Polynomial processes and applications to option pricing

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A tractable class of Markov processes

- Polynomial processes
- Theorem - Characterization
- Polynomial Feller semimartingales

Examples

- Affine processes
- Lévy models
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- Dunkl Process

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- Pricing and sensitivities
- Variance reduction
Introduction - Aim of this talk

We consider a class of time-homogeneous Markov processes $X,...$

- ...with the property that the expected value of any polynomial of the process is again a polynomial of same or lower degree in the initial value $X_0$. 
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- The class of processes which we describe ranges between 2 and 3 since European option prices can be calculated explicitly (up to matrix exponentials) for a dense set of claims.

⇒ Variance reduction techniques.
Setting and notation

- $X := (X^x_t)_{t \geq 0, x \in S}$: time-homogeneous Markov process with state space $S \subseteq \mathbb{R}^n$, a closed subset of $\mathbb{R}^n$. 
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- \((P_t)_{t \geq 0}\): associated semigroup

\[
P_t f(x) := \mathbb{E}[f(X_t^x)] = \int_{S} f(\xi) p_t(x, d\xi), \quad x \in S,
\]

defined on functions \( f : S \to \mathbb{R} \) where \( \mathbb{E}[f(X_t^x)] < \infty \).
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defined on functions $f : S \to \mathbb{R}$ where $\mathbb{E}[f(X^x_t)] < \infty$.
- $\mathcal{A}$: infinitesimal generator,

$$\mathcal{A}f = \lim_{t \to 0} \frac{P_t f - f}{t}.$$
Definition of polynomial processes

- **Pol**$_{\leq m}(S)$: finite dimensional vector space of polynomials up to degree $m \geq 0$ on $S$, that is the restriction of polynomials on $\mathbb{R}^n$ to $S$.
- Pol$_{\leq m}(S)$ is endowed with some norm $\| \cdot \|_{\text{Pol}_{\leq m}}$ and its dimension is denoted by $N < \infty$. 
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**Definition**

We call an $S$-valued time-homogeneous Markov process $m$-polynomial if,

$$P_t f(x) \in \text{Pol}_{\leq m}(S)$$

for all $f \in \text{Pol}_{\leq m}(S)$ and $t \geq 0$. If $X$ is $m$-polynomial for all $m \geq 0$, then it is called polynomial.
Characterization of polynomial processes

Theorem (1)

Let $X$ be a time-homogeneous Markov process with state space $S$ and semigroup $(P_t)$, pointwise continuous at $t = 0$. Then, the following assertions are equivalent:

(a) $X$ is $m$-polynomial for some $m \geq 0$.

(b) There exists a linear map $A$ on $\text{Pol}_{\leq m}(S)$, such that $(P_t)$ restricted to $\text{Pol}_{\leq m}(S)$ can be written as

$$P_t|_{\text{Pol}_{\leq m}(S)} = e^{tA}$$

for all $t \geq 0$. 
Characterization of polynomial processes

Theorem (Continuation)

(c) *The infinitesimal generator* $\mathcal{A}$ *is well defined on* $\text{Pol}_{\leq m}(S)$ *and maps* $\text{Pol}_{\leq m}(S)$ *to itself.*

(d) *The Kolmogorov backward equation for an initial value* $f(\cdot, 0) \in \text{Pol}_{\leq m}(S)$

$$\partial_t f(x, t) = \mathcal{A}f(x, t)$$

*has a real analytic solution for all times* $t \in \mathbb{R}$. *In particular,* $f(\cdot, t) \in \text{Pol}_{\leq m}(S)$. 
Sketch of the proof

(a)⇒(b) - The semigroup

\[ P(\cdot) : \mathbb{R}_+ \to \mathcal{L}(\text{Pol}_{\leq m}(S)) \]  \hspace{1cm} (1)

satisfies for all \( t, s \geq 0 \) the Cauchy functional equation

\[
\begin{cases}
P_{t+s} = P_t P_s, \\
P_0 = \text{Id}
\end{cases}
\]

