Stochastic Target Problems with controlled Loss

B. Bouchard, R. Elie, C. Imbert *
and N. Touzi†

*CREST and Ceremade, Paris-Dauphine
†CMAP, Polytechnique

Motivation

- **Stock price:** (with large investor's strategy π)

$$\frac{dS^{\pi}(u)}{S^{\pi}(u)} = \mu\left(u, S^{\pi}(u), \pi_u\right) du + \sigma\left(u, S^{\pi}(u), \pi_u\right) dW_u$$

Motivation

- **Stock price:** (with large investor's strategy π)

$$\frac{dS^{\pi}(u)}{S^{\pi}(u)} = \mu\left(u, S^{\pi}(u), \pi_u\right) du + \sigma\left(u, S^{\pi}(u), \pi_u\right) dW_u$$

- **Wealth process:** (risk free interest rate $r = 0$)

$$dX^{\pi}(u) = \pi_u \frac{dS^{\pi}(u)}{S^{\pi}(u)} = \pi_u \left[\mu\left(u, S^{\pi}(u), \pi_u\right) du + \sigma\left(u, S^{\pi}(u), \pi_u\right) dW_u\right]$$

Motivation

- **Stock price:** (with large investor's strategy π)

$$\frac{dS^\pi(u)}{S^\pi(u)} = \mu\left(u, S^\pi(u), \pi_u\right) du + \sigma\left(u, S^\pi(u), \pi_u\right) dW_u$$

- **Wealth process:** (risk free interest rate $r = 0$)

$$dX^\pi(u) = \pi_u \frac{dS^\pi(u)}{S^\pi(u)} = \pi_u \left[\mu\left(u, S^\pi(u), \pi_u\right) du + \sigma\left(u, S^\pi(u), \pi_u\right) dW_u\right]$$

- **Super Hedging problem of claim $g(S^\pi(T))$:**

$$v(0, S_0) := \inf\left\{x \geq 0 : \; X_x^\pi(T) \geq g\left(S^\pi(T)\right) \; \mathbb{P}-\text{ps}, \quad \text{for some } \pi \in \mathcal{A}\right\} .$$

Motivation

- **Stock price:** (with large investor's strategy π)

$$\frac{dS^{\pi}(u)}{S^{\pi}(u)} = \mu\left(u, S^{\pi}(u), \pi_u\right) du + \sigma\left(u, S^{\pi}(u), \pi_u\right) dW_u$$

- **Wealth process:** (risk free interest rate $r = 0$)

$$dX^{\pi}(u) = \pi_u \frac{dS^{\pi}(u)}{S^{\pi}(u)} = \pi_u \left[\mu\left(u, S^{\pi}(u), \pi_u\right) du + \sigma\left(u, S^{\pi}(u), \pi_u\right) dW_u\right]$$

- **Super Hedging problem of claim $g(S^{\pi}(T))$:**

$$v(0, S_0) := \inf\left\{x \geq 0 : \; X_x^{\pi}(T) \geq g\left(S^{\pi}(T)\right) \; \mathbb{P}-\text{ps}, \quad \text{for some } \pi \in \mathcal{A}\right\} .$$

- **Quantile Hedging problem:** Given $p \in (0, 1)$, find

$$v(0, S_0; p) := \inf\left\{x \geq 0 : \; \mathbb{P}\left[X_x^{\pi}(T) \geq g\left(S^{\pi}(T)\right)\right] \geq p, \quad \text{for some } \pi \in \mathcal{A}\right\} .$$

Motivation

- **Stock price:** (with large investor's strategy π)

$$\frac{dS^\pi(u)}{S^\pi(u)} = \mu\left(u, S^\pi(u), \pi_u\right) du + \sigma\left(u, S^\pi(u), \pi_u\right) dW_u$$

- **Wealth process:** (risk free interest rate $r = 0$)

$$dX^\pi(u) = \pi_u \frac{dS^\pi(u)}{S^\pi(u)} = \pi_u \left[\mu\left(u, S^\pi(u), \pi_u\right) du + \sigma\left(u, S^\pi(u), \pi_u\right) dW_u\right]$$

- **Super Hedging problem of claim $g(S^\pi(T))$:**

$$v(0, S_0) := \inf \left\{x \geq 0 : \; X_x^\pi(T) \geq g\left(S^\pi(T)\right) \; \mathbb{P}-\text{ps}, \quad \text{for some } \pi \in \mathcal{A}\right\} .$$

- **Quantile Hedging problem:** Given $p \in (0, 1)$, find

$$v(0, S_0; p) := \inf \left\{x \geq 0 : \; \mathbb{P}\left[X_x^\pi(T) \geq g\left(S^\pi(T)\right)\right] \geq p, \quad \text{for some } \pi \in \mathcal{A}\right\} .$$

- **Asset management under Quantile hedging constraint:**

$$\sup_\pi \mathbb{E}\left[U(X_T^{x,\pi})\right] \qquad \text{for } \pi \;\; s.t. \;\; \mathbb{P}\left[X_x^\pi(T) \geq g\left(S^\pi(T)\right)\right] \geq p .$$

Explicit Solution in Complete Market

- **Stock price under the (unique) Risk Neutral Measure $\mathbb{Q}$:**

$$\frac{dS(u)}{S(u)} = \sigma\left(u, S(u)\right) dW_u^{\mathbb{Q}} \quad \text{(independent on } \pi\text{)}$$

Explicit Solution in Complete Market

- **Stock price under the (unique) Risk Neutral Measure $\mathbb{Q}$:**

$$\frac{dS(u)}{S(u)} = \sigma\left(u, S(u)\right) dW_u^{\mathbb{Q}} \quad \text{(independent on } \pi\text{)}$$

- **Wealth process:**

$$dX^{\pi}(u) = \pi_u \sigma\left(u, S(u)\right) \, dW_u^{\mathbb{Q}}$$

Explicit Solution in Complete Market

- **Stock price under the (unique) Risk Neutral Measure $\mathbb{Q}$:**

$$\frac{dS(u)}{S(u)} = \sigma(u, S(u))\, dW_u^{\mathbb{Q}} \quad \text{(independent on } \pi)$$

- **Wealth process:**

$$dX^{\pi}(u) = \pi_u \sigma(u, S(u))\, dW_u^{\mathbb{Q}}$$

- **Problem Reformulation:**

Maximize the Probability of Hedge for a given starting wealth x

$$\Updownarrow$$

$$\max_{\pi \in \mathcal{A}} \mathbb{P}\left[X_x^{\pi}(T) \geq g(S(T))\right]$$

Explicit Solution in Complete Market

- **Problem Reformulation:**

Maximize the Probability of Hedge for a given starting wealth x

$$\Updownarrow$$

$$\max_{\pi \in \mathcal{A}} \mathbb{P}\left[X_x^\pi(T) \geq g\left(S(T)\right)\right]$$

Explicit Solution in Complete Market

- **Problem Reformulation:**

Maximize the Probability of Hedge for a given starting wealth x

$$\Updownarrow$$

$$\max_{\pi \in \mathcal{A}} \mathbb{P}\left[X_x^\pi(T) \geq g\left(S(T)\right)\right]$$

$$\Updownarrow$$

$$\max_{X \in L_T^0} \mathbb{P}\left[X \geq g\left(S(T)\right)\right] \quad \text{under} \quad \mathbb{E}^{\mathbb{Q}}\left[X\right] \leq x$$

Explicit Solution in Complete Market

- **Problem Reformulation:**

Maximize the Probability of Hedge for a given starting wealth x

$$\Updownarrow$$

$$\max_{\pi \in \mathcal{A}} \mathbb{P}\left[X_x^{\pi}(T) \geq g\left(S(T)\right)\right]$$

$$\Updownarrow$$

$$\max_{X \in L_T^0} \mathbb{P}\left[X \geq g\left(S(T)\right)\right] \quad \text{under} \quad \mathbb{E}^{\mathbb{Q}}\left[X\right] \leq x$$

$$A = \{X \geq g\left(S(T)\right)\} \qquad \Updownarrow \qquad X = g\left(S(T)\right)\mathbf{1}_A$$

$$\max_{A \in \mathcal{F}_T} \mathbb{P}\left[A\right] \quad \text{under} \quad \mathbb{E}^{\mathbb{Q}}\left[g\left(S(T)\right)\mathbf{1}_A\right] \leq x$$

