Dynamic risk assessment:

martingale aspects of
time-consistency
and the appearance of
bubbles

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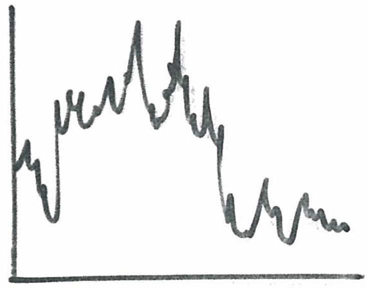
standard setting: 

$$X = (X_t)_{t \geq 0}$$

discounted <u>price process</u> of some liquid asset on

$$(\Omega, \mathcal{F}, P) \qquad (\mathcal{F}_t)_{t \geq 0}$$

$\nexists$ arbitrage

"$\Longleftrightarrow$" $\exists\, P^* \approx P$:

X is <u>martingale</u> under P^*

$$Z_t = \frac{dP^*}{dP}\bigg|_{\mathcal{F}_t} \qquad \text{via Girsanov}$$

$Z = (Z_t)_{t \geq 0}$ <u>martingale</u> under P

Two departures
= "*bubble phenomena*" :

① X is *local* martingale
(not martingale)

Jarrow, Protter (2006, 2009)
Cox, Hobson (2005)

e.g. violation of
put-call parity

F., Protter:
effects of filtration shrinkage

② Z is *local* martingale
(not martingale)

$\Longrightarrow_{\text{FTAP}}$ $\exists$ arbitrage

e.g. R. Fernholz: via "market diversity"
[illegible]

Y. Karatzas, R. Fernholz, D. Fernholz:
INTECH (Princeton)

<u>optimal arbitrage</u>:

$$u(T) := \inf \{ w > 0 \mid \exists \pi : V_T^{\pi, w} \geq 1 \ \ P\text{-a.s.} \}$$

$$= E[Z_T]$$

$$= P^Z[\xi > T]$$

$$< 1 \quad \text{for some } T$$

where

P^Z = measure on

$\bar{\Omega} := \Omega \times (0, \infty]$

$\bar{\mathcal{P}}$:= predictable σ-field

$\xi(\omega, t) := t$ "life time"

(F.: 1972, 1973, 2006 (with A. Gundel)

"<u>bubbles</u>"

$\approx$ <u>local</u> martingales

?

illusionary vs. realistic expectations !

Back to textbook finance/econom.
(in discrete time):

Price of liquid asset (stock) yielding an uncertain cash-flow of future pay-offs (dividends)

$\hat{C}_t \qquad (t = 0, 1, \ldots) \qquad$ adapted, ≥ 0

?

Locally risk-free interest rates:

$r_t \qquad (t = 0, 1, \ldots) \qquad$ predictable

$$B_t = \prod_{k=1}^{t} (1 + r_k) \qquad (t = 0, 1, \ldots)$$

money market account = benchmark

<u>minimal</u> solution of (*) (≥ 0):

$$X_t^0 := E_P\left[\sum_{k=t+1}^{\infty} c_k \,\middle|\, \mathcal{F}_t\right]$$

= "fundamental" component
= "potential" generated by cash-flow

general solution:

$$X_t = X_t^0 + N_t$$

(N_t) = <u>martingale</u> ≥ 0
" <u>bubble</u> "
= "illusionary" component
(sustained / evaporating)
unif. int. $\longrightarrow 0$ P-a.s.

Jean Tirole (MIT / Toulouse): REE
Econometrica 1982

risk-neutral valuation:

global view (long-run):

$$\hat{X}_t = E_P\left[\sum_{k=t+1}^{\infty} \hat{C}_k \underbrace{\prod_{l=t+1}^{k} (1+r_l)^{-1}}_{= \frac{B_t}{B_k}} \,\middle|\, \mathcal{F}_t\right]$$

= expected sum of discounted future cash-flow

in discounted terms $X_t := \frac{\hat{X}_t}{B_t}$, $C_t = \frac{\hat{C}_t}{B_t}$

$$X_t = E_P\left[\sum_{k=t+1}^{\infty} C_k \,\middle|\, \mathcal{F}_t\right]$$

local view (myopic):

$$\hat{X}_t = \frac{1}{1+r_{t+1}} E_P\left[\hat{C}_{t+1} + \hat{X}_{t+1} \,\middle|\, \mathcal{F}_t\right]$$

or

$$(*) \quad X_t = E_P\left[C_{t+1} + X_{t+1} \,\middle|\, \mathcal{F}_t\right]$$

"[illegible]"

HUMBLY SPEAKING

"Our ability to forecast is limited. We have to have some humility."