- Finite dimensionality of \( \text{Pol}_{\leq m}(S) \) and continuity of (1) at \( t = 0 \) imply \( P_t = e^{tA} \).
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(b)⇒(c)  - For \( f \in \text{Pol}_{\leq m}(S) \) the generator is given by

\[
A f = \lim_{t \to 0} \frac{P_t f - f}{t} = \lim_{t \to 0} \frac{e^{tA} f - f}{t} = Af,
\]

which is obviously well defined with respect to \( \| \cdot \|_{\text{Pol}_{\leq m}} \) and in \( \text{Pol}_{\leq m}(S) \).
Sketch of the proof

(c) ⇒ (d) - Since $\mathcal{A}$ maps $\text{Pol}_{\leq m}(S)$ to itself, the Kolmogorov backward equation can be understood as a linear ODE in the classical sense whose solution is $e^{t\mathcal{A}}f(\cdot,0)$. 
Sketch of the proof

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(d)⇒(a) - For any initial value $f$ in an appropriate Banach space, $P_t f$ is the unique solution of the Kolmogorov backward equation.
- On $\text{Pol}_{\leq m}(S)$ it must therefore be equal to $e^{tA}f$.
  $\Rightarrow P_t f \in \text{Pol}_{\leq m}(S) \Rightarrow X$ is $m$-polynomial.
Corollary

Let $X$ be an $m$-polynomial process with semigroup $(P_t)$, continuous at $t = 0$ and let $f \in \text{Pol}_{\leq m}(S)$ be fixed. Then there exists a unique function $Q : \mathbb{R} \times S \rightarrow \mathbb{R}$, being real analytic in time and $Q(t, \cdot) \in \text{Pol}_{\leq m}(S)$ for all $t \in \mathbb{R}$, such that

(a) $Q(0, x) = f(x)$ and

(b) $Q(t - s, X_s)$ is a martingale for $s \geq 0$.

Moreover, $Q(-s, X_s)$ is a time space harmonic function for the $m$-polynomial process $X$. 
Feller semimartingales

Aim: Find sufficient conditions for \( m \)-polynomial processes in terms of the infinitesimal generator of Feller processes.

- Conservative Feller semigroup \((P_t)\) with \( C^\infty_c(S) \subset D(A) \).
- There exist functions \( a_{kl}, b_k \) and a kernel \( K(x, d\xi) \) such that for \( u \in C^2_c(S) \) the infinitesimal generator \( A \) is given by

\[
Au(x) = \frac{1}{2} \sum_{k,l=1}^{n} a_{kl}(x) \frac{\partial^2 u(x)}{\partial x_k \partial x_l} + \sum_{k=1}^{n} b_k(x) \frac{\partial u(x)}{\partial x_k} \\
+ \int_{\mathbb{R}^n \setminus \{0\}} \left( u(x + \xi) - u(x) - \sum_{k=1}^{n} \chi_k(\xi) \frac{\partial u(x)}{\partial x_k} \right) K(x, d\xi).
\]

(2)

- \( \chi : \mathbb{R}^n \to \mathbb{R}^n \) some truncation function.
- The parameters satisfy admissibility conditions guaranteeing the existence of the process in \( S \).
Feller semimartingales

- If $X$ is additionally a semimartingale, then its characteristics $(B, C, \nu)$ associated with the truncation function $\chi(\xi)$ are given by

$$B_t = \int_0^t b(X_s) \, ds, \quad C_t = \int_0^t a(X_s) \, ds,$$

$$\nu(dt, d\xi) = K(X_t, d\xi) \, dt.$$

- $(b, a, K)$ are referred as differential characteristics of $X$ (see Kallsen [5]).

