Forward Dynamic Utility

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with the financial support of
the "Fondation du Risque"
and the Fédération des banques Françaises

18 May 2009

Plan

1. Utility forward
   - Framework and definition
2. Forward Stochastic Utilities
   - Définition
3. Non linear Stochastic PDE
   - Utility Volatility
4. Change of numeraire

Investment Banking and Utility Theory I

Some remarks on utility functions and their dynamic properties
from M.Musiela, T.Zariphopoulo, C.Rogers +alii (2005-2009)

- No clear idea how to specify the utility function
- Classical or recursive utility are defined in isolation to the investment opportunities given to an agent
- Explicit solutions to optimal investment problems can only be derived under very restrictive model and utility assumptions - dependence on the Markovian assumption and HJB equations
- In non-Markovian framework, theory is concentrated on the problem of existence and uniqueness of an optimal solution, often via the dual representation of utility.

Investment Banking and Utility Theory II

Main Drawbacks

- Not easy to develop pratical intuition on asset allocation
- Creates potential intertemporal inconsistency

The classical formulation I

Different steps

1. Choose a utility function, $U(x)$ (concave et strictly increasing) for a fixed investment horizon T
2. Specify the investment universe, i.e. the dynamics of assets would be traded, and investment constraints.
3. Solve for a self-financing strategy selection which maximizes the expected utility of the terminal wealth
4. Analyze properties of the optimal solution

Shortcomings I

Intertemporality

1. The investor may want to use intertemporal diversification, i.e., implement short, medium and long term strategies
2. Can the same utility function be used for all time horizons ?
3. No- in fact the investor gets more value (in terms of the value function) from a longer term investment.
4. At the optimum the investor should become indifferent to the investment horizon. .

Dynamic programming and Intertemporality I

1. In the classical formulation the utility refers to the utility for the last rebalancing period
2. The mathematical version is the Dynamic programming principle (in Markovian setup for simplicity) :
   Let V(t,x,U,T) be the maximal expected utility for a initial wealth x at time t, and a terminal utility function $U(x, T)$, then

   $$V(t, x, U, T) = V(t, x, V(t+h, ., U, T), t+h)$$

   The value function $V(t+h, ., U, T)$ is the implied utility for the maturity $t+h$

Dynamic programming and Intertemporality II

3. To be indifferent to investment horizon, it needs to maintain a intertemporal consistency
4. Only at the optimum the investor achieves on the average his performance objectives. Sub optimally he experiences decreasing future expected performance.
5. Need to be stable with respect of classical operation in the market as change of numéraire.

Forward Dynamic Utility

Investment Universe I

- Asset dynamics

$$d\xi_t^i = \xi_t^i[b_t^i dt + \sum_{i=1}^{d} \sigma_t^{i,j} dW_t^j], \qquad d\xi_t^0 = \xi_t^0 r_t dt$$

- Risk premium vector, $\eta(t)$ with $b(t) - r(t)\mathbf{1} = \sigma_t \eta(t)$
- Self-financing strategy starting from x at time r

$$dX_t^\pi = r_t X_t^\pi dt + \pi_t^* \sigma_t (dW_t + \eta_t dt) \quad , X_r^\pi = x$$

- The set of admissible strategies is a vector space (cone) denoted by $\mathcal{A}$.

Classical optimization problem I

Classical problem

Given a utility function $U(T, x)$, maximize :

$$V(r, x) = sup_{\pi \in \mathcal{A}} \mathbf{E}(U(X_T^\pi)) \tag{1}$$

- The choice of numéraire is not really discussed
- Backward problem since the solution is obtained by recursive procedure from the horizon.

In the forward point of view, a given utility function is randomly diffused, but with the constrained to be at any time a utility function.

Forward Utility I

Definition (**Forward Utility**)

A forward dynamic utility process starting from the given utility $U(r,x)$, is an adapted process $u(t,x)$ s.t.

i) Concavity assumption $u(r,.) = U(r.)$, and for $t \geq r$, $x \mapsto u(t,x)$ is increasing concave function,

ii) Consistency with the investment universe For any admissible strategy $\pi in \mathcal{A}$

$$\mathbb{E}_{\mathbb{P}}(u(t, X_t^{\pi})/\mathcal{F}_s) \leq u(s, X_s^{\pi}), \ \forall s \leq t$$

or equivalently $(u(t, X_t^{\pi}); t \geq r)$ is a supermartingale.

