d\xi)$ such that for $u \in C_c^2(S)$ the infinitesimal generator $\mathcal{A}$ is given by

$$\mathcal{A}u(x) = \frac{1}{2}\sum_{k,l=1}^{n} a_{kl}(x)\frac{\partial^2 u(x)}{\partial x_k \partial x_l} + \sum_{k=1}^{n} b_k(x)\frac{\partial u(x)}{\partial x_k} + \int_{\mathbb{R}^n\setminus\{0\}} \left(u(x+\xi) - u(x) - \sum_{k=1}^{n} \chi_k(\xi)\frac{\partial u(x)}{\partial x_k} \right) K(x, d\xi). \tag{2}$$

  - $\chi : \mathbb{R}^n \to \mathbb{R}^n$ some truncation function.
  - The parameters satisfy admissibility conditions guaranteeing the existence of the process in S.

Feller semimartingales

- If X is additionally a semimartingale, then its characteristics (B, C, ν) associated with the truncation function $\chi(\xi)$ are given by

$$B_t = \int_0^t b(X_s)ds, \quad C_t = \int_0^t a(X_s)ds,$$
$$\nu(dt, d\xi) = K(X_t, d\xi)dt.$$

- (b, a, K) are referred as differential characteristics of X (see Kallsen [5]).

$\Rightarrow$ Specify the form of a, b and K such that $\mathcal{A}$ generates an m-polynomial process.

Conditions on the kernel $K(x, d\xi)$

Condition A

The kernel $K(x, d\xi)$ is of the form

$$K(X_t, d\xi) := \mu_{00}(d\xi) + \sum_{i \in I} X_{s,i}\mu_{i0}(d\xi) + \sum_{(i,j) \in J} X_{s,i}X_{s,j}\mu_{ij}(d\xi),$$

where all μ_{ij} are Lévy measures on $\mathbb{R}^n$ with

$$\int_{\|\xi\|>1} \|\xi\|^m \mu_{ij}(d\xi) < \infty.$$

The index sets I and J are defined by $I = \{1 \leq i \leq n | S_i \subseteq \mathbb{R}_+\}$ and $J = \{(i,j), i \leq j | S_i \times S_j \subseteq \mathbb{R}^2_+ \text{ or } S_i \times S_j \subseteq \mathbb{R}^2_-\}$, where S_i stands for the projection on the i^{th} component.

Conditions on the kernel $K(x, d\xi)$

Condition B

The kernel $K(x, d\xi)$ satisfies

$$K(X_t, d\xi) := g_*^{X_t}\mu(d\xi),$$

where for each $x \in S$, $g_*^x\mu$ denotes the pushforward of the measure μ under the map g^x. Moreover, $g^x(y) = g(x, y)$ is affine in x, that is

$$g : S \times \mathbb{R}^d \to \mathbb{R}^n, \ (x, y) \mapsto H(y)x + h(y),$$

where $H : \mathbb{R}^d \to \mathbb{R}^{n\times n}$ and $h : \mathbb{R}^d \to \mathbb{R}^n$ some measurable functions. Furthermore, μ is a Lévy measure on $\mathbb{R}^d$ integrating

$$\int_{\mathbb{R}^d\setminus\{0\}} \left(\|H(y)\|^k + \|h(y)\|^k\right) \mu(dy) \text{ for } 1 \leq k \leq m.$$

Conditions for polynomial Feller semimartingales

Theorem (2)

Let $m \geq 2$ and X_t^x be a Feller semimartingale on S whose infinitesimal generator on $C_c^\infty(S)$ is of form (2). Assume furthermore that $\mathbb{E}\left[\|(X_t^x)\|^m\right] < \infty$ for all $t \in [0,1]$. Then, X is m-polynomial if its differential characteristics (b, a, K) associated with the "truncation function" $\chi(\xi) = \xi$ are of the form

$$b_t = b + \sum_{i=1}^{n} X_{t,i}\beta_i, \quad b,\ \beta_i \in \mathbb{R}^n,$$

$$a_t = a + \sum_{i=1}^{n} X_{t,i}\alpha_{i0} + \sum_{i \leq j} X_{t,i}X_{t,j}\alpha_{ij}, \quad a,\ \alpha_{ij} \in \mathbb{R}^{n \times n},$$

with K satisfying either Condition A or Condition B.

Remark on the truncation function

The characteristic b must be adapted to the choice of the truncation function χ:

$$\left(b_t(\chi(\xi)) + \int_{\mathbb{R}^n \setminus \{0\}} (\xi - \chi(\xi))\, K(X_t, d\xi) \right) \in \mathrm{Pol}_{\leq 1}(S)$$

is an equivalent condition guaranteeing that X is m-polynomial.