Explicit Solution in Complete Market

- **Problem Reformulation:**

Maximize the Probability of Hedge for a given starting wealth x

$$\Updownarrow$$

$$\max_{\pi \in \mathcal{A}} \mathbb{P}\left[X_x^\pi(T) \geq g\left(S(T)\right)\right]$$

$$\Updownarrow$$

$$\max_{X \in L_T^0} \mathbb{P}\left[X \geq g\left(S(T)\right)\right] \quad \text{under} \quad \mathbb{E}^{\mathbb{Q}}\left[X\right] \leq x$$

$$A = \{X \geq g\left(S(T)\right)\} \qquad \Updownarrow \qquad X = g\left(S(T)\right) \mathbf{1}_A$$

$$\max_{A \in \mathcal{F}_T} \mathbb{P}\left[A\right] \quad \text{under} \quad \mathbb{E}^{\mathbb{Q}}\left[g\left(S(T)\right) \mathbf{1}_A\right] \leq x$$

$$\Updownarrow$$

$$\max_{A \in \mathcal{F}_T} \mathbb{P}\left[A\right] \quad \text{under} \quad \mathbb{Q}^g[A] \leq \frac{x}{\mathbb{E}^{\mathbb{Q}}\left[g\left(S(T)\right)\right]},$$

with $\mathbb{Q}^g$ the risk neutral measure under the contingent claim numeraire

$$\frac{d\mathbb{Q}^g}{d\mathbb{Q}} = \frac{g\left(S(T)\right)}{\mathbb{E}^{\mathbb{Q}}\left[g\left(S(T)\right)\right]}$$

Explicit Solution in Complete Market

Maximize the Probability of Hedge for a given starting wealth x

$$\Updownarrow$$

$$\max_{A \in \mathcal{F}_T} \mathbb{P}[A] \text{ under } \mathbb{Q}^g[A] \leq \frac{x}{\mathbb{E}^{\mathbb{Q}}[g(S(T))]},$$

- **Foellmer and Leukert's solution:**

A interprets as a critical region when testing $\mathbb{Q}^g$ against $\mathbb{P}$.

Explicit Solution in Complete Market

Maximize the Probability of Hedge for a given starting wealth x

$$\Updownarrow$$

$$\max_{A \in \mathcal{F}_T} \mathbb{P}[A] \text{ under } \mathbb{Q}^g[A] \leq \frac{x}{\mathbb{E}^{\mathbb{Q}}[g(S(T))]},$$

- **Foellmer and Leukert's solution:**

A interprets as a critical region when testing $\mathbb{Q}^g$ against $\mathbb{P}$.

By Neyman-Pearson Lemma,

$$A^*(x) = \left\{ \frac{d\mathbb{P}}{d\mathbb{Q}} > a^* \frac{d\mathbb{Q}^g}{d\mathbb{Q}} \right\}, \text{ with } a^* := \inf \left\{ a : \mathbb{Q}^g \left[\frac{d\mathbb{P}}{d\mathbb{Q}} > a \frac{d\mathbb{Q}^g}{d\mathbb{Q}} \right] = \frac{x}{\mathbb{E}^{\mathbb{Q}}[g(S(T))]} \right\}$$

Explicit Solution in Complete Market

Maximize the Probability of Hedge for a given starting wealth x

$$\Updownarrow$$

$$\max_{A \in \mathcal{F}_T} \mathbb{P}\,[A] \text{ under } \mathbb{Q}^g[A] \leq \frac{x}{\mathbb{E}^{\mathbb{Q}}\,[g\,(S(T))]},$$

- **Foellmer and Leukert's solution:**

A interprets as a critical region when testing $\mathbb{Q}^g$ against $\mathbb{P}$.

By Neyman-Pearson Lemma,

$$A^*(x) = \left\{\frac{d\mathbb{P}}{d\mathbb{Q}} > a^*\,\frac{d\mathbb{Q}^g}{d\mathbb{Q}}\right\}, \text{ with } a^* := \inf\left\{a :\ \mathbb{Q}^g\left[\frac{d\mathbb{P}}{d\mathbb{Q}} > a\,\frac{d\mathbb{Q}^g}{d\mathbb{Q}}\right] = \frac{x}{\mathbb{E}^{\mathbb{Q}}\,[g\,(S(T))]}\right\}$$

and the success region $A^*(x) = \{X_x^{\pi^*(x)}(T) \geq g\,(S(T))\}$ with

$$\pi^*(x) \text{ the hedging strategy of } g(S(T))\mathbf{1}_{A^*(x)}$$

Explicit Solution in Complete Market

Maximize the Probability of Hedge for a given starting wealth x

$$\Updownarrow$$

$$\max_{A \in \mathcal{F}_T} \mathbb{P}\,[A] \text{ under } \mathbb{Q}^g[A] \leq \frac{x}{\mathbb{E}^{\mathbb{Q}}\,[g\,(S(T))]},$$

- **Foellmer and Leukert's solution:**

A interprets as a critical region when testing $\mathbb{Q}^g$ against $\mathbb{P}$.

By Neyman-Pearson Lemma,

$$A^*(x) = \left\{ \frac{d\mathbb{P}}{d\mathbb{Q}} > a^* \frac{d\mathbb{Q}^g}{d\mathbb{Q}} \right\}, \text{ with } a^* := \inf \left\{ a : \mathbb{Q}^g \left[\frac{d\mathbb{P}}{d\mathbb{Q}} > a \frac{d\mathbb{Q}^g}{d\mathbb{Q}} \right] = \frac{x}{\mathbb{E}^{\mathbb{Q}}\,[g\,(S(T))]} \right\}$$

and the success region $A^*(x) = \{X_x^{\pi^*(x)}(T) \geq g\,(S(T))\}$ with

$$\pi^*(x) \quad \text{the hedging strategy of} \quad g(S(T))\mathbf{1}_{A^*(x)}$$

$$\Rightarrow \text{ Find } x^*(p) \text{ such that } \mathbb{P}[A^*(x^*(p))] = p$$

Solution in General Case

- **Pros:**

- Explicit solution in some simple (but important) cases.

- Generic solution of the form:

$$X_x^\pi(T) = g\left(S_{t,s}(T)\right)\mathbf{1}_A \quad \text{or} \quad X_x^\pi(T) = g\left(S_{t,s}(T)\right)\zeta \text{ with } \zeta \in L^0[0,1]\,.$$

- Similar structure in incomplete markets.

Explicit Solution in General Case

- **Pros:**

- Explicit solution in some simple (but important) cases.

- Generic solution of the form:

$X_x^\pi(T) = g\left(S_{t,s}(T)\right)\mathbf{1}_A$ or $X_x^\pi(T) = g\left(S_{t,s}(T)\right)\zeta$ with $\zeta \in L^0[0,1]$.

- Similar structure in incomplete markets.

- **Cons:**

- Explicit solution not known in general (numerics)

- Dual problem in incomplete markets is a control problem: how to solve it ?

- Relies heavily on the duality between super-hedgeable claims and risk neutral measures.

Comparison with the super-hedging problem

$$v(t,s;1) \quad := \quad \inf\left\{x \geq 0 : \; \exists\, \pi \in \mathcal{A} \text{ s.t. } \mathbb{P}\left[X^{\pi}_{t,s,x}(T) \geq g\left(S_{t,s}(T)\right)\right] = 1\right\}$$

- **Dual approach:**

$$v(t,s;1) \quad = \quad \sup_{\mathbb{Q}} \mathbb{E}^{\mathbb{Q}}\left[g\left(S_{t,s}(T)\right)\right]$$

Comparison with the super-hedging problem

$$v(t,s;1) \quad := \quad \inf\left\{x \geq 0 : \; \exists\, \pi \in \mathcal{A} \text{ s.t. } \mathbb{P}\left[X^{\pi}_{t,s,x}(T) \geq g\left(S_{t,s}(T)\right)\right] = 1\right\}$$

- **Dual approach:**

$$v(t,s;1) \quad = \quad \sup_{\mathbb{Q}} \mathbb{E}^{\mathbb{Q}}\left[g\left(S_{t,s}(T)\right)\right]$$

- **Direct approach of Soner and Touzi:**

- (DP1): $x > v(t,s;1) \Rightarrow \exists\, \pi \in \mathcal{A}$ s.t. for all stopping time $\tau \leq T$

$$X^{\pi}_{t,s,x}(\tau) \geq v(\tau, S^{\pi}_{t,s}(\tau); 1)$$

Comparison with the super-hedging problem

$$v(t,s;1) \ := \ \inf\left\{x \geq 0 : \ \exists\, \pi \in \mathcal{A} \text{ s.t. } \mathbb{P}\left[X^{\pi}_{t,s,x}(T) \geq g\left(S_{t,s}(T)\right)\right] = 1\right\}$$

- **Dual approach:**

$$v(t,s;1) \ = \ \sup_{\mathbb{Q}} \mathbb{E}^{\mathbb{Q}}\left[g\left(S_{t,s}(T)\right)\right]$$