Definition

iii) Existence of optimal There exists an optimal admissible self-financing strategy π^*, for which the utility of the optimal wealth is a martingale :

$$\mathbb{E}_{\mathbb{P}}(u(t, X_t^{\pi^*})/\mathcal{F}_s) = u(s, X_s^{\pi^*}), \ \forall s \leq t$$

iv) In short for any admissible strategy, $u(t, X_t^{\pi})$ is a supermartingale, and a martingale for the optimal strategy π^* and then :
$u(r, x)$ is the value function of the optimization program with terminal random utility function $u(T, x)$,

$$u(r, x) = \sup_{\pi \in \mathcal{A}(r,x)} \mathbf{E}(u(T, X_T^{r,x,\pi})/\mathcal{F}_r), \ \forall T \geq r$$

where $\mathcal{A}(r, x)$ is the set of admissible strategies

Change of probability in standard utility function I

Let v be $\mathcal{C}^2$- utility function and Z a positive semimartingale, with drift λ_t and volatility γ_t.

Change of probability

Let u be the adapted process defined by $u(t,x) \stackrel{def}{=} Z_t v(x)$.
$u(t,x)$ is an adapted concave and increasing random field

- Consistency with Investment Universe The supermartingale property for $u(t,X_t^\pi)$ holds true when Z is the discounted density of martingale measure $H_t = \exp(-\int_0^t (r_s ds + \eta_s^* dW_s + \frac{1}{2}||\eta_s||^2 ds)$. or the discounted density of any equivalent martingale measure.

The condition is not necessary, since by standard calculation, if

$$\frac{v}{xv_x(t,x)}\mu_t + r_t - \frac{v_x(t,x)}{2xv_{xx}(t,x)}||\mathrm{Proj}_{\mathcal{A}}(\eta_t+\gamma_t)||^2 = 0$$

The property holds true

- If $v(x) = x^{1-\alpha}/1-\alpha$ (Power utility) and $\mu_t/(\alpha-1) + r_t - \frac{1}{2\alpha}||||\eta_t+\gamma_t||_{\mathcal{A}}||^2$, then u is a forward utility.
- If $v(x) = \exp -c\,x$ is a forward utility if $r=0$,and $\mu_t = \frac{1}{2}||||\eta_t+\gamma_t||_{\mathcal{A}}||^2$
- In the other cases, the martingale is the only solution....

Change of Numéraire I

Let Y a positive process with return α_t and volatility δ_t.

Change of numeraire

Let u be $u(t,x) \stackrel{def}{=} v(x/Y_t)$. $u(t,x)$ is an adapted concave and increasing random field

- The supermartingale property holds true if Y is the inverse of discounted density of martingale measure, known as Market Numéraire, or Growth optimal portfolio.
- We have $r_t = \alpha_t - < \delta_t, \eta_t >, \quad , \eta - \delta \in (\mathcal{K}\sigma_t)^{\perp}, \delta \in (\mathcal{K}\sigma_t)$
- By Itô's formula, the volatility of the forward utility is $\Gamma(t,x) = -x\, u_x(t,x)\delta$

Markovian case I

We first consider the Markovian case where

- all parameters are functions of the time and of the state variables.
- The diffusion generator is the elliptic operator $\mathcal{L}^{\xi}$ w.r. ξ.
- Admissible portfolios are stable w. r. to the initial condition

$$X_{t+h}^{r,x,\pi} = X_{t+h}^{t,X_t^{r,x,\pi},\pi}, \quad \pi \in \mathcal{A}(t, X_t^{r,x,\pi})$$

What is HJB equation for Markovian forward utility ?

Example (HJB PDE)

Let $u(t,.,\xi)$ be a Markov forward utility with initial condition $u(r,x)$, concave w.r.t. to x. Then

$$\begin{aligned} u_t(t,x,\xi) &+ \mathcal{L}^{\xi}u(t,x,\xi) \\ &+ \mathbf{H}(t,x,\xi,u',u'',\Delta^{\sigma}_{x,\xi}u)(t,x)) = 0 \end{aligned}$$

- The Hamiltonian is defined for $w < 0$ by

$$\mathbf{H}(t,x,\xi,p,p',w) = \sup_{\pi\in\mathcal{A}_t} \left(< \sigma^*\pi, p\eta + p' > +1/2\,\pi^*\sigma\sigma^*\pi \right.$$

- $\mathbf{H}(t,x,\xi,p,p',w) = -\frac{1}{2w}||\mathrm{Proj}_{\mathcal{K}_t}(\eta p + p')||^2$ is the Hamiltonian taken at the optimal

Optimal Portfolio and Volatility I

Optimal portfolio

$$\tilde{\pi}\tilde{\sigma}(t,x,\xi) = -\frac{1}{u_{xx}(t,x,\xi)}\mathrm{Proj}_{\mathcal{K}_t}(\eta u' + \Delta^{\sigma}_{x,\xi}u).$$

Volatility Parameters The utility volatility is $\Gamma(t,x,\xi_t)$ is

$$\Gamma(t,x,\xi) = (\nabla_{\xi}u(t,x,\xi))^{*}\sigma(t,\xi), \quad \Gamma_x = \frac{\partial}{\partial x}\Gamma^{u}.$$

Theorem (Non Linear Dynamics, $u(r,x) = U(r,x)$)

$$du(t,x,\xi_t) = \frac{||\mathrm{Proj}_{\mathcal{K}_t}(u_x(t,x)\eta_t + \Gamma_x(t,x,\xi_t))||^2}{2u_{xx}(t,x,\xi_t)}dt + \Gamma(t,x,\xi_t)dW_t$$

Stochastic PDE I

In the case of forward utility, we apply Itô-Ventzell-Kunita to the random field $u(t,x)$ in place of Ito formula.