Sketch of the proof of Theorem (2)

- Denote the right side of the integro-differential operator (2) literally by $\mathcal{A}^\sharp$.
- Observe that under the above conditions $\mathcal{A}^\sharp f \in \mathrm{Pol}_{\leq m}(S)$ for every $f \in \mathrm{Pol}_{\leq m}(S)$.
- Showing that the process

$$M_t^f := f(X_t^x) - f(x) - \int_0^t \mathcal{A}^\sharp f(X_s^x) ds$$

is a well-defined martingale for every $f \in \mathrm{Pol}_{\leq m}(S)$ yields $\mathcal{A} = \mathcal{A}^\sharp$ on $\mathrm{Pol}_{\leq m}(S)$.
- Theorem (1) then yields the assertion.

Affine processes

$\Rightarrow$ Every conservative affine process X on $S = \mathbb{R}_+^p \times \mathbb{R}^{n-p}$ (see Duffie et al. [1]) is m-polynomial if $\mathbb{E}\left[\|(X_t^x)\|^m\right] < \infty$ for $t \in [0, 1]$.

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Proof.

- Every conservative affine process is a Feller semimartingale.
- On $C_c^2(S)$, the generator of an affine process is given by

$$\mathcal{A}u(x) = \frac{1}{2}\sum_{k,l=1}^{n}\left(a_{kl} + \sum_{i=1}^{p} x_i \alpha_{i0,kl}\right)\frac{\partial^2 u(x)}{\partial x_k \partial x_l} + \left\langle b + \sum_{i=1}^{n} x_i \beta_i, \nabla u(x)\right\rangle$$
$$+ \int_{S\setminus\{0\}} \left(u(x+\xi) - u(x) - \langle \chi(\xi), \nabla u(x)\rangle\right)\left(\mu_{00}(d\xi) + \sum_{i=1}^{p} x_i \mu_{i0}(d\xi)\right).$$

□

Lévy driven SDEs

- L_t: Lévy process on $\mathbb{R}^d$ with triplet (b, a, μ).
- $V_1, \ldots, V_d$: affine functions $V_i : S \to \mathbb{R}^n$, $x \mapsto H_i x + h_i$ with $H_i \in \mathbb{R}^{n \times n}$ and $h_i \in \mathbb{R}^n$.

$\Rightarrow$ A process X, which solves the stochastic differential equation of type

$$dX_t = \sum_{i=1}^{d} V_i(X_{t-}) dL_t^i, \quad X_0 = x \in S,$$

in S, is m-polynomial ($m \geq 2$) as soon as L admits finite m^{th} moment, that is $\int_{\|\xi\|>1} \|\xi\|^m \mu(d\xi) < \infty$.

Remark

- *The moment condition* $\mathbb{E}\left[\|(X_t^x)\|^m\right] < \infty$ *for* $t \in [0, 1]$ *is automatically satisfied due to the assumption on* μ*.*

Exponential Lévy models

- L_t: Lévy process on $\mathbb{R}$ with triplet (b, a, μ).

$\Rightarrow$ An exponential Lévy model $X_t^x = xe^{L_t}$ is m-polynomial if $\int_{|y|>1} e^{my}\mu(dy) < \infty$.

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Sketch of the proof

- *The infinitesimal generator is given by*

$$\mathcal{A}u(x) = \frac{ax^2}{2}\frac{d^2u(x)}{dx^2} + \left(b + \frac{a}{2} + \int_{\mathbb{R}} (e^y - 1 - \chi(y))\,\mu(dy)\right) x\frac{du(x)}{dx}$$
$$+ \int_{\mathbb{R}} \left(u\left(xe^y\right) - u(x) - x\left(e^y - 1\right)\frac{du(x)}{dx}\right)\mu(dy).$$

- *We are in the situation of Condition B with* $g(x, y) = H(y)x = (e^y - 1)x$.

Jacobi process

- The Jacobi process is the solution of

$$dX_t = -\beta(X_t - \theta)dt + \sigma\sqrt{X_t(1 - X_t)}dB_t, \quad X_0 = x \in [0,1],$$

on $S = [0,1]$, where $\theta \in [0,1]$ and $\beta, \sigma > 0$ (see for example Gourieroux [4]).

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- Extension by adding jumps:
  - Jump times correspond to those of a Poisson process N_t with intensity λ.
  - Jump size h(x)=1-2x, i.e. if a jump occurs, the process is reflected at $\frac{1}{2}$.

$$dX_t = -\beta(X_t - \theta)dt + \sigma\sqrt{X_t(1-X_t)}dB_t + (1-2X_t)dN_t.$$

  - Generator:

$$\mathcal{A}u = \frac{1}{2}\sigma^2(x(1-x))\frac{d^2u(x)}{dx^2} - \beta(x-\theta)\frac{du(x)}{dx} + \lambda(u(1-x) - u(x)).$$

  - In terms of Condition B, we have here $g(x,y) = -2yx + y$ and $\mu(dy) = \lambda\delta_1(dy)$.

Dunkl process

- Dunkl processes (see for example Gallardo and Yor [3]) are Feller processes parametrized by $k \geq 0$ whose infinitesimal generator for $u \in C^2(\mathbb{R})$ is given by

$$\mathcal{A}u = \frac{1}{2}\frac{d^2u(x)}{dx^2} + k\left(\frac{1}{x}\frac{du(x)}{dx} - \frac{u(x) - u(-x)}{2x^2}\right).$$

- For every $f \in \mathrm{Pol}_{\leq m}(\mathbb{R})$, we therefore have $\mathcal{A}f \in \mathrm{Pol}_{\leq m}(\mathbb{R})$.
- Dunkl processes lie in the class of polynomial processes.
- Example that the conditions of Theorem (2) are only sufficient and not necessary.

Moment calculation

- There exists a linear map A such that moments of m-polynomial processes can simply be calculated by computing e^{tA}.
- Choose a basis $\langle e_1, \ldots, e_N \rangle$ of $\mathrm{Pol}_{\leq m}(S)$.
- $A = (a_{ij})_{i,j=1,\ldots N}$ is obtained by

$$\mathcal{A} e_i = \sum_{j=1}^{N} a_{ij} e_j.$$

- Writing f as $\sum_{k=1}^{N} \alpha_k e_k$, yields

$$P_t f = (\alpha_1, \ldots, \alpha_N) e^{tA} (e_1, \ldots, e_N)^\top.$$

Generalized Method of Moments

- Let $\theta \in \Theta \subseteq \mathbb{R}^p$ be the vector of parameters to be estimated and $g : S \times \Theta \to \mathbb{R}^q$ a function such that $\mathbb{E}[g(X_t, \theta_0)] = 0$ for the true value of the parameter θ_0.
- The Generalized Method of Moments estimator is the value of θ which minimizes
  $Q_T(\theta) = \left(\sum_{t=1}^T g(X_t, \theta)\right)^\top W_T \left(\sum_{t=1}^T g(X_t, \theta)\right)$,
  where W_T is a positive semi-definite $q \times q$ matrix.
- A typical moment condition is given by

$$g(X_t, \theta) = \begin{pmatrix} X_t^{n_1} X_{t+s}^{m_1} - \mathbb{E}[X_t^{n_1} X_{t+s}^{m_1}] \\ \vdots \\ X_t^{n_q} X_{t+s}^{m_q} - \mathbb{E}[X_t^{n_q} X_{t+s}^{m_q}] \end{pmatrix}, \quad n_i, m_i \in \mathbb{N}.$$

- $\mathbb{E}\left[X_t^n X_{t+s}^m\right] = \mathbb{E}\left[X_t^n \mathbb{E}\left[X_{t+s}^m \middle| X_t\right]\right]$ can be easily computed (see Zhou [6], Forman [2] in the one dimensional diffusion case).

European option pricing - setting

- X: m-polynomial process.
- $G : S \to \mathbb{R}^n$ deterministic measurable map such that the price processes are given through

$$S_t = G(X_t)$$

  under a martingale measure.
- $F = \phi(S_T)$: bounded measurable European claim for some maturity $T > 0$ whose price at $t \geq 0$ is given by

$$p_t^F = \mathbb{E}[\phi(S_T)|\mathcal{F}_t] = \mathbb{E}[(\phi \circ G)(X_T)|\mathcal{F}_t].$$

Analytically tractable claims

- Claims of the form $F = f \circ G^{-1}(S_T)$ for $f \in \mathrm{Pol}_{\leq m}(S)$ are analytically tractable:

$$\begin{aligned} p_t^F &= \mathbb{E}[(f \circ G^{-1})(S_T)|\mathcal{F}_t] = P_{T-t} f(G^{-1}(S_t)) \\ &= e^{(T-t)A} f(G^{-1}(S_t)) \end{aligned}$$

  for $0 \leq t \leq T$.