- **Direct approach of Soner and Touzi:**

- (DP1): $x > v(t,s;1) \Rightarrow \exists\ \pi \in \mathcal{A}$ s.t. for all stopping time $\tau \leq T$

$$X^{\pi}_{t,s,x}(\tau) \geq v(\tau, S^{\pi}_{t,s}(\tau); 1)$$

- (DP2): $x < v(t,s;1) \Rightarrow$ for all stopping time $\tau \leq T$ and $\pi \in \mathcal{A}$

$$\mathbb{P}\left[X^{\pi}_{t,s,x}(\tau) > v(\tau, S^{\pi}_{t,s}(\tau); 1)\right] < 1$$

Comparison with the super-hedging problem

$$v(t,s;1) \;:=\; \inf\left\{x \geq 0 : \;\exists\, \pi \in \mathcal{A} \text{ s.t. } \mathbb{P}\left[X^{\pi}_{t,s,x}(T) \geq g\left(S_{t,s}(T)\right)\right] = 1\right\}$$

- **Dual approach:**

$$v(t,s;1) \;=\; \sup_{\mathbb{Q}} \mathbb{E}^{\mathbb{Q}}\left[g\left(S_{t,s}(T)\right)\right]$$

- **Direct approach of Soner and Touzi:**

- (DP1): $x > v(t,s;1) \Rightarrow \exists\, \pi \in \mathcal{A}$ s.t. for all stopping time $\tau \leq T$

$$X^{\pi}_{t,s,x}(\tau) \geq v(\tau, S^{\pi}_{t,s}(\tau); 1)$$

- (DP2): $x < v(t,s;1) \Rightarrow$ for all stopping time $\tau \leq T$ and $\pi \in \mathcal{A}$

$$\mathbb{P}\left[X^{\pi}_{t,s,x}(\tau) > v(\tau, S^{\pi}_{t,s}(\tau); 1)\right] < 1$$

$\Rightarrow$ is sufficient to derive PDEs associated to $v(\cdot;1)$.

Direct approach for quantile hedging ?

- **Form of the DP:**

$$x > v(t, s; p) \quad \Rightarrow \quad \exists\, \pi \in \mathcal{A} \ \text{ s.t. } \forall \tau \le T, \quad X^{\pi}_{t,s,x}(\tau) \ge v(\tau, S^{\pi}_{t,s}(\tau); P_\tau)$$

$$\text{where } \ P_\tau := \mathbb{P}\left[X^{\pi}_{t,s,x}(T) \ge g(S^{\pi}_{t,s}(T)) \mid X^{\pi}_{t,s,x}(\tau)\right]$$

Direct approach for quantile hedging ?

- **Form of the DP:**

$$x > v(t,s;p) \quad \Rightarrow \quad \exists\, \pi \in \mathcal{A} \text{ s.t. } \forall \tau \le T, \quad X^{\pi}_{t,s,x}(\tau) \ge v(\tau, S^{\pi}_{t,s}(\tau); P_\tau)$$

$$\text{where } \; P_\tau := \mathbb{P}\left[X^{\pi}_{t,s,x}(T) \ge g(S^{\pi}_{t,s}(T)) \mid X^{\pi}_{t,s,x}(\tau)\right]$$

$$P \;\text{ is a martingale and } \mathbb{E}_t\left[P_\tau\right] \ge p \qquad \Rightarrow \qquad P_\tau \ge p + \int_t^\tau \alpha_u dW_u$$

Direct approach for quantile hedging ?

- **Form of the DP:**

$$x > v(t,s;p) \quad \Rightarrow \quad \exists\, \pi \in \mathcal{A} \text{ s.t. } \forall \tau \le T, \quad X^{\pi}_{t,s,x}(\tau) \ge v(\tau, S^{\pi}_{t,s}(\tau); P_\tau)$$

$$\text{where } \; P_\tau := \mathbb{P}\left[X^{\pi}_{t,s,x}(T) \ge g(S^{\pi}_{t,s}(T)) \mid X^{\pi}_{t,s,x}(\tau)\right]$$

$$P \text{ is a martingale and } \mathbb{E}_t\left[P_\tau\right] \ge p \qquad \Rightarrow \qquad P_\tau \ge p + \int_t^\tau \alpha_u dW_u$$

- **Dynamic Programming:** Set $P^{\alpha}_{t,p} = p + \int_t^{\cdot} \alpha_u dW_u$.

Direct approach for quantile hedging ?

- **Form of the DP:**

$$x > v(t,s;p) \quad \Rightarrow \quad \exists\, \pi \in \mathcal{A} \text{ s.t. } \forall \tau \le T, \quad X^{\pi}_{t,s,x}(\tau) \ge v(\tau, S^{\pi}_{t,s}(\tau); P_\tau)$$

$$\text{where } \; P_\tau := \mathbb{P}\left[X^{\pi}_{t,s,x}(T) \ge g(S^{\pi}_{t,s}(T)) \mid X^{\pi}_{t,s,x}(\tau)\right]$$

$$P \;\text{ is a martingale and } \mathbb{E}_t\left[P_\tau\right] \ge p \qquad \Rightarrow \qquad P_\tau \ge p + \int_t^\tau \alpha_u dW_u$$

- **Dynamic Programming:** Set $P^{\alpha}_{t,p} = p + \int_t^{\cdot} \alpha_u dW_u$.

- (DP1): $x > v(t,s;p) \Rightarrow \exists\; (\pi, \alpha) \in \mathcal{A} \times L^2$ s.t. for all stop. time $\tau \le T$

$$X^{\pi}_{t,s,x}(\tau) \ge v(\tau, S^{\pi}_{t,s}(\tau); P^{\alpha}_{t,p}(\tau))$$

Direct approach for quantile hedging ?

- **Form of the DP:**

$$x > v(t,s;p) \quad \Rightarrow \quad \exists\, \pi \in \mathcal{A} \text{ s.t. } \forall \tau \leq T, \quad X^{\pi}_{t,s,x}(\tau) \geq v(\tau, S^{\pi}_{t,s}(\tau); P_{\tau})$$

$$\text{where } \; P_{\tau} := \mathbb{P}\left[X^{\pi}_{t,s,x}(T) \geq g(S^{\pi}_{t,s}(T)) \mid X^{\pi}_{t,s,x}(\tau)\right]$$

P is a martingale and $\mathbb{E}_t\left[P_{\tau}\right] \geq p \qquad \Rightarrow \qquad P_{\tau} \geq p + \int_t^{\tau} \alpha_u dW_u$

- **Dynamic Programming:** Set $P^{\alpha}_{t,p} = p + \int_t^{\cdot} \alpha_u dW_u$.

- (DP1): $x > v(t,s;p) \Rightarrow \exists\, (\pi, \alpha) \in \mathcal{A} \times L^2$ s.t. for all stop. time $\tau \leq T$

$$X^{\pi}_{t,s,x}(\tau) \geq v(\tau, S^{\pi}_{t,s}(\tau); P^{\alpha}_{t,p}(\tau))$$

- (DP2): $x < v(t,s;p) \Rightarrow$ for all stop. time $\tau \leq T$ and $(\pi, \alpha) \in \mathcal{A} \times L^2$,

$$\mathbb{P}\left[X^{\pi}_{t,s,x}(\tau) > v(\tau, S^{\pi}_{t,s}(\tau); P^{\alpha}_{t,p}(\tau))\right] < 1$$

PDE derivation (formally)

- Dynamics of the wealth

$$dX^{\pi}_{t,s,x}(u) \;=\; \pi_u \left[\mu\left(u, S^{\pi}(u), \pi_u\right) du + \sigma\left(u, S^{\pi}(u), \pi_u\right) dW_u\right]$$

- Dynamics of the quantile price at point $Y_u = (u, S^{\pi}_{t,s}(u); P^{\alpha}_{t,p}(u))$

$$dv(Y_u) \;=\; \mathcal{L}^{\pi,\alpha} v(Y_u) du + \left[D_s v(Y_u) \sigma\left(u, S^{\pi}(u), \pi_u\right) + D_p v(Y_u) \alpha_u\right] dW_u$$

PDE derivation (formally)

- Dynamics of the wealth

$$dX^{\pi}_{t,s,x}(u) \;=\; \pi_u \left[\mu\left(u, S^{\pi}(u), \pi_u\right) du + \sigma\left(u, S^{\pi}(u), \pi_u\right) dW_u\right]$$

- Dynamics of the quantile price at point $Y_u = (u, S^{\pi}_{t,s}(u); P^{\alpha}_{t,p}(u))$

$$dv(Y_u) \;=\; \mathcal{L}^{\pi,\alpha} v(Y_u) du + \left[D_s v(Y_u) \sigma\left(u, S^{\pi}(u), \pi_u\right) + D_p v(Y_u) \alpha_u\right] dW_u$$

- Take $x \sim v(t, s; p)$.