— **Fed chairman Alan Greenspan**

Photo: REUTERS Design: LOH JAHAN

$$P = ?$$

role of model uncertainty

("Knightian uncertainty")

?

Axiomatics of monetary valuation / risk measures

$$\varphi = -\rho$$

(Artzner, Delbaen, Eber, Heath (1999)
F., Schied (2002), Frittelli, Rosazza-Gianin (2002))
...

on <u>random variables</u>
("financial positions")

$$C_T \in L^\infty(\Omega, \mathcal{F}_T, P):$$

$$\rho_t : L^\infty \longrightarrow L^\infty_t$$

convex monetary risk measure
Fatou property

<u>robust (dual) representation</u>

$$\rho_t(C_T) = \operatorname*{ess.sup}_{Q \ll P} \left(E_Q[-C_T \mid \mathcal{F}_t] - \alpha_t(Q) \right)$$

with

<u>penalty</u> $\alpha_t(Q) = \operatorname*{ess.sup}_{X \text{ "acceptable"},\ \text{i.e. } \rho_t(X) \le 0} E_Q[-X \mid \mathcal{F}_t]$

"robust" view:

not one model Q,
but a whole

class $\mathcal{Q}$

of probability measures Q
on $(\Omega, \mathcal{F})$

cf.

- convex risk measures
- microeconomic theory of preferences (Gilboa, Schmeidler)
- robust statistics (Huber, Hampel)

preferences

$\succ$

on $\mathcal{X}$ = a space of "financial positions" $X : \Omega \to \mathbb{R}$

or, more generally, on

$\tilde{\mathcal{X}}$:= all stochastic kernels $\tilde{X}(\omega, dx)$ from $(\Omega, \mathcal{F})$ to $\mathbb{R}$

contains

a) "lotteries" $\tilde{X}(\omega, \cdot) \equiv \mu$

b) "positions" $\tilde{X}(\omega, \cdot) = \delta_{X(\omega)}$

<u>numerical representation:</u>

$$\tilde{X} \prec \tilde{Y} \iff U(\tilde{X}) < U(\tilde{Y})$$

Axioms of "Rationality":

(1) von Neumann – Morgenstern

(2) Savage, Anscombe, Aumann

$$U(\hat{X}) = E_Q\left[\int u(x)\, \hat{X}(\cdot, dx)\right]$$

"expected utility"

(3) Gilboa, Schmeidler (1989)

(4) Maccheroni, Marinacci, Rustichini (Econometrica 2006)

$\Downarrow$ F., Schied, Weber (2009)

$$U(\tilde{X}) = -\rho\left(\int u(x)\,\tilde{X}(\cdot, dx)\right)$$

for some <u>convex risk measure</u> ρ

$$\underset{\text{above}}{=} \inf_{Q \in \mathcal{Q}} \left(E_Q[u(X)] + \alpha(Q) \right)$$

= "worst case" expected utility over some class of probabilistic models, suitably penalized

(1,2) : $\rho = -E_Q$

(3) : ρ coherent

(4) : ρ convex

Samuel Drapeau: conditional version

Example: <u>entropic risk measure</u>

$$\rho_t(X) = \frac{1}{\beta} \log E_P[e^{-\beta X} \mid \mathcal{F}_t]$$

$\Leftrightarrow$

$$\alpha_t(Q) = \frac{1}{\beta} \underline{\text{remaining relative entropy}} \text{ of } Q \text{ w.r.t. } P$$

$$= \frac{1}{\beta} H(Q(\cdot \mid \mathcal{F}_t) \mid P(\cdot \mid \mathcal{F}_t))$$

$$= \frac{1}{\beta} E_Q\left[\log \frac{dQ(\cdