A Monte Carlo method for fully nonlinear parabolic PDEs with applications to financial mathematics

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- Price: $\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$.
- Bank account with interest rate $= r$.
- Choose an admissible portfolio strategy. Self-financing portfolio: $dX^{x,\theta} = \theta_t dS_t + (X^{x,\theta} - S_t\theta_t)rdt$.
- Change of numéraire: $dX^{x,\theta} = \theta_t \left((\mu - r)dt + \sigma dW_t\right)$.
- To maximize $\mathbb{E}\left[U(X_T^\theta)|X_0 = x\right]$ over all admissible portfolio.
- DPP approach:

$$v(t,x) := \sup_{\theta.} \mathbb{E}\left[U(X_T^\theta)|X_t = x\right].$$

- H-J-B equation:

$$-\frac{\partial v}{\partial t} - \sup_{\theta}\left\{\theta(\mu - r)\frac{\partial v}{\partial x} + \frac{1}{2}\theta^2\sigma^2\frac{\partial^2 v}{\partial x^2}\right\} = 0$$

$$v(T,\cdot) = U(\cdot).$$

- The equation is parabolic and fully nonlinear.

$$\begin{aligned} -\frac{\partial v}{\partial t} + \frac{(\mu - r)^2 \left(\frac{\partial v}{\partial s}\right)^2}{2\sigma^2 \frac{\partial^2 v}{\partial s^2}} &= 0 \\ v(T, x) &= U(x). \end{aligned}$$

- Jump in price: $\frac{dS_t}{S_{t-}} = \mu dt + \sigma dW_t + \int_{\mathbb{R}_*} \eta(z)\tilde{J}(dt, dz)$.
- Fully nonlinear nonlocal parabolic PDE:

$$\begin{aligned} -\frac{\partial v}{\partial t} - \sup_{\theta} \left\{ \theta(\mu - r)\frac{\partial v}{\partial x} + \frac{1}{2}\theta^2\sigma^2\frac{\partial^2 v}{\partial x^2} + \int_{\{|z|\geq 1\}} (v(t, x + \theta\eta(z)) - v(t, x))\, d\nu(z) \right\} &= 0 \\ v(T, \cdot) &= U(\cdot) \end{aligned}$$

- Non financial motivation: coupled FBSDE (Markovian case).

$$
\begin{aligned}
X_t &= x + \int_0^t b(s, X_s, Y_s)ds + \int_0^t \sigma(s, X_s, Y_s)dW_s \\
Y_t &= g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s
\end{aligned}
$$

- Simulation:
  - 4 step scheme ([Ma, Protter, Yong, 1994]).
  - Picard type iteration ([Zhang, Bender] or [Menozzi, Delarue]).

4 step scheme

- Quasi-linear PDE:

$$\begin{aligned} -\frac{\partial v}{\partial t} - \mathrm{tr}[\sigma\sigma^T(t,x,v)D^2 v] + Dv \cdot b(t,x,v) + f(t,x,v,Dv\sigma(t,x,v)) &= 0 \\ v(T,\cdot) &= g(\cdot). \end{aligned}$$

- $Y_t = v(t, X_t)$ and $Z_t = Dv(t, X_t)\sigma(t, x, Y_t)$.
- A Monte Carlo method could be used in 4 step scheme.
- Despite iterative methods, 4 step scheme is not practically implementable in great dimensions.

Fully nonlinear parabolic PDE

$$\begin{aligned} -\mathcal{L}^X v(t,x) - F\left(t,x,v(t,x),Dv(t,x),D^2v(t,x)\right) &= 0, \text{on}[0,T)\times\mathbb{R}^d, \\ v(T,\cdot) &= g, \end{aligned}$$

where

$$\mathcal{L}^X \varphi(t,x) := \left(\frac{\partial\varphi}{\partial t} + \mu\cdot D\varphi + \frac{1}{2}a\cdot D^2\varphi\right)(t,x)$$

Separation of linear and non linear part is to some extent arbitrary.

Scheme idea

Let X be the diffusion corresponding to $\mathcal{L}^X$. If the solution v is smooth enough by Itô lemma:

$$\begin{aligned}\mathbb{E}_{t_i,x}\left[v\left(t_{i+1},X_{t_{i+1}}\right)\right] &= v\left(t_i,x\right)+\mathbb{E}_{t_i,x}\left[\int_{t_i}^{t_{i+1}}\mathcal{L}^X v(t,X_t)dt\right]\\ &= v(t_i,x)-\mathbb{E}_{t_i,x}\left[\int_{t_i}^{t_{i+1}}F(\cdot,v,Dv,D^2v)(t,X_t)dt\right]\end{aligned}$$

$$\begin{aligned}v(t_i,x) &\approx \mathbb{E}_{t_i,x}\left[v\left(t_{i+1},X_{t_{i+1}}\right)\right]\\ &+ hF\left(\cdot,\mathbb{E}_{t_i,x}v(t_{i+1},X_{t_{i+1}}),\mathbb{E}_{t_i,x}Dv(t_{i+1},X_{t_{i+1}}),\mathbb{E}_{t_i,x}D^2v(t_{i+1},X_{t_{i+1}})\right).\end{aligned}$$

The choice of linear operator = the choice of diffusion process

Scheme

$\hat{X}$ is Euler discretization of X.

$$v^h(T,.) := g \quad \text{and} \quad v^h(t_i,x) := \mathbf{T}_h[v^h](t_i,x),$$

where

$$\mathbf{T}_h[v^h]\ (t_i,x) := \mathbb{E}_{t_i,x}\left[v\left(t_{i+1},\hat{X}_{t_{i+1}}\right)\right]$$
$$+ \quad hF\left(\cdot,\mathbb{E}_{t_i,x}v(t_{i+1},\hat{X}_{t_{i+1}}),\mathbb{E}_{t_i,x}Dv(t_{i+1},\hat{X}_{t_{i+1}}),\mathbb{E}_{t_i,x}D^2v(t_{i+1},\hat{X}_{t_{i+1}})\right).$$

Integration by part

Let

$$H_h^0 = 1, \quad H_h^1 = \left(\sigma^{\mathrm{T}}\right)^{-1} \frac{W_h}{h}, \quad H_h^2 = \left(\sigma^{\mathrm{T}}\right)^{-1} \frac{W_h W_h^{\mathrm{T}} - h\mathbf{I}_d}{h^2} \sigma^{-1}.$$

For every function $\varphi : Q_T \to \mathbb{R}$ with exponential growth, we have:

$$\mathbb{E}[D^{(k)}\varphi(t_{i+1}, \hat{X}_h^{t_i,x})] = \mathbb{E}[\varphi(t_{i+1}, \hat{X}_h^{t_i,x}) H_h^k(t_i, x)],$$

- This result gives an Monte Carlo approximation for the derivatives.

2BSDE I

- If the solution v is smooth enough by applying Itô lemma to $Y_t := v(t, X_t)$ and $Z_t := Dv(t, X_t)$:

$$
\begin{aligned}
Y_t &= g(X_T) + \int_t^T F(s, X_s, Y_s, Z_s, \Gamma_s) sd - \int_t^T Z_s dX_s \\
dZ_t &= A_t dt + \Gamma_t dW_t
\end{aligned}
$$

- $(Y_t, Z_t, \Gamma_t, A_t) = (v(t, X_t), Dv(t, X_t), D^2 v(t, X_t), \mathcal{L} Dv(t, X_t))$.
- No existance result. Uniqueness due to [Cheridito-Soner-Touzi-Victoir].

2BSDE II

- $t_i = ih$. Discretization gives:

$$Z_i = \frac{1}{h}\mathbb{E}_i\left[Y_{i+1}\Delta W_{i+1}\right],\ \Gamma_i = \frac{1}{h}\mathbb{E}_i\left[Z_{i+1}\Delta W_{i+1}\right]$$
$$Y_i = \mathbb{E}_i\left[Y_{i+1}h + F(t_i, X_i, Y_i, Z_i, \Gamma_s)\right],\ A_i = ...$$

- Suggested by [Cheridito-Soner-Touzi-Victoir]. Different from $\mathbf{T}_h-$scheme.

$$\mathbf{T}_h\text{-scheme}:\ \ D^{(2)}\varphi \approx \mathbb{E}[\varphi(t_{i+1}, \hat{X}_h^{t_i,x})H_h^2]$$
$$\text{CSTV-scheme}:\ \ D^{(k)}\varphi \approx \mathbb{E}[\varphi(t_{i+1}, \hat{X}_h^{t_i,x})H_{\frac{h}{2}}^1 H_{\frac{h}{2}}^1]$$

Convergence I

Assumption for convergence

The nonlinearity F is Lipschitz-continuous with respect to (r, p, γ) uniformly in (t, x), Lipschitz continuous on x uniformly on other variables, $1/2-$Hölder on t uniformly on other variables and $|F(t, x, 0, 0, 0)|_\infty < \infty$. Moreover, F is uniformly elliptic and dominated by the diffusion of the linear operator $\mathcal{L}^X$, i.e.

$$\varepsilon I_d \leq \nabla_\gamma F \leq a \text{ on } \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d \quad \text{for some} \quad \varepsilon > 0.$$

Convergence II

Convergence Theorem(F., Touzi, Warin)

Assume that the fully nonlinear PDE has comparison for bounded functions. Then for every bounded Lipschitz function g, there exists a bounded function v so that

$$v^h \longrightarrow v \qquad \text{locally uniformly.}$$

In addition, v is the unique bounded viscosity solution of the problem.

Regularity

v is lipschitz on x and $1/2-$Hölder on t.

Rate of convergence I

Assumption for rate of convergence: HJB

The Lipschitz nonlinearity F is of the Hamilton-Jacobi-Bellman type:

$$
\begin{aligned}
F(t,x,r,p,\gamma) &= \inf_{\alpha\in\mathcal{A}}\{\mathcal{L}^\alpha(t,x,r,p,\gamma)\} \\
\mathcal{L}^\alpha(t,x,r,p,\gamma) &:= \frac{1}{2}Tr[\sigma^\alpha\sigma^{\alpha\mathrm{T}}(t,x)\gamma]+b^\alpha(t,x)p+c^\alpha(t,x)r+f^\alpha(t,x)
\end{aligned}
$$

where the functions μ, σ, σ^α, b^α, c^α and f^α satisfy:

$$
|\mu|_\infty+|\sigma|_\infty+\sup_{\alpha\in\mathcal{A}}\left(|\sigma^\alpha|_1+|b^\alpha|_1+|c^\alpha|_1+|f^\alpha|_1\right) \;<\; \infty.
$$

Rate of convergence II

HJB+

The nonlinearity F satisfies **HJB**, and for any $\delta > 0$, there exists a finite set $\{\alpha_i\}_{i=1}^{M_\delta}$ such that for any $\alpha \in \mathcal{A}$:

$$\inf_{1 \le i \le M_\delta} |\sigma^\alpha - \sigma^{\alpha_i}|_\infty + |b^\alpha - b^{\alpha_i}|_\infty + |c^\alpha - c^{\alpha_i}|_\infty + |f^\alpha - f^{\alpha_i}|_\infty \quad \le \quad \delta.$$

Rate of convergence Theorem(F., Touzi, Warin)

Assume that the final condition g is bounded Lipschitz-continuous. Then, there is a constant $C > 0$ such that:

(i) under Assumption HJB, we have $v - v^h \le Ch^{1/4}$,

(ii) under the stronger condition $\mathrm{HJB}+$, we have $-Ch^{1/10} \le v - v^h \le Ch^{1/4}$.

Conditional expectation

- Method is backward. We can calculate $v^h(t,x)$ if we know $v^h(t+h,\cdot)$ at any point.
- Shoud we do simulate many paths at each piont to estimate $v^h(t,x)$? [Touzi, Bouchard] and [Longstaff, Schwartz]
- $\mathbb{E}[\varphi(X_h^x)] = \mathbb{E}[\varphi(X_{t_{i+1}})|X_{t_i} = x]$.
- For the second conditional expectation, we do not need to use the paths which satisfies $X_{t_i} = x$.
- Weighting the sample paths with their t_i time distance from x.