(DP1) $\Rightarrow$ $\exists(\pi, \alpha)$ s.t. $\quad X^{\pi}_{t,s,x}(\tau) \geq v(\tau, S^{\pi}_{t,s}(\tau); P^{\alpha}_{t,p}(\tau))$

(DP2) $\Rightarrow$ $\forall(\pi, \alpha)$, $\quad \mathbb{P}\left[X^{\pi}_{t,s,x}(\tau) \geq v(\tau, S^{\pi}_{t,s}(\tau); P^{\alpha}_{t,p}(\tau))\right] < 1$

PDE derivation (formally)

- Dynamics of the wealth

$$dX^{\pi}_{t,s,x}(u) \;=\; \pi_u \left[\mu\left(u, S^{\pi}(u), \pi_u\right) du + \sigma\left(u, S^{\pi}(u), \pi_u\right) dW_u\right]$$

- Dynamics of the quantile price at point $Y_u = (u, S^{\pi}_{t,s}(u); P^{\alpha}_{t,p}(u))$

$$dv(Y_u) \;=\; \mathcal{L}^{\pi,\alpha} v(Y_u) du + \left[D_s v(Y_u)\sigma\left(u, S^{\pi}(u), \pi_u\right) + D_p v(Y_u)\alpha_u\right] dW_u$$

- Take $x \sim v(t, s; p)$.

(DP1) $\Rightarrow \exists(\pi, \alpha)$ s.t. $\quad X^{\pi}_{t,s,x}(\tau) \geq v(\tau, S^{\pi}_{t,s}(\tau); P^{\alpha}_{t,p}(\tau))$

(DP2) $\Rightarrow \forall(\pi, \alpha)$, $\quad \mathbb{P}\left[X^{\pi}_{t,s,x}(\tau) \geq v(\tau, S^{\pi}_{t,s}(\tau); P^{\alpha}_{t,p}(\tau))\right] < 1$

- This formally leads to the PDE

$$\max_{(\pi,\alpha)\in\mathcal{G}(t,s,p)} \pi\mu\left(t, s, \pi\right) - \mathcal{L}^{\pi,\alpha} v(t, s; p) = 0$$

where $\mathcal{G}(t, s, p) := \{(\pi, \alpha) : \; \pi\sigma\left(t, s, \pi\right) = D_s v(t, s; p)\sigma\left(t, s, \pi\right) + D_p v(t, s; p)\alpha\}$

PDE derivation (rigourous)

- The expected PDE is

$$\max_{(\pi,\alpha)\in\mathcal{G}(t,s,p)} \pi\mu(t,s,\pi) - \mathcal{L}^{\pi,\alpha}v(t,s;p) = 0$$

where $\mathcal{G}(t,s,p) := \{(\pi,\alpha) : \ \pi\sigma(t,s,\pi) = D_s v(t,s;p)\sigma(t,s,\pi) + D_p v(t,s;p)\alpha\}$

- Viscosity approach

PDE derivation (rigourous)

- The expected PDE is

$$\max_{(\pi,\alpha)\in\mathcal{G}(t,s,p)} \pi\mu\,(t,s,\pi) - \mathcal{L}^{\pi,\alpha}v(t,s;p) = 0$$

where $\mathcal{G}(t,s,p) := \{(\pi,\alpha) : \ \pi\sigma\,(t,s,\pi) = D_s v(t,s;p)\sigma\,(t,s,\pi) + D_p v(t,s;p)\alpha\}$

- Viscosity approach

- α is not bounded

PDE derivation (rigourous)

- The expected PDE is

$$\max_{(\pi,\alpha)\in\mathcal{G}(t,s,p)} \pi\mu(t,s,\pi) - \mathcal{L}^{\pi,\alpha}v(t,s;p) = 0$$

where $\mathcal{G}(t,s,p) := \{(\pi,\alpha) : \pi\sigma(t,s,\pi) = D_s v(t,s;p)\sigma(t,s,\pi) + D_p v(t,s;p)\alpha\}$

- Viscosity approach

- α is not bounded

- Behaviour at the Boundary of the domain

Boundary in p

$v(t,s,0^+) = 0$ and $v(t,s,1^-)$ is the super replication price

PDE derivation (rigourous)

- The expected PDE is

$$\max_{(\pi,\alpha)\in\mathcal{G}(t,s,p)} \pi\mu\,(t,s,\pi) - \mathcal{L}^{\pi,\alpha}v(t,s;p) = 0$$

where $\mathcal{G}(t,s,p) := \{(\pi,\alpha) :\ \pi\sigma\,(t,s,\pi) = D_s v(t,s;p)\sigma\,(t,s,\pi) + D_p v(t,s;p)\alpha\}$

- Viscosity approach

- α is not bounded

- Behaviour at the Boundary of the domain

Boundary in p

$$v(t,s,0^+) = 0 \quad \text{and} \quad v(t,s,1^-) \text{ is the super replication price}$$

Boundary in time

$$v(T^-,s,p) = p\,g(s)$$

Example: Quantile Hedging in Black Scholes

- **The Dynamics:**

$$dS_{t,s}(r) = S_{t,s}(r)\left(\mu dt + \sigma dW_r\right) \quad \text{and} \quad dX^{\pi}_{t,x,s}(r) = \pi_r dS_{t,s}(r)$$

- **The Problem:**

$$v(t,s;p) \; := \; \inf\left\{x \in \mathbb{R}_+ \; : \; \exists\, \pi \in \mathcal{A} \text{ s.t. } \mathbb{P}\left[X^{\pi}_{t,x,s}(T) \geq g(S_{t,s}(T))\right] \geq p\right\} \; .$$

Example: Quantile Hedging in Black Scholes

- **The Dynamics:**

$$dS_{t,s}(r) = S_{t,s}(r)\left(\mu dt + \sigma dW_r\right) \quad \text{and} \quad dX^{\pi}_{t,x,s}(r) = \pi_r dS_{t,s}(r)$$

- **The Problem:**

$$v(t,s;p) \;:=\; \text{inf}\left\{x \in \mathbb{R}_+ \;:\; \exists\, \pi \in \mathcal{A} \text{ s.t. } \mathbb{P}\left[X^{\pi}_{t,x,s}(T) \geq g(S_{t,s}(T))\right] \geq p\right\}\;.$$

- **Associated PDE:**

$$0 = \sup_{\pi\sigma s = \sigma s v_s + \alpha v_p}\left(\pi\mu s - \mu s v_s - \frac{1}{2}\sigma^2 s^2 v_{ss} - \alpha\sigma s v_{sp} - \alpha^2 v_{pp}\right)$$

Example: Quantile Hedging in Black Scholes

- **The Dynamics:**

$$dS_{t,s}(r) = S_{t,s}(r)\left(\mu dt + \sigma dW_r\right) \quad \text{and} \quad dX^{\pi}_{t,x,s}(r) = \pi_r dS_{t,s}(r)$$

- **The Problem:**

$$v(t,s;p) \;:=\; \inf\left\{x \in \mathbb{R}_+ \;:\; \exists\, \pi \in \mathcal{A} \text{ s.t. } \mathbb{P}\left[X^{\pi}_{t,x,s}(T) \geq g(S_{t,s}(T))\right] \geq p\right\}\;.$$

- **Associated PDE:**

$$0 = \sup_{\pi\sigma s = \sigma s v_s + \alpha v_p} \left(\pi\mu s - \mu s v_s - \frac{1}{2}\sigma^2 s^2 v_{ss} - \alpha\sigma s v_{sp} - \alpha^2 v_{pp}\right)$$

$$\Rightarrow \quad 0 = -v_t - \frac{1}{2}\sigma^2 s^2 v_{ss} + \frac{1}{2}\frac{\left(\frac{\mu}{\sigma}v_p - \sigma s v_{sp}\right)^2}{v_{pp}}$$

with the controls

$$\hat{\pi} := v_s + \frac{\hat{\alpha}}{s\sigma}v_p \quad \text{and} \quad \hat{\alpha} := \frac{\frac{\mu}{\sigma}v_p - \sigma s v_{sp}}{v_{pp}}\;.$$

Verification in the quantile hedging problem

- **Associated PDE:** $0 = -v_t - \frac{1}{2}\sigma^2 s^2 v_{ss} + \frac{1}{2}\frac{\left(\frac{\mu}{\sigma}v_p - \sigma s v_{sp}\right)^2}{v_{pp}}$