⇒ Specify the form of $a$, $b$ and $K$ such that $A$ generates an $m$-polynomial process.
Conditions on the kernel $K(x, d\xi)$

**Condition A**

The kernel $K(x, d\xi)$ is of the form

$$K(X_t, d\xi) := \mu_{00}(d\xi) + \sum_{i \in I} X_{s,i} \mu_{i0}(d\xi) + \sum_{(i,j) \in J} X_{s,i} X_{s,j} \mu_{ij}(d\xi),$$

where all $\mu_{ij}$ are Lévy measures on $\mathbb{R}^n$ with

$$\int_{\|\xi\| > 1} \|\xi\|^m \mu_{ij}(d\xi) < \infty.$$ 

The index sets $I$ and $J$ are defined by $I = \{1 \leq i \leq n | S_i \subseteq \mathbb{R}_+ \}$ and $J = \{(i,j), i \leq j | S_i \times S_j \subseteq \mathbb{R}_+^2 \text{ or } S_i \times S_j \subseteq \mathbb{R}_-^2 \}$, where $S_i$ stands for the projection on the $i^{th}$ component.
Conditions on the kernel $K(x, d\xi)$

**Condition B**

The kernel $K(x, d\xi)$ satisfies

$$K(X_t, d\xi) := g^X_t \mu(d\xi),$$

where for each $x \in S$, $g^x_* \mu$ denotes the pushforward of the measure $\mu$ under the map $g^x$. Moreover, $g^x(y) = g(x, y)$ is affine in $x$, that is

$$g : S \times \mathbb{R}^d \to \mathbb{R}^n, \ (x, y) \mapsto H(y)x + h(y),$$

where $H : \mathbb{R}^d \to \mathbb{R}^{n \times n}$ and $h : \mathbb{R}^d \to \mathbb{R}^n$ some measurable functions. Furthermore, $\mu$ is a Lévy measure on $\mathbb{R}^d$ integrating

$$\int_{\mathbb{R}^d \setminus \{0\}} (\|H(y)\|^k + \|h(y)\|^k) \mu(dy) \text{ for } 1 \leq k \leq m.$$
Conditions for polynomial Feller semimartingales

Theorem (2)

Let $m \geq 2$ and $X^x_t$ be a Feller semimartingale on $S$ whose infinitesimal generator on $C^\infty_c(S)$ is of form (2). Assume furthermore that

$$\mathbb{E} \left[ \| (X^x_t) \|^m \right] < \infty \text{ for all } t \in [0, 1].$$

Then, $X$ is $m$-polynomial if its differential characteristics $(b, a, K)$ associated with the “truncation function” $\chi(\xi) = \xi$ are of the form

$$b_t = b + \sum_{i=1}^n X_{t,i} \beta_i, \quad b, \beta_i \in \mathbb{R}^n,$$

$$a_t = a + \sum_{i=1}^n X_{t,i} \alpha_{i0} + \sum_{i \leq j} X_{t,i} X_{t,j} \alpha_{ij}, \quad a, \alpha_{ij} \in \mathbb{R}^{n \times n},$$

with $K$ satisfying either Condition A or Condition B.
Remark on the truncation function

The characteristic $b$ must be adapted to the choice of the truncation function $\chi$:

$$
\left( b_t(\chi(\xi)) + \int_{\mathbb{R}^n \setminus \{0\}} (\xi - \chi(\xi)) K(X_t, d\xi) \right) \in \text{Pol}_{\leq 1}(S)
$$

is an equivalent condition guaranteeing that $X$ is $m$-polynomial.
Sketch of the proof of Theorem (2)

- Denote the right side of the integro-differential operator (2) literally by $\mathcal{A}^\#$.
- Observe that under the above conditions $\mathcal{A}^\# f \in \text{Pol}_{\leq m}(S)$ for every $f \in \text{Pol}_{\leq m}(S)$.
- Showing that the process

$$M_t^f := f(X_t^x) - f(x) - \int_0^t \mathcal{A}^\# f(X_s^x) \, ds$$

is a well-defined martingale for every $f \in \text{Pol}_{\leq m}(S)$ yields $\mathcal{A} = \mathcal{A}^\#$ on $\text{Pol}_{\leq m}(S)$.
- Theorem (1) then yields the assertion.
Affine processes

⇒ Every conservative affine process $X$ on $S = \mathbb{R}_+^p \times \mathbb{R}^{n-p}$ (see Duffie et al. [1]) is $m$-polynomial if $\mathbb{E} \left[ \| (X^x_t) \|^m \right] < \infty$ for $t \in [0, 1]$. 
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Proof.