$$\mathbb{E}[\varphi(X_{t_{i+1}})|X_{t_i} = x] \approx \frac{\sum_{l=1}^{N} \varphi(X_{t_{i+1}}^l)\kappa(X_{t_i}^l - x)}{\sum_{l=1}^{N} \kappa(X_{t_i}^l - x)}.$$

Mean curvature I

Mean curvature flow

$$v_t - \Delta v + \frac{Dv \cdot D^2 v Dv}{|Dv|^2} = 0 \quad \text{and} \quad v(0,x) = g(x)$$

Mean curvature II

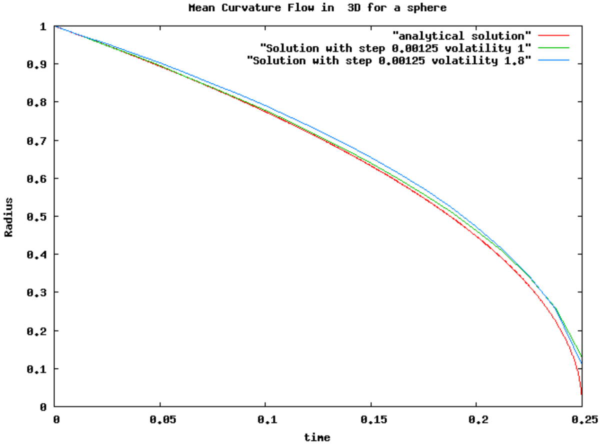


Figure: Solution of the mean curvature flow for the sphere problem

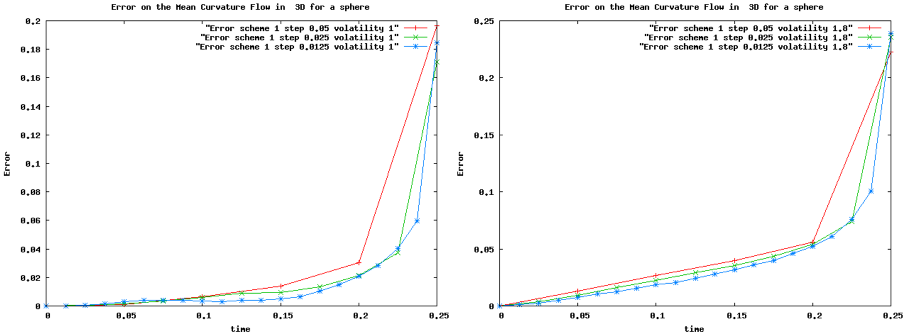


Figure: Mean curvature flow problem for different time step and diffusion: scheme 1

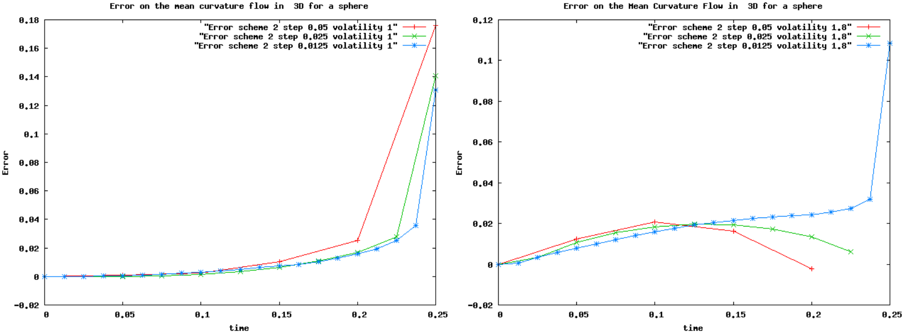


Figure: Mean curvature flow problem for different time step and diffusions: scheme 2