- The sensitivities of the price process with respect to the factors of X can be calculated by

$$\nabla p_t^F = \nabla P_{T-t} f(G^{-1}(S_t)) \nabla G^{-1}(S_t).$$

Variance reduction for Monte Carlo simulation using polynomial control variates

- $X^1, \ldots, X^L$: L sample paths of the polynomial process X.
- Standard estimator in Monte Carlo simulation: $\pi_0^F = \frac{1}{L}\sum_{i=1}^L (\phi \circ G)(X_T^i)$.
- Approximation of $\phi \circ G$ by a polynomial f, $\mathbb{E}[f(X_T)]$ is explicitly known.
- Estimator with control variate f:

$$\hat{\pi}_0^F = \frac{1}{L}\sum_{i=1}^L \left((\phi \circ G)(X_T^i) - (f(X_T^i) - \mathbb{E}[f(X_T)])\right).$$

$$\Rightarrow \hat{\pi}_0^F \xrightarrow{L\to\infty} p_0^F \text{ and } Var(\hat{\pi}_0^F) < Var(\pi_0^F).$$

Method illustration: exponential Lévy model

- Price process: $S_t = S_0 e^{X_t}$.
- X_t: Lévy process with Lévy triplet (b, a, ν).
- r: interest rate, $e^{-rt} S_t$ is a martingale.

$$\Rightarrow \int_{|y|>1} e^y \nu(dy) < \infty \text{ and } b = r - \frac{a}{2} - \int_{\mathbb{R}} \left(e^y - 1 - y 1_{|y| \le 1} \right) \nu(dy).$$

- Generator of the Lévy process:

$$\mathcal{A}u(x) = \frac{a}{2} \frac{d^2 u(x)}{dx^2} + \left(r - \frac{a}{2} \right) \frac{du(x)}{dx} + \int_{\mathbb{R}} \left(u(x+y) - u(x) - (e^y - 1) \frac{du(x)}{dx} \right) \nu(dy).$$

- Example: constant jump intensity λ, exponential jumps size distribution with parameter $\frac{1}{c}$.
- Applying $\mathcal{A}$ to $(x^0, x^1, \ldots, x^m)$ yields the following $(m+1) \times (m+1)$ matrix

Method illustration: exponential Lévy model

$$A = \begin{pmatrix} 0 & \dots & & & & & & \\ \kappa & 0 & \dots & & & & & \\ a+c^2 & 2\kappa & 0 & \dots & & & & \\ 6c^3 & 3(a+c^2) & 3\kappa & 0 & \dots & & & \\ & & & & \ddots & & & \\ & & & & & & & \\ m!c^m & \dots & \dots & \frac{m!}{(m-i)!}c^i & \dots & \frac{m(m-1)}{2}(a+c^2) & m\kappa & 0 \end{pmatrix},$$

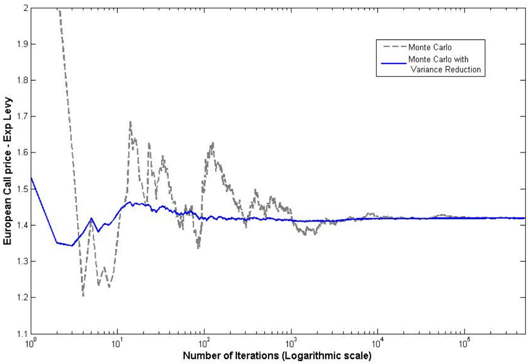
where $\kappa = r - \frac{a}{2} - \frac{c^2}{1-c}$.

$\Rightarrow \mathbb{E}[\sum_{k=0}^m \alpha_k (X_t^x)^k] = (\alpha_0, \dots, \alpha_m) e^{tA} (x^0, \dots, x^m)^\top$.

- Approximate $(S_0 e^x - K)^+$ by a polynomial f.
- Use f as control variate for variance reduction in the MC simulation.

Method illustration: exponential Lévy model

- Comparison: Monte Carlo simulation for European call prices with and without variance reduction.
- Ratio of the variance of the uncontrolled estimator to that of the controlled estimator: around 100 (depending on the polynomial approximation).



Example: Heston type model with volatility dependent jumps (Bates 2000)

- Price process: $S_t = S_0 e^{X_t}$, where

$$dX_t = \left(r - \frac{V_t}{2} - \lambda V_t \int_{\mathbb{R}} (e^y - 1) F(dy)\right) dt + \sqrt{V_t} dW_t^1 + dJ_t(V_t),$$
$$dV_t = -\beta(V_t - \theta) + \sigma \sqrt{V_t} dW_t^2.$$

  - $J_t = \sum_{i=1}^{N_t} Z_i$, $(Z_i)_{i \geq 1}$ i.i.d sequence of random variables with probability distribution F.
  - N_t: Poisson process with intensity λV_t.
  - Correlation between Brownian motions ρ.