- **Boundary conditions:**

$$v(t, s, 0) = 0 \quad \text{and} \quad v(T, s, p) = pg(s)$$

Verification in the quantile hedging problem

- **Associated PDE:** $0 = -v_t - \frac{1}{2}\sigma^2 s^2 v_{ss} + \frac{1}{2}\frac{\left(\frac{\mu}{\sigma}v_p - \sigma s v_{sp}\right)^2}{v_{pp}}$

- **Boundary conditions:**

$$v(t,s,0) = 0 \quad \text{and} \quad v(T,s,p) = pg(s)$$

- **Legendre-Fenchel transform of v with respect to the $p-$variable:**

$$u(t,s,q) := \sup_{p\in[0,1]} \{pq - v(t,s,p)\} \ .$$

Verification in the quantile hedging problem

- **Associated PDE:** $0 = -v_t - \frac{1}{2}\sigma^2 s^2 v_{ss} + \frac{1}{2}\frac{\left(\frac{\mu}{\sigma}v_p - \sigma s v_{sp}\right)^2}{v_{pp}}$

- **Boundary conditions:**

$$v(t,s,0) = 0 \quad \text{and} \quad v(T,s,p) = pg(s)$$

- **Legendre-Fenchel transform of v with respect to the $p-$variable:**

$$u(t,s,q) := \sup_{p\in[0,1]} \{pq - v(t,s,p)\} \ .$$

a- Associated PDE:

$$-u_t - \frac{1}{2}\sigma^2 u_{ss} - (\mu/\sigma)q\sigma s u_{sq} - \frac{1}{2}(\mu/\sigma)^2 q^2 u_{qq} = 0$$

Verification in the quantile hedging problem

- **Associated PDE:** $0 = -v_t - \frac{1}{2}\sigma^2 s^2 v_{ss} + \frac{1}{2}\frac{\left(\frac{\mu}{\sigma}v_p - \sigma s v_{sp}\right)^2}{v_{pp}}$

- **Boundary conditions:**

$$v(t,s,0) = 0 \quad \text{and} \quad v(T,s,p) = pg(s)$$

- **Legendre-Fenchel transform of v with respect to the $p-$variable:**

$$u(t,s,q) := \sup_{p\in[0,1]} \{pq - v(t,s,p)\} \ .$$

a- Associated PDE:

$$-u_t - \frac{1}{2}\sigma^2 u_{ss} - (\mu/\sigma)q\sigma s u_{sq} - \frac{1}{2}(\mu/\sigma)^2 q^2 u_{qq} = 0$$

b- Boundary conditions: $u(T,s,q) = (q - g(s))^+$

Verification in the quantile hedging problem

- **Associated PDE (bis):** $0 = -v_t - \frac{1}{2}\sigma^2 s^2 v_{ss} + \frac{1}{2}\frac{\left(\frac{\mu}{\sigma}v_p - \sigma s v_{sp}\right)^2}{v_{pp}}$
- **Boundary conditions:**

$$v(t,s,0) = 0 \quad \text{and} \quad v(T,s,p) = pg(s)$$

- **Legendre-Fenchel transform of v with respect to the $p-$variable:**

$$u(t,s,q) := \sup_{p\in[0,1]} \{pq - v(t,s,p)\} \ .$$

a- Associated PDE:

$$-u_t - \frac{1}{2}\sigma^2 u_{ss} - (\mu/\sigma) q\sigma s u_{sq} - \frac{1}{2}(\mu/\sigma)^2 q^2 u_{qq} = 0$$

b- Boundary conditions: $u(T,s,q) = (q - g(s))^+$

c- Feynman-Kac:

$$u(t,s,q) = \mathbb{E}_t^{\mathbb{Q}}\left[\left(Q_{t,q}(T) - g\left(S_{t,s}(T)\right)\right)^+\right] \quad \text{where} \quad \frac{dQ(r)}{Q(r)} = (\mu/\sigma) dW_r^{\mathbb{Q}}$$

Extensions

- **On the Dynamics:**

$$
\begin{aligned}
S^{\pi} &= s + \int_t^{\cdot} \mu\left(S^{\pi}(u), \pi_u\right) du + \int_t^{\cdot} \sigma\left(S^{\pi}(u), \pi_u\right) dW_u \\
X^{\pi} &= x + \int_t^{\cdot} \rho\left(S^{\pi}(u), X^{\pi}(u), \pi_u\right) du + \int_t^{\cdot} \beta\left(S^{\pi}(u), X^{\pi}(u), \pi_u\right) dW_u
\end{aligned}
$$

Extensions

- **On the Dynamics:**

$$\begin{aligned} S^\pi &= s + \int_t^\cdot \mu\left(S^\pi(u), \pi_u\right) du + \int_t^\cdot \sigma\left(S^\pi(u), \pi_u\right) dW_u \\ X^\pi &= x + \int_t^\cdot \rho\left(S^\pi(u), X^\pi(u), \pi_u\right) du + \int_t^\cdot \beta\left(S^\pi(u), X^\pi(u), \pi_u\right) dW_u \end{aligned}$$

- **On the Problems:** Given ℓ from $\mathbb{R}^d \times \mathbb{R}$ into $\mathbb{R}$ and $p \in \text{Im}(\ell)$,

$$v(t,s;p) \;:=\; \inf\left\{x \in \mathbb{R} \;:\; \exists\, \pi \in \mathcal{A} \text{ s.t. } \mathbb{E}\left[\ell\left(S^\pi_{t,s}(T), X^\pi_{t,x,s}(T)\right)\right] \geq p\right\} \;.$$

Extensions

- **On the Dynamics:**

$$S^\pi = s + \int_t^\cdot \mu\left(S^\pi(u), \pi_u\right) du + \int_t^\cdot \sigma\left(S^\pi(u), \pi_u\right) dW_u$$

$$X^\pi = x + \int_t^\cdot \rho\left(S^\pi(u), X^\pi(u), \pi_u\right) du + \int_t^\cdot \beta\left(S^\pi(u), X^\pi(u), \pi_u\right) dW_u$$

- **On the Problems:** Given ℓ from $\mathbb{R}^d \times \mathbb{R}$ into $\mathbb{R}$ and $p \in \text{Im}(\ell)$,

$$v(t, s; p) := \inf\left\{x \in \mathbb{R} \ : \ \exists\, \pi \in \mathcal{A} \text{ s.t. } \mathbb{E}\left[\ell\left(S^\pi_{t,s}(T), X^\pi_{t,x,s}(T)\right)\right] \geq p\right\} .$$

- **Applications**

$\ell(s, x) = \mathbf{1}\{x \geq g(s)\}$ $\Rightarrow$ Quantile Hedging

$\ell(s, x) = U([x - g(s)]^+)$ with $U \nearrow$ concave $\Rightarrow$ Loss function

$\ell(s, x) = U(x - g(s))$ with $U \nearrow$ concave $\Rightarrow$ Indifference prices

Extensions

- **On the Dynamics:**

$$S^\pi = s + \int_t^\cdot \mu\left(S^\pi(u), \pi_u\right) du + \int_t^\cdot \sigma\left(S^\pi(u), \pi_u\right) dW_u$$

$$X^\pi = x + \int_t^\cdot \rho\left(S^\pi(u), X^\pi(u), \pi_u\right) du + \int_t^\cdot \beta\left(S^\pi(u), X^\pi(u), \pi_u\right) dW_u$$

- **On the Problems:** Given ℓ from $\mathbb{R}^d \times \mathbb{R}$ into $\mathbb{R}$ and $p \in \text{Im}(\ell)$,

$$v(t,s;p) := \text{inf}\left\{x \in \mathbb{R} \ : \ \exists\, \pi \in \mathcal{A} \text{ s.t. } \mathbb{E}\left[\ell\left(S^\pi_{t,s}(T), X^\pi_{t,x,s}(T)\right)\right] \geq p\right\} \ .$$

- **Applications**

$\ell(s,x) = 1\{x \geq g(s)\}$ $\Rightarrow$ Quantile Hedging

$\ell(s,x) = U([x-g(s)]^+)$ with $U \nearrow$ concave $\Rightarrow$ Loss function

$\ell(s,x) = U(x-g(s))$ with $U \nearrow$ concave $\Rightarrow$ Indifference prices

- **Dynamic programming based on the reformulation**

$$v(t,s;p) = \text{inf}\left\{x \in \mathbb{R}_+ : \ \exists\, (\pi,\alpha) \in \mathcal{A} \times L^2 \text{ s.t. } \ell\left(S^\pi_{t,s}(T), X^\pi_{t,x,s}(T)\right) \geq P^\alpha_{t,p}(T)\right\}$$