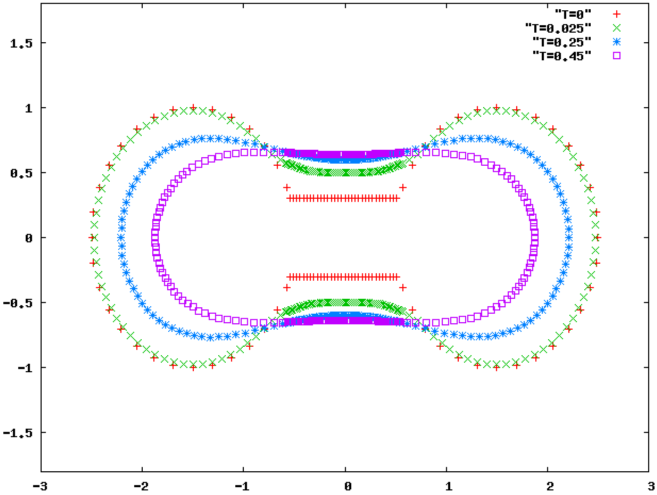


Figure: Mean curvature flow problem in 2D

Portfolio optimization

Two dimensional

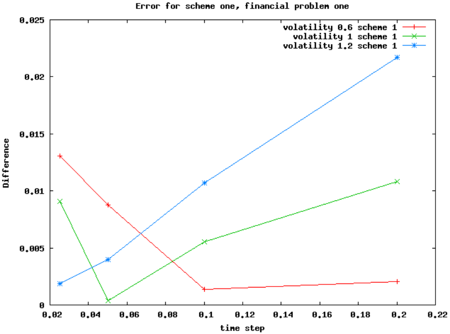
Heston model:

$$\begin{aligned} dS_t &= \mu S_t dt + \sqrt{Y_t} S_t dW_t^{(1)} \\ dY_t &= k(m - Y_t)dt + c\sqrt{Y_t}\left(\rho dW_t^{(1)} + \sqrt{1-\rho^2} dW_t^{(2)}\right). \end{aligned}$$

HJB PDE

$$v(T, x, y) = -e^{-\eta x}$$

$$\begin{aligned} 0 &= -v_t - k(m-y)v_y - \frac{1}{2}c^2 y v_{yy} - \sup_{\theta\in\mathbb{R}}\left(\frac{1}{2}\theta^2 y v_{xx} + \theta(\mu v_x + \rho c y v_{xy}\right) \\ &= -v_t - k(m-y)v_y - \frac{1}{2}c^2 y v_{yy} + \frac{(\mu v_x + \rho c y v_{xy})^2}{2 y v_{xx}}. \end{aligned}$$



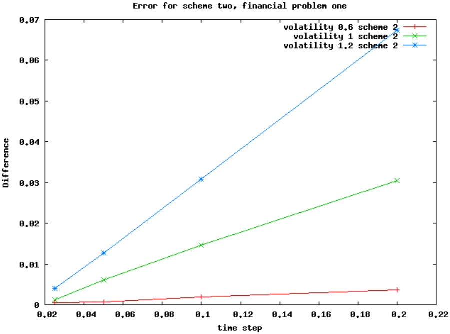


Figure: Difference between calculation and reference for scheme one and two

Portfolio optimization

Five dimensional

Vasicek model:

$$dr_t = \kappa(b - r_t)dt + \zeta dW_t^{(0)}.$$

CEV-SV model and Heston model.

$$\begin{aligned} dS_t^{(i)} &= \mu_i S_t^{(i)} dt + \sigma_i \sqrt{Y_t^{(i)}} {S_t^{(i)}}^{\beta_i} dW_t^{(i,1)}, \quad \beta_2 = 1, \\ dY_t^{(i)} &= k_i \left(m_i - Y_t^{(i)}\right) dt + c_i \sqrt{Y_t^{(i)}} dW_t^{(i,2)} \end{aligned}$$

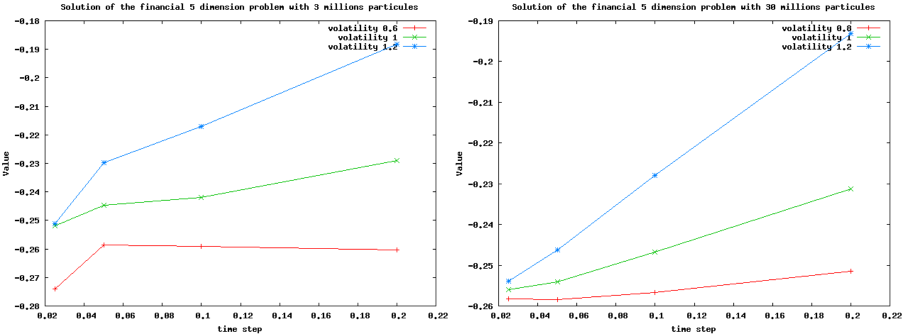


Figure: Five dimensional financial problem and its results for different volatilities with 3 millions and 30 millions particles