- Every conservative affine process is a Feller semimartingale.
- On $C_c^2(S)$, the generator of an affine process is given by

$$
\mathcal{A}u(x) = \frac{1}{2} \sum_{k,l=1}^n \left( a_{kl} + \sum_{i=1}^p x_i \alpha_{i0,kl} \right) \frac{\partial^2 u(x)}{\partial x_k \partial x_l} + \left\langle b + \sum_{i=1}^n x_i \beta_i, \nabla u(x) \right\rangle \\
+ \int_{S \setminus \{0\}} \left( u(x + \xi) - u(x) - \langle \chi(\xi), \nabla u(x) \rangle \right) \left( \mu_{00}(d\xi) + \sum_{i=1}^p x_i \mu_{i0}(d\xi) \right).
$$


Lévy driven SDEs

- $L_t$: Lévy process on $\mathbb{R}^d$ with triplet $(b, a, \mu)$.

- $V_1, \ldots, V_d$: affine functions $V_i : S \to \mathbb{R}^n$, $x \mapsto H_i x + h_i$ with $H_i \in \mathbb{R}^{n \times n}$ and $h_i \in \mathbb{R}^n$.

$\Rightarrow$ A process $X$, which solves the stochastic differential equation of type

$$dX_t = \sum_{i=1}^{d} V_i(X_{t-})dL^i_t, \quad X_0 = x \in S,$$

in $S$, is $m$-polynomial $(m \geq 2)$ as soon as $L$ admits finite $m^{th}$ moment, that is $\int_{\|\xi\| > 1} \|\xi\|^m \mu(d\xi) < \infty$.

**Remark**

- *The moment condition $\mathbb{E} \left[ \|X^t\|^m \right] < \infty$ for $t \in [0, 1]$ is automatically satisfied due to the assumption on $\mu$.***
Exponential Lévy models

- $L_t$: Lévy process on $\mathbb{R}$ with triplet $(b, a, \mu)$.

$\Rightarrow$ An exponential Lévy model $X_t^x = xe^{Lt}$ is $m$-polynomial if

$$\int_{|y|>1} e^{my} \mu(dy) < \infty.$$
Exponential Lévy models

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⇒ An exponential Lévy model $X_t^x = xe^{Lt}$ is $m$-polynomial if $\int_{|y|>1} e^{my} \mu(dy) < \infty$.

Sketch of the proof

- The infinitesimal generator is given by

$$
\mathcal{A}u(x) = \frac{ax^2}{2} \frac{d^2 u(x)}{dx^2} + \left( b + \frac{a}{2} + \int_{\mathbb{R}} (e^y - 1 - \chi(y)) \mu(dy) \right) x \frac{du(x)}{dx} \\
+ \int_{\mathbb{R}} \left( u(xe^y) - u(x) - x (e^y - 1) \frac{du(x)}{dx} \right) \mu(dy).
$$

- We are in the situation of Condition B with $g(x, y) = H(y)x = (e^y - 1)x$. 
Jacobi process

- The Jacobi process is the solution of

\[ dX_t = -\beta (X_t - \theta) dt + \sigma \sqrt{X_t(1 - X_t)} dB_t, \quad X_0 = x \in [0, 1], \]

on \( S = [0, 1] \), where \( \theta \in [0, 1] \) and \( \beta, \sigma > 0 \) (see for example Gourieroux [4]).
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Extension by adding jumps:

- Jump times correspond to those of a Poisson process \( N_t \) with intensity \( \lambda \).
- Jump size \( h(x) = 1 - 2x \), i.e. if a jump occurs, the process is reflected at \( \frac{1}{2} \).

\[
dX_t = -\beta(X_t - \theta)dt + \sigma \sqrt{X_t(1 - X_t)} dB_t + (1 - 2X_t) dN_t.
\]