Extensions

- **On the Dynamics:**

$$
\begin{aligned}
S^\pi &= s + \int_t^{\cdot} \mu\left(S^\pi(u), \pi_u\right) du + \int_t^{\cdot} \sigma\left(S^\pi(u), \pi_u\right) dW_u \\
X^\pi &= x + \int_t^{\cdot} \rho\left(S^\pi(u), X^\pi(u), \pi_u\right) du + \int_t^{\cdot} \beta\left(S^\pi(u), X^\pi(u), \pi_u\right) dW_u
\end{aligned}
$$

- **On the Problems:** Given ℓ from $\mathbb{R}^d \times \mathbb{R}$ into $\mathbb{R}$ and $p \in \text{Im}(\ell)$,

$$
v(t,s;p) \;:=\; \inf\left\{x \in \mathbb{R} \;:\; \exists\, \pi \in \mathcal{A} \text{ s.t. } \mathbb{E}\left[\ell\left(S^\pi_{t,s}(T), X^\pi_{t,x,s}(T)\right)\right] \geq p\right\} .
$$

- **Applications**

$\ell(s,x) = \mathbf{1}\{x \geq g(s)\}$ $\Rightarrow$ Quantile Hedging

$\ell(s,x) = U([x-g(s)]^+)$ with $U \nearrow$ concave $\Rightarrow$ Loss function

$\ell(s,x) = U(x-g(s))$ with $U \nearrow$ concave $\Rightarrow$ Indifference prices

- **Dynamic programming based on the reformulation**

$$
v(t,s;p) = \inf\left\{x \in \mathbb{R}_+ :\; \exists\,(\pi,\alpha) \in \mathcal{A} \times L^2 \text{ s.t. } \ell\left(S^\pi_{t,s}(T), X^\pi_{t,x,s}(T)\right) \geq P^\alpha_{t,p}(T)\right\}
$$

$\Rightarrow$ **Stochastic Target problems (with unbounded controls)**

Optimal Control with Stochastic Target Constraints

General framework

- **Dynamics:**

$$
\begin{aligned}
S^\pi &= s + \int_t^\cdot \mu\left(S^\pi(u), \pi_u\right) du + \int_t^\cdot \sigma\left(S^\pi(u), \pi_u\right) dW_u \\
X^\pi &= x + \int_t^\cdot \rho\left(S^\pi(u), X^\pi(u), \pi_u\right) du + \int_t^\cdot \beta\left(S^\pi(u), X^\pi(u), \pi_u\right) dW_u
\end{aligned}
$$

- **Problems:** Given $F, \bar{\ell}$ from $\mathbb{R}^d \times \mathbb{R}$ into $\mathbb{R}$:

$$
V(t,s,x) \;:=\; \sup_{\pi \in \bar{\mathcal{A}}^{\bar{\ell}}_{t,s,x}} \mathbb{E}\left[F\left(S^\pi_{t,s}(T), X^\pi_{t,x,s}(T)\right)\right]
$$

where

$$
\bar{\mathcal{A}}^{\bar{\ell}}_{t,s,x} := \left\{\pi \in \mathcal{A} \text{ s.t. } \bar{\ell}\left(S^\pi_{t,s}(T), X^\pi_{t,x,s}(T)\right) \geq 0\right\} .
$$

Example 1: Moment constraints

- **Problems:** Given F, ℓ from $\mathbb{R}^d \times \mathbb{R}$ into $\mathbb{R}$:

$$V(t,s,x;p) \ := \sup_{\pi \in \mathcal{A}^{\ell}_{t,s,x,p}} \mathbb{E}\left[F\left(S^{\pi}_{t,s}(T), X^{\pi}_{t,x,s}(T)\right)\right]$$

where $\mathcal{A}^{\ell}_{t,s,x,p} := \left\{\pi \in \mathcal{A} \text{ s.t. } \mathbb{E}\left[\ell\left(S^{\pi}_{t,s}(T), X^{\pi}_{t,x,s}(T)\right)\right] \geq p\right\}$.

Example 1: Moment constraints

- **Problems:** Given F, ℓ from $\mathbb{R}^d \times \mathbb{R}$ into $\mathbb{R}$:

$$V(t,s,x;p) \;\; := \sup_{\pi \in \mathcal{A}^{\ell}_{t,s,x,p}} \mathbb{E}\left[F\left(S^{\pi}_{t,s}(T), X^{\pi}_{t,x,s}(T)\right)\right]$$

where $\mathcal{A}^{\ell}_{t,s,x,p} := \left\{\pi \in \mathcal{A} \text{ s.t. } \mathbb{E}\left[\ell\left(S^{\pi}_{t,s}(T), X^{\pi}_{t,x,s}(T)\right)\right] \geq p\right\}$.

- **Reformulation:** We have

$\mathcal{A}^{\ell}_{t,s,x,p} := \left\{\pi \in \mathcal{A} \text{ s.t. } \exists\, \alpha \in L^2 \text{ s.t. } \ell\left(S^{\pi}_{t,s}(T), X^{\pi}_{t,x,s}(T)\right) \geq P^{\alpha}_{t,p}(T)\right\}$,
with $P^{\alpha}_{t,p}(r) := p + \int_t^r \alpha_u dW_u$.

Example 1: Moment constraints

- **Problems:** Given F, ℓ from $\mathbb{R}^d \times \mathbb{R}$ into $\mathbb{R}$:

$$V(t,s,x;p) \;:=\; \sup_{\pi \in \mathcal{A}^{\ell}_{t,s,x,p}} \mathbb{E}\left[F\left(S^{\pi}_{t,s}(T), X^{\pi}_{t,x,s}(T)\right)\right]$$

where $\mathcal{A}^{\ell}_{t,s,x,p} := \left\{\pi \in \mathcal{A} \text{ s.t. } \mathbb{E}\left[\ell\left(S^{\pi}_{t,s}(T), X^{\pi}_{t,x,s}(T)\right)\right] \geq p\right\}$.

- **Reformulation:** We have

$$\mathcal{A}^{\ell}_{t,s,x,p} := \left\{\pi \in \mathcal{A} \text{ s.t. } \exists\, \alpha \in L^2 \text{ s.t. } \ell\left(S^{\pi}_{t,s}(T), X^{\pi}_{t,x,s}(T)\right) \geq P^{\alpha}_{t,p}(T)\right\} .$$

with $P^{\alpha}_{t,p}(r) := p + \int_t^r \alpha_u dW_u$.

- Setting $\bar{\ell}(s,x,p) := \ell(s,x) - p$, we get

then $V(t,s,x;p) \;:=\; \sup_{(\pi,\alpha) \in \bar{\mathcal{A}}^{\bar{\ell}}_{t,s,x,p}} \mathbb{E}\left[F\left(S^{\pi}_{t,s}(T), X^{\pi}_{t,x,s}(T)\right)\right]$.

where $\bar{\mathcal{A}}^{\bar{\ell}}_{t,s,x,p} := \left\{(\pi,\alpha) \in \mathcal{A} \times L^2 \text{ s.t. } \bar{\ell}\left(S^{\pi}_{t,s}(T), X^{\pi}_{t,x,s}(T), P^{\alpha}_{t,p}(T)\right) \geq 0\right\}$

Example 2: Constraints in probability

- **Problems:** Given F, ℓ from $\mathbb{R}^d \times \mathbb{R}$ into $\mathbb{R}$:

$$V(t,s,x;p) \;:=\; \sup_{\pi \in \mathcal{A}^{\ell}_{t,s,x,p}} \mathbb{E}\left[F\left(S^{\pi}_{t,s}(T), X^{\pi}_{t,x,s}(T)\right)\right]$$

where $\mathcal{A}^{\ell}_{t,s,x,p} := \left\{\pi \in \mathcal{A} \text{ s.t. } \mathbb{P}\left[X^{\pi}_{t,x,s}(T) \geq g(S^{\pi}_{t,s}(T))\right] \geq p\right\}$,

for $\ell(s,x) := \mathbf{1}_{x \geq g(x)}$.

(see Boyle and Tian 07 for dual approach in complete market)

Example 3: Index tracking constraint

- $F(s,x) = U(x)$: utility function.
- $S^{\pi,1}$ an index. and X^π: wealth process.
- Portfolio optimization problem

$$V(t,s,x) \;:=\; \sup_{\pi \in \bar{\mathcal{A}}^{\bar{\ell}}_{t,s,x}} \mathbb{E}\left[U\left(X^{\pi}_{t,x,s}(T)\right)\right]$$

where

$$\bar{\mathcal{A}}^{\bar{\ell}}_{t,s,x} := \left\{\pi \in \mathcal{A} \text{ s.t. } X^{\pi}_{t,x,s}(T)/x_0 \geq 90\% \times S^{\pi,1}_{t,s}(T)/s^1_0\right\} .$$

Here, $\bar{\ell}(s,x) := x/x_0 - 90\% \times s/s_0$.

Example 4: Mean variance

$$V(t,s,x;p) \;:=\; \inf_{\pi\in\mathcal{A}_{t,s,x,p}} \mathbb{E}\left[\left(X^{\pi}_{t,x,s}(T)\right)^2\right]$$

where $\mathcal{A}_{t,s,x,p} := \left\{\pi \in \mathcal{A} \text{ s.t. } \mathbb{E}\left[X^{\pi}_{t,x,s}(T)\right] \geq p\right\}$.

PDE Derivation

- **Dynamics:**

$$S^\pi = s + \int_t^{\cdot} \mu\left(S^\pi(u), \pi_u\right) du + \int_t^{\cdot} \sigma\left(S^\pi(u), \pi_u\right) dW_u$$

$$X^\pi = x + \int_t^{\cdot} \rho\left(S^\pi(u), X^\pi(u), \pi_u\right) du + \int_t^{\cdot} \beta\left(S^\pi(u), X^\pi(u), \pi_u\right) dW_u$$

- **Problems:** Given $F, \bar{\ell}$ from $\mathbb{R}^d \times \mathbb{R}$ into $\mathbb{R}$:

$$V(t,s,x) := \sup_{\pi \in \bar{\mathcal{A}}^{\bar{\ell}}_{t,s,x}} \mathbb{E}\left[F\left(S^\pi_{t,s}(T), X^\pi_{t,x,s}(T)\right)\right]$$

where $\bar{\mathcal{A}}^{\bar{\ell}}_{t,s,x} := \left\{\pi \in \mathcal{A} \text{ s.t. } \bar{\ell}\left(S^\pi_{t,s}(T), X^\pi_{t,x,s}(T)\right) \geq 0\right\}$.

PDE Derivation

- **Dynamics:**

$$
\begin{aligned}
S^\pi &= s + \int_t^\cdot \mu\left(S^\pi(u), \pi_u\right) du + \int_t^\cdot \sigma\left(S^\pi(u), \pi_u\right) dW_u \\
X^\pi &= x + \int_t^\cdot \rho\left(S^\pi(u), X^\pi(u), \pi_u\right) du + \int_t^\cdot \beta\left(S^\pi(u), X^\pi(u), \pi_u\right) dW_u
\end{aligned}
$$

- **Problems:** Given $F, \bar{\ell}$ from $\mathbb{R}^d \times \mathbb{R}$ into $\mathbb{R}$:

$$
V(t,s,x) := \sup_{\pi \in \bar{\mathcal{A}}^{\bar{\ell}}_{t,s,x}} \mathbb{E}\left[F\left(S^\pi_{t,s}(T), X^\pi_{t,x,s}(T)\right)\right]
$$

where $\bar{\mathcal{A}}^{\bar{\ell}}_{t,s,x} := \left\{\pi \in \mathcal{A} \text{ s.t. } \bar{\ell}\left(S^\pi_{t,s}(T), X^\pi_{t,x,s}(T)\right) \geq 0\right\}$.

- Set $D := \{(t,s,x) : \ \bar{\mathcal{A}}^{\bar{\ell}}_{t,s,x} \neq \emptyset\}$ and $v(t,s) := \inf\{x \in \mathbb{R} : \ (t,s,x) \in D\}$.

PDE Derivation

- **Dynamics:**

$$S^\pi = s + \int_t^\cdot \mu\left(S^\pi(u), \pi_u\right) du + \int_t^\cdot \sigma\left(S^\pi(u), \pi_u\right) dW_u$$

$$X^\pi = x + \int_t^\cdot \rho\left(S^\pi(u), X^\pi(u), \pi_u\right) du + \int_t^\cdot \beta\left(S^\pi(u), X^\pi(u), \pi_u\right) dW_u$$

- **Problems:** Given $F, \bar{\ell}$ from $\mathbb{R}^d \times \mathbb{R}$ into $\mathbb{R}$:

$$V(t,s,x) := \sup_{\pi \in \bar{\mathcal{A}}^{\bar{\ell}}_{t,s,x}} \mathbb{E}\left[F\left(S^\pi_{t,s}(T), X^\pi_{t,x,s}(T)\right)\right]$$

where $\bar{\mathcal{A}}^{\bar{\ell}}_{t,s,x} := \left\{\pi \in \mathcal{A} \text{ s.t. } \bar{\ell}\left(S^\pi_{t,s}(T), X^\pi_{t,x,s}(T)\right) \geq 0\right\}$.

- Set $D := \{(t,s,x) : \ \bar{\mathcal{A}}^{\bar{\ell}}_{t,s,x} \neq \emptyset\}$ and $v(t,s) := \inf\{x \in \mathbb{R} : \ (t,s,x) \in D\}$.

- If $\bar{\ell}$ is non-decreasing in x and v is smooth, then

$$\text{cl}(D) = \text{int}_p D \cup \partial_p D \cup \partial_T D \quad \text{with} \quad \begin{array}{l} \text{int}_p D := \{t < T \ , \ x > v(t,s)\}, \\ \partial_p D := \{t < T \ , \ x = v(t,s)\} \ , \\ \partial_T D := \{t = T \ , \ x \geq v(T,s)\} \ . \end{array}$$

PDE Derivation

- **Dynamics:**

$$S^\pi = s + \int_t^\cdot \mu\left(S^\pi(u), \pi_u\right) du + \int_t^\cdot \sigma\left(S^\pi(u), \pi_u\right) dW_u$$
$$X^\pi = x + \int_t^\cdot \rho\left(S^\pi(u), X^\pi(u), \pi_u\right) du + \int_t^\cdot \beta\left(S^\pi(u), X^\pi(u), \pi_u\right) dW_u$$

- **Problems:** Given $F, \bar\ell$ from $\mathbb{R}^d \times \mathbb{R}$ into $\mathbb{R}$:

$$V(t,s,x) := \sup_{\pi \in \bar{\mathcal{A}}^{\bar\ell}_{t,s,x}} \mathbb{E}\left[F\left(S^\pi_{t,s}(T), X^\pi_{t,x,s}(T)\right)\right]$$

where $\bar{\mathcal{A}}^{\bar\ell}_{t,s,x} := \left\{\pi \in \mathcal{A} \text{ s.t. } \bar\ell\left(S^\pi_{t,s}(T), X^\pi_{t,x,s}(T)\right) \geq 0\right\}$.

- Set $D := \{(t,s,x) : \ \bar{\mathcal{A}}^{\bar\ell}_{t,s,x} \neq \emptyset\}$ and $v(t,s) := \inf\{x \in \mathbb{R} : \ (t,s,x) \in D\}$.

- If $\bar\ell$ is non-decreasing in x, then

$$\mathrm{cl}(D) = \mathrm{int}_p D \cup \partial_p D \cup \partial_T D \quad \text{with} \quad \begin{array}{l} \mathrm{int}_p D := \{t < T\ ,\ x > v^*(t,s)\}, \\ \partial_p D := \{t < T\ ,\ x \in [v_*(t,s), v^*(t,s)]\}\ , \\ \partial_T D := \{t = T\ ,\ x \geq v_*(T,s)\}\ . \end{array}$$

PDE in the domain $\text{int}_p D$

- Recall that

$$\text{int}_p D := \{t < T\ ,\ x > v^*(t,s)\} \quad \text{with} \quad v(t,s) := \inf\{x \in \mathbb{R}\ :\ \bar{\mathcal{A}}^{\bar{\ell}}_{t,s,x} \neq \emptyset\}$$

- $x > v^*(t,s) \Rightarrow X^\pi_{t,x,s}(\tau) > v^*(\tau, S^\pi_{t,s}(\tau))$ for $\tau > t$ well chosen and $\pi \in \mathcal{A}$ given.

- Locally can choose any control !

- Associated PDE

$$\inf_{\pi \in A} \left(-\mathcal{L}^\pi_{(S,X)} V(t,s,x)\right) = 0$$

On the boundary $\partial_T D$

- Recall that

$$\partial_T D := \{t = T\ ,\ x \geq v_*(t,s)\} \quad \text{with} \quad v(t,s) := \inf\{x \in \mathbb{R}\ :\ \bar{\mathcal{A}}^{\bar{\ell}}_{t,s,x} \neq \emptyset\}$$

- We have the natural boundary condition: $V(T-, s, x) = F(s, x)$.