Theorem

The general case *:Drift Constraint*

Assume that $du(t,x) = \beta(t,x)dt + \Gamma(t,x)dW_t, \quad u(r,x) = u(x)$, *then*

$$\beta(t,x) = u_x(t,x).\frac{u_x(t,x)}{2u_{xx}(t,x)}||\mathrm{Proj}_{\mathcal{K}_t}(\eta_t + \frac{\Gamma_x(t,x)}{u_x(t,x)})||^2$$

Open Questions ? I

- What about the volatility of the utility ?
- Under which assumptions, how can be sure that solutions are concave and increasing, with Inada condition and asymptotic elasticity constraint.

Decreasing forward Utility I

Zariphopoulo, C.Rogers and alii

Decreasing forward Utility II

Theorem

Assume the volatility $\forall t, \ \forall x \Gamma(t,x) = 0$. *Then u is decreasing in time,*

$$du(t,x) = \frac{u_x^2}{2u_{xx}(t,x)} ||\eta_t||^2 dt$$

- *u is a forward utility iff there exist C and* ν *, a finite measure with support in* $[0,+\infty)$ $(\nu(0)=0)$*, such that the Fenchel transform of u,* $v(t,x)$ *verifies*

$$u(t,y) = \int \frac{1}{1-r}(1 - y^{1-r} e^{\frac{r(1-r)}{2} \int_0^t |\eta_s|^2 ds} \nu(dr) + C$$

- *This result is based on the result of Widder (1963) characterizing positive space-time harmonic function.*

The new market I

New "hat" equations

$$d\hat{X}^\pi_t = [\gamma^*_t \eta_t]\hat{X}^\pi_t dt + [\frac{\pi^*_t \sigma_t}{y_t} - \hat{X}^\pi_t \gamma_t]^*(dW_t + (\eta_t - \gamma_t)dt)$$

$$\frac{d\hat{\xi}^i_t}{\hat{\xi}^i_t} = b^i_t dt + (\sigma^i_t - \gamma_t)^*(dW_t - \gamma_t dt)\ 0 \le i \le d$$

Let $\bar{\xi}$ be $(\hat{\xi}, y)$ et par $\bar{\sigma}$ la matrice$((\sigma^i - \gamma)_{i=1..d}, \gamma)$, et on supposera que les utilités forward dans ce marché sont fonctions régulières du temps t, de la richesse $\hat{x}$ et de $\bar{\xi}$

Change of numéraire I

Let $y > 0$ be a new numéraire such that $\frac{dy_t}{y_t} = \gamma_t dW_t$

In the new market,

$$\hat{X}_t^\pi := \frac{X_t^\pi}{y_t},\ \hat{\xi}_t^i := \frac{\xi_t^i}{y_t}$$

and

$$\frac{d\hat{\xi}_t^i}{\hat{\xi}_t^i} = b_t^i dt + (\sigma_t^i - \delta_t)^*(dW_t - \delta_t dt)$$

$$d\hat{X}_t^\pi = [\delta_t^* \eta_t]\hat{X}_t^\pi dt + [\frac{\pi_t^* \sigma_t}{y_t} - \hat{X}_t^\pi \gamma_t]^*(dW_t + (\eta_t - \delta_t)dt)$$

Change of numéraire II

Theorem (Stability by change of numeraire)

Let u(t,x) be a forward utility and Y_t a numeraire.
Then $\hat{u}(t, \hat{x}) = u(t, x/Y_t)$ is a forward utility with the investment universe associated with the change of numeraire (X_t/Y_t), with initial condition $\hat{u}(0, \hat{x}) = u(0, y\hat{x})$

Volatility Interpretation I

By change of numéraire, we can still assume that the market has no risk premium.
The volatility of u may the optimization still no trivial.

Theorem (Volatility and risk premium)

With the market numeraire as numeraire, the volatility of the forward utility sans prime de risque, is the transform of the first utility. The ration $\frac{\Gamma_x}{u_x}$ play the rôleof a risk premium associated with the wealth x at time t.

Change of numeraire argument permits also to characterize forward utility with given optimal portfolio (Work in progress) thank you for your attention

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