Nonlocal parabolic fully nonlinear PDE

$$
\begin{aligned}
&-\mathcal{L}^X v(t,x) - F\left(t,x,v(t,x),Dv(t,x),D^2v(t,x),v(t,\cdot)\right) = 0, \quad \text{on } [0,T)\times\mathbb{R}^d,\\
&v(T,\cdot) = g, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } \in \mathbb{R}^d.
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}^X\varphi(t,x) \quad := \quad & \Big(\frac{\partial\varphi}{\partial t} + \mu\cdot D\varphi + \frac{1}{2}a\cdot D^2\varphi\Big)(t,x)\\
& + \int_{\mathbb{R}^d_*} \left(\varphi(t,x+\eta(t,x,z)) - \varphi(t,x) - \mathbb{1}_{\{|z|\le 1\}} D\varphi(t,x)\eta(t,x,z)\right) d\nu(z).
\end{aligned}
$$

Discretization

$$\hat{X}_h^{t,x,\kappa} = x + \mu(t,x)h + \sigma(t,x)W_h + \int_{\{|z|\geq\kappa\}} \eta(z)\tilde{J}([0,h],dz),$$

$$\hat{X}_{t_{i+1}}^{x,\kappa} = \hat{X}_h^{t_i,\hat{X}_{t_i}^{x,\kappa},\kappa} \quad \text{and} \quad \hat{X}_0^{x,\kappa} = x.$$

$$N_t^\kappa \;=\; \int_{\{|z|\geq\kappa\}} \tilde{J}([0,T],dz).$$

Notice that N_t^κ has intensity equal to $\lambda_\kappa := \int_{\{|z|\geq\kappa\}} \nu(dz)$.

$$\hat{X}_h^{t,x,\kappa} \;=\; x + (\mu - \lambda_\kappa)\, h + \sigma W_h + \sum_{i=1}^{N_h} \eta(Y_i),$$

where Y_is are i.i.d. $\mathbb{R}_*^d-$valued random variables, independent of W distributed as $\frac{1}{\lambda_\kappa}\nu(dz)$.

Integration by part

We can choose κ dependent on h such that $h\lambda_\kappa = \lambda$ for some constant λ.

$$\hat{\nu}_h^\kappa(\psi)(t,x) := \mathbb{E}\left[\int_{\{|z|\geq\kappa\}} \psi(t+h, \hat{X}_h^x + \eta(z))d\nu(z)\right].$$

Lemma

For every function $\varphi : \mathbb{R}^d \to \mathbb{R}$ with exponential growth, we have:

$$\hat{\nu}_h^\kappa(\psi)(t,x) \quad = \quad \frac{1}{h}\mathbb{E}[\psi(t+h, \hat{X}_h^{t,x,\kappa})N_h^\kappa].$$

N_h^κ is a Poisson process with parameter λ.

Integral operator

$$\mathcal{I}[t,x,r,p,\psi] \;:=\; \int_{\mathbb{R}^d_*} \big(\psi(t,x+\eta(t,x,z)) - \psi(t,x) - \mathbb{1}_{\{|z|\le 1\}} D\psi(t,x)\eta(t,x,z)\big)\, d\nu(z)$$

$$\mathcal{I}^\kappa_h[t,x,r,p,\psi] \;:=\; \hat{\nu}^\kappa_h(\psi)(t,x) - \lambda r - \lambda p \cdot \mathbb{E}\left[\mathbb{1}_{\{|Y|\le 1\}}\eta(Y)\right].$$

Final remarks

- The convergence is in $L_\infty-$norm.
- despite the convergence is established on uniform ellipticity assumption, the numerical tests shows the convergence in not uniformly elliptic cases.
- The rate of convergence may not be optimal. It is shown in [F., Touzi, Warin] that the rate for linear equation is $1/2$ from both sides.
- Rate of convergence is derived for convex(concave) nonlinearities.
- The proofs in nonlocal case do not change dramatically.
- Possible extensions: Elliptic problems, Obstacle problem.