Example: Heston type model with volatility dependent jumps (Bates 2000)

Usual method:

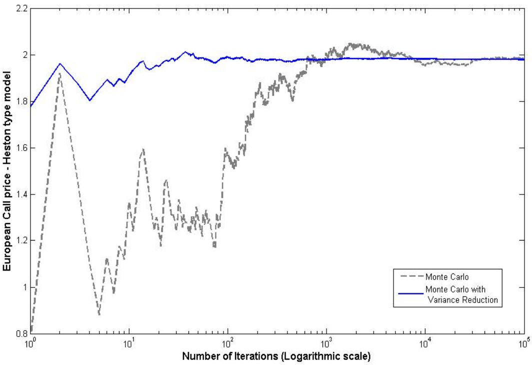
- Solve the Ricatti equations:

$$\begin{aligned}
\partial_t \phi(t,x,v) &= rx + \beta\theta\psi(t,x,v), \\
\partial_t \psi(t,x,v) &= \frac{1}{2}(x^2 - x) - \beta\psi(t,x,v) + \frac{1}{2}\sigma^2\psi^2(t,x,v) \\
&\quad + \rho\sigma x\psi(t,x,v) \\
&\quad + \lambda\Big(\int_{\mathbb{R}} (e^{xy} - 1)dF(y) - x\int_{\mathbb{R}} (e^{y} - 1)dF(y)\Big).
\end{aligned}$$

- Analytic solutions for ψ might be difficult to find $\Rightarrow$ Numerical solutions must be used.
- Application of Fourier pricing methods to obtain the European call price.

Example: Heston type model

- Comparison: Monte Carlo simulation for European call prices with and without variance reduction.
- Ratio of the variance of the uncontrolled estimator to that of the controlled estimator: between 300 and 400 (depending on the polynomial approximation).



Example: Basket option under a Heston type model

- Payoff function: $e^{-rT}(S^1_T + S^2_T - K)^+$.
- Price processes: $S^1_t = S^1_0 e^{X_t}$ and $S^2_t = S^2_0 e^{Y_t}$.

$$dX_t = \Big(r - \frac{V_t}{2} - \lambda^1 V_t \int_{\mathbb{R}} (e^y - 1) F^1(dy)\Big) dt + \sqrt{V_t} dW^1_t + dJ^1_t(V_t),$$

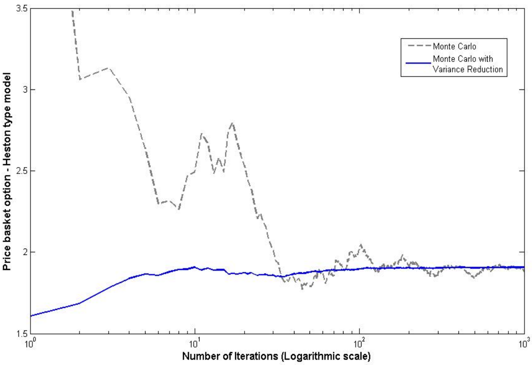
$$dY_t = \Big(r - \frac{\gamma^2 V_t}{2} - \lambda^2 V_t \int_{\mathbb{R}} (e^y - 1) F^2(dy)\Big) dt + \gamma\sqrt{V_t} dW^2_t + dJ^2_t(V_t),$$

$$dV_t = -\beta(V_t - \theta) + \sigma\sqrt{V_t} dW^3_t.$$

- $J^j_t = \sum_{i=1}^{N^j_t} Z^j_i$, $(Z^j_i)_{i\geq 1}$ i.i.d sequences of random variables with probability distribution F^j, $j = 1, 2$.
- N^j_t: Poisson processes with intensity $\lambda^j V_t$.
- Brownian motions are correlated.

Example: Basket option under a Heston type model

- Comparison: Monte Carlo simulation for European call prices with and without variance reduction.
- Ratio of the variance of the uncontrolled estimator to that of the controlled estimator: around 150 (depending on the polynomial approximation).



Conclusion

- Under certain moment conditions, processes with
  - affine drift,
  - quadratic diffusion function and
  - jump compensator which is either quadratic or the pushforward of a Lévy measure under an affine function,

  are polynomial processes.
- The calculation of moments of polynomial processes only requires the computation of matrix exponentials.
- Polynomial claims can therefore be priced analytically, which leads to variance reduction techniques by using polynomial control variates approximating the payoff function well.

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Thank you for your attention!