PDE on the spacial boundary $\partial_p D$

- Recall that

$$\partial_p D := \{t < T \;,\; x \in [v_*(t,s), v^*(t,s)]\} \text{ with } v(t,s) := \inf\{x \in \mathbb{R} : \bar{\mathcal{A}}^{\bar{\ell}}_{t,s,x} \neq \emptyset\}$$

- Assume v is smooth.

If $x = v(t,s)$, we should have $dX^{\pi}_{t,x,s}(t) \geq dv(t, S^{\pi}_{t,s}(t))$.

This implies that

$$\begin{aligned} \pi_t \in \mathcal{N}(t,s,x,v) \;\; := \;\; \{\pi \in A \;:\; & \beta(s,x,\pi) = \sigma(s,\pi) Dv(t,s) \;, \\ & \rho(s,x,\pi) - \mathcal{L}^{\pi}_S v(t,s) \geq 0\} \;\; . \end{aligned}$$

- PDE on $\partial_p D$

$$\inf_{\pi \in \mathcal{N}(t,s,x,v)} \left(-\mathcal{L}^{\pi}_{(S,X)} V(t,s,x) \right) = 0 \; .$$

PDE formulation: sum up

- On $\text{int}_p D := \{t < T\ ,\ x > v^*(t,s)\}$:

$$\inf_{\pi \in A} \left(-\mathcal{L}^{\pi}_{(S,X)} V(t,s,x) \right) = 0\ .$$

PDE formulation: sum up

- On $\text{int}_p D := \{t < T \ , \ x > v^*(t,s)\}$:

$$\inf_{\pi \in A} \Big(-\mathcal{L}^{\pi}_{(S,X)} V(t,s,x) \Big) = 0 \ .$$

- On $\partial_T D := \{t = T \ , \ x \geq v_*(t,s)\}$: $\quad V(T-,s,x) = F(s,x)$.

PDE formulation: sum up

- On $\text{int}_p D := \{t < T\ ,\ x > v^*(t,s)\}$:

$$\inf_{\pi \in A} \left(-\mathcal{L}^{\pi}_{(S,X)} V(t,s,x) \right) = 0\ .$$

- On $\partial_T D := \{t = T\ ,\ x \geq v_*(t,s)\}$: $\quad V(T-,s,x) = F(s,x)$.

- On $\partial_p D := \{t < T\ ,\ x \in [v_*(t,s), v^*(t,s)]\}$

$$\inf_{\pi \in \mathcal{N}(t,s,x,v)} \left(-\mathcal{L}^{\pi}_{(S,X)} V(t,s,x) \right) = 0\ .$$

with $\mathcal{N}(t,s,x,v) :=$
$\{\pi \in A\ :\ \beta(s,x,\pi) = \sigma(s,\pi) Dv(t,s)\ ,\ \rho(s,x,\pi) - \mathcal{L}^{\pi}_{S} v(t,s) \geq 0\}$.

PDE formulation: sum up

- On $\text{int}_p D := \{t < T\ ,\ x > v^*(t,s)\}$:

$$\inf_{\pi \in A} \left(-\mathcal{L}^{\pi}_{(S,X)} V(t,s,x) \right) = 0\ .$$

- On $\partial_T D := \{t = T\ ,\ x \geq v_*(t,s)\}$: $\quad V(T-,s,x) = F(s,x)$.

- On $\partial_p D := \{t < T\ ,\ x \in [v_*(t,s), v^*(t,s)]\}$

$$\inf_{\pi \in \mathcal{N}(t,s,x,v)} \left(-\mathcal{L}^{\pi}_{(S,X)} V(t,s,x) \right) = 0\ .$$

with $\mathcal{N}(t,s,x,v) :=$
$\{\pi \in A\ :\ \beta(s,x,\pi) = \sigma(s,\pi) Dv(t,s)\ ,\ \rho(s,x,\pi) - \mathcal{L}^{\pi}_{S} v(t,s) \geq 0\}$.

- Already proved:

On $\text{int}_p D$ after relaxing the operator (A may be unbounded).

PDE formulation: sum up

• On $\text{int}_p D := \{t < T \ , \ x > v^*(t,s)\}$:

$$\inf_{\pi \in A} \left(-\mathcal{L}^{\pi}_{(S,X)} V(t,s,x) \right) = 0 \ .$$

• On $\partial_T D := \{t = T \ , \ x \geq v_*(t,s)\}$: $\quad V(T-,s,x) = F(s,x)$.

• On $\partial_p D := \{t < T \ , \ x \in [v_*(t,s), v^*(t,s)]\}$

$$\inf_{\pi \in \mathcal{N}(t,s,x,v)} \left(-\mathcal{L}^{\pi}_{(S,X)} V(t,s,x) \right) = 0 \ .$$

with $\mathcal{N}(t,s,x,v) :=$
$\{\pi \in A \ : \ \beta(s,x,\pi) = \sigma(s,\pi) Dv(t,s) \ , \ \rho(s,x,\pi) - \mathcal{L}^{\pi}_{S} v(t,s) \geq 0\}$.

• Already proved:
On $\partial_T D$ after relaxing the operator (A may be unbounded).

PDE formulation: sum up

- On $\text{int}_p D := \{t < T\ ,\ x > v^*(t,s)\}$:

$$\inf_{\pi \in A} \left(-\mathcal{L}^{\pi}_{(S,X)} V(t,s,x) \right) = 0\ .$$

- On $\partial_T D := \{t = T\ ,\ x \geq v_*(t,s)\}$: $\quad V(T-,s,x) = F(s,x)$.

- On $\partial_p D := \{t < T\ ,\ x \in [v_*(t,s), v^*(t,s)]\}$

$$\inf_{\pi \in \mathcal{N}(t,s,x,v)} \left(-\mathcal{L}^{\pi}_{(S,X)} V(t,s,x) \right) = 0\ .$$

with $\mathcal{N}(t,s,x,v) :=$
$\{\pi \in A\ :\ \beta(s,x,\pi) = \sigma(s,\pi) Dv(t,s)\ ,\ \rho(s,x,\pi) - \mathcal{L}^{\pi}_{S} v(t,s) \geq 0\}$.

- Already proved:

On $\partial_p D$ when v is continuous (need to express the constraint $\mathcal{N}$ in terms of test functions for v).

PDE formulation: sum up

- On $\text{int}_p D := \{t < T\ ,\ x > v^*(t,s)\}$:

$$\inf_{\pi \in A} \left(-\mathcal{L}^{\pi}_{(S,X)} V(t,s,x) \right) = 0\ .$$

- On $\partial_T D := \{t = T\ ,\ x \geq v_*(t,s)\}$: $\quad V(T-,s,x) = F(s,x)$.

- On $\partial_p D := \{t < T\ ,\ x \in [v_*(t,s), v^*(t,s)]\}$

$$\inf_{\pi \in \mathcal{N}(t,s,x,v)} \left(-\mathcal{L}^{\pi}_{(S,X)} V(t,s,x) \right) = 0\ .$$

with $\mathcal{N}(t,s,x,v) :=$
$\{\pi \in A\ :\ \beta(s,x,\pi) = \sigma(s,\pi) Dv(t,s)\ ,\ \rho(s,x,\pi) - \mathcal{L}^{\pi}_{S} v(t,s) \geq 0\}$.

- Already proved:

On $\partial_p D$ when v is not continuous: the constraint does not appear in the subsolution property.

PDE formulation: sum up

- On $\text{int}_p D := \{t < T\ ,\ x > v^*(t,s)\}$:

$$\inf_{\pi \in A} \left(-\mathcal{L}^{\pi}_{(S,X)} V(t,s,x)\right) = 0\ .$$

- On $\partial_T D := \{t = T\ ,\ x \geq v_*(t,s)\}$: $\quad V(T-,s,x) = F(s,x)$.

- On $\partial_p D := \{t < T\ ,\ x \in [v_*(t,s), v^*(t,s)]\}$

$$\inf_{\pi \in \mathcal{N}(t,s,x,v)} \left(-\mathcal{L}^{\pi}_{(S,X)} V(t,s,x)\right) = 0\ .$$

with $\mathcal{N}(t,s,x,v) :=$
$\{\pi \in A\ :\ \beta(s,x,\pi) = \sigma(s,\pi)Dv(t,s)\ ,\ \rho(s,x,\pi) - \mathcal{L}^{\pi}_S v(t,s) \geq 0\}$.

- Already proved:

On $\partial_p D$ when v is $C^{1,2}$: after a change of variable, the boundary is characterized, and leads to a Dirichlet condition on the boundary.

Remaining points to study

1. Comparison principle

2. Numerical schemes on PDE

3. Examples