Generator:

\[
\mathcal{A}u = \frac{1}{2} \sigma^2 (x(1 - x)) \frac{d^2 u(x)}{dx^2} - \beta(x - \theta) \frac{du(x)}{dx} + \lambda(u(1 - x) - u(x)).
\]

In terms of Condition B, we have here \( g(x, y) = -2yx + y \) and \( \mu(dy) = \lambda \delta_1(dy) \).
Dunkl process

- Dunkl processes (see for example Gallardo and Yor [3]) are Feller processes parametrized by \( k \geq 0 \) whose infinitesimal generator for \( u \in C^2(\mathbb{R}) \) is given by

\[
Au = \frac{1}{2} \frac{d^2 u(x)}{dx^2} + k \left( \frac{1}{x} \frac{du(x)}{dx} - \frac{u(x) - u(-x)}{2x^2} \right).
\]

- For every \( f \in \text{Pol}_{\leq m}(\mathbb{R}) \), we therefore have \( Af \in \text{Pol}_{\leq m}(\mathbb{R}) \).
- Dunkl processes lie in the class of polynomial processes.
- Example that the conditions of Theorem (2) are only sufficient and not necessary.
Moment calculation

- There exists a linear map $A$ such that moments of $m$-polynomial processes can simply be calculated by computing $e^{tA}$.
- Choose a basis $\langle e_1, \ldots, e_N \rangle$ of $\text{Pol}_{\leq m}(S)$.
- $A = (a_{ij})_{i,j=1,\ldots,N}$ is obtained by

$$A e_i = \sum_{j=1}^{N} a_{ij} e_j.$$

- Writing $f$ as $\sum_{k=1}^{N} \alpha_k e_k$, yields

$$P_t f = (\alpha_1, \ldots, \alpha_N) e^{tA}(e_1, \ldots, e_N)^\top.$$
Generalized Method of Moments

- Let $\theta \in \Theta \subseteq \mathbb{R}^p$ be the vector of parameters to be estimated and $g : S \times \Theta \to \mathbb{R}^q$ a function such that $\mathbb{E}[g(X_t, \theta_0)] = 0$ for the true value of the parameter $\theta_0$.

- The Generalized Method of Moments estimator is the value of $\theta$ which minimizes
  
  
  $Q_T(\theta) = \left( \sum_{t=1}^T g(X_t, \theta) \right)^\top W_T \left( \sum_{t=1}^T g(X_t, \theta) \right),$

  where $W_T$ is a positive semi-definite $q \times q$ matrix.

- A typical moment condition is given by
  
  
  $g(X_t, \theta) = \begin{pmatrix} X_t^{n_1} X_{t+s}^{m_1} - \mathbb{E}[X_t^{n_1} X_{t+s}^{m_1}] \\ \vdots \\ X_t^{n_q} X_{t+s}^{m_q} - \mathbb{E}[X_t^{n_q} X_{t+s}^{m_q}] \end{pmatrix}, \quad n_i, m_i \in \mathbb{N}.$

- $\mathbb{E}[X_t^n X_{t+s}^m] = \mathbb{E}[X_t^n \mathbb{E}[X_{t+s}^m | X_t]]$ can be easily computed (see Zhou [6], Forman [2] in the one dimensional diffusion case).
European option pricing - setting

- \( X \): \( m \)-polynomial process.
- \( G : S \to \mathbb{R}^n \) deterministic measurable map such that the price processes are given through

\[
S_t = G(X_t)
\]

under a martingale measure.

- \( F = \phi(S_T) \): bounded measurable European claim for some maturity \( T > 0 \) whose price at \( t \geq 0 \) is given by

\[
p_t^F = \mathbb{E}[\phi(S_T)|\mathcal{F}_t] = \mathbb{E}[(\phi \circ G)(X_T)|\mathcal{F}_t].
\]
Analytically tractable claims

- Claims of the form $F = f \circ G^{-1}(S_T)$ for $f \in \text{Pol}_{\leq m}(S)$ are analytically tractable:

$$
p_t^F = \mathbb{E}[(f \circ G^{-1})(S_T)|\mathcal{F}_t] = P_{T-t}f(G^{-1}(S_t))
= e^{(T-t)A}f(G^{-1}(S_t))
$$

for $0 \leq t \leq T$.

- The sensitivities of the price process with respect to the factors of $X$ can be calculated by

$$
\nabla p_t^F = \nabla P_{T-t}f(G^{-1}(S_t))\nabla G^{-1}(S_t).
$$
Variance reduction for Monte Carlo simulation using polynomial control variates

- $X^1, \ldots, X^L$: $L$ sample paths of the polynomial process $X$.
- Standard estimator in Monte Carlo simulation:
  \[ \pi_0^F = \frac{1}{L} \sum_{i=1}^{L} (\phi \circ G)(X^i_T). \]
- Approximation of $\phi \circ G$ by a polynomial $f$, $\mathbb{E}[f(X_T)]$ is explicitly known.
- Estimator with control variate $f$:
  \[ \hat{\pi}_0^F = \frac{1}{L} \sum_{i=1}^{L} \left( (\phi \circ G)(X^i_T) - (f(X^i_T) - \mathbb{E}[f(X_T)]) \right). \]

  \[ \Rightarrow \hat{\pi}_0^F \xrightarrow{L \to \infty} \rho_0^F \text{ and } \text{Var}(\hat{\pi}_0^F) < \text{Var}(\pi_0^F). \]
Method illustration: exponential Lévy model

- Price process: \( S_t = S_0 e^{X_t} \).
- \( X_t \): Lévy process with Lévy triplet \((b, a, \nu)\).
- \( r \): interest rate, \( e^{-rt} S_t \) is a martingale.

\[ \Rightarrow \int_{|y|>1} e^y \nu(dy) < \infty \text{ and } b = r - \frac{a}{2} - \int_{\mathbb{R}} \left( e^y - 1 - y 1_{|y| \leq 1} \right) \nu(dy). \]

- Generator of the Lévy process:

\[
\mathcal{A} u(x) = \frac{a}{2} \frac{d^2 u(x)}{dx^2} + \left( r - \frac{a}{2} \right) \frac{du(x)}{dx} \\
+ \int_{\mathbb{R}} \left( u(x + y) - u(x) - (e^y - 1) \frac{du(x)}{dx} \right) \nu(dy).
\]

- Example: constant jump intensity \( \lambda \), exponential jumps size distribution with parameter \( \frac{1}{c} \).
- Applying \( \mathcal{A} \) to \((x^0, x^1, \ldots, x^m)\) yields the following \((m + 1) \times (m + 1)\) matrix
Method illustration: exponential Lévy model

\[ A = \begin{pmatrix} 0 & \ldots & \\ \kappa & 0 & \ldots \\ a + c^2 & 2\kappa & 0 & \ldots \\ 6c^3 & 3(a + c^2) & 3\kappa & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots \\ m!c^m & \ldots & \ldots & \frac{m!}{(m-i)!}c^i & \ldots & \frac{m(m-1)}{2}(a + c^2) & m\kappa & 0 \end{pmatrix}, \]

where \( \kappa = r - \frac{a}{2} - \frac{c^2}{1-c}. \)

\[ \Rightarrow \mathbb{E}[\sum_{k=0}^{m} \alpha_k (X_t^x)^k] = (\alpha_0, \ldots, \alpha_m)e^{tA}(x^0, \ldots, x^m)^{\top}. \]

- Approximate \((S_0e^x - K)^+\) by a polynomial \(f\).
- Use \(f\) as control variate for variance reduction in the MC simulation.
Method illustration: exponential Lévy model

- Comparison: Monte Carlo simulation for European call prices with and without variance reduction.
- Ratio of the variance of the uncontrolled estimator to that of the controlled estimator: around 100 (depending on the polynomial approximation).
Example: Heston type model with volatility dependent jumps (Bates 2000)

- Price process: $S_t = S_0 e^{X_t}$, where
  
  
  $$dX_t = \left( r - \frac{V_t}{2} - \lambda V_t \int_{\mathbb{R}} (e^y - 1) F(\text{d}y) \right) dt + \sqrt{V_t} \text{d}W_t^1 + \text{d}J_t(V_t),$$
  
  $$dV_t = -\beta (V_t - \theta) + \sigma \sqrt{V_t} \text{d}W_t^2.$$  

- $J_t = \sum_{i=1}^{N_t} Z_i$, $(Z_i)_{i \geq 1}$ i.i.d sequence of random variables with probability distribution $F$.
- $N_t$: Poisson process with intensity $\lambda V_t$.
- Correlation between Brownian motions $\rho$. 

Example: Heston type model with volatility dependent jumps (Bates 2000)

Usual method:

- Solve the Ricatti equations:

\[
\begin{align*}
\partial_t \phi(t, x, v) &= r x + \beta \theta \psi(t, x, v), \\
\partial_t \psi(t, x, v) &= \frac{1}{2} (x^2 - x) - \beta \psi(t, x, v) + \frac{1}{2} \sigma^2 \psi^2(t, x, v) \\
&\quad + \rho \sigma x \psi(t, x, v) \\
&\quad + \lambda \left( \int_\mathbb{R} (e^{xy} - 1) dF(y) - x \int_\mathbb{R} (e^y - 1) dF(y) \right).
\end{align*}
\]

- Analytic solutions for \( \psi \) might be difficult to find \( \Rightarrow \) Numerical solutions must be used.
- Application of Fourier pricing methods to obtain the European call price.
Example: Heston type model

- Comparison: Monte Carlo simulation for European call prices with and without variance reduction.
- Ratio of the variance of the uncontrolled estimator to that of the controlled estimator: between 300 and 400 (depending on the polynomial approximation).
Example: Basket option under a Heston type model

- Payoff function: \( e^{-rT}(S^1_T + S^2_T - K)^+ \).
- Price processes: \( S^1_t = S^1_0 e^{X_t} \) and \( S^2_t = S^2_0 e^{Y_t} \).

\[
dX_t = \left( r - \frac{V_t}{2} - \lambda^1 V_t \int_{\mathbb{R}} (e^y - 1)F^1(dy) \right) dt + \sqrt{V_t} dW^1_t + dJ^1_t(V_t),
\]
\[
dY_t = \left( r - \frac{\gamma^2 V_t}{2} - \lambda^2 V_t \int_{\mathbb{R}} (e^y - 1)F^2(dy) \right) dt + \gamma \sqrt{V_t} dW^2_t + dJ^2_t(V_t),
\]
\[
dV_t = -\beta (V_t - \theta) + \sigma \sqrt{V_t} dW^3_t.
\]

- \( J^j_t = \sum_{i=1}^{N^j_t} Z^j_i \), \( (Z^j_i)_{i \geq 1} \) i.i.d sequences of random variables with probability distribution \( F^j \), \( j = 1, 2 \).
- \( N^j_t \): Poisson processes with intensity \( \lambda^j V_t \).
- Brownian motions are correlated.
Example: Basket option under a Heston type model

- Comparison: Monte Carlo simulation for European call prices with and without variance reduction.
- Ratio of the variance of the uncontrolled estimator to that of the controlled estimator: around 150 (depending on the polynomial approximation).

![Graph showing comparison between Monte Carlo and variance reduction methods for basket option pricing.](image-url)
Conclusion

Under certain moment conditions, processes with
- affine drift,
- quadratic diffusion function and
- jump compensator which is either quadratic or the pushforward of a Lévy measure under an affine function,
are polynomial processes.

The calculation of moments of polynomial processes only requires the computation of matrix exponentials.

Polynomial claims can therefore be priced analytically, which leads to variance reduction techniques by using polynomial control variates approximating the payoff function well.
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Thank you for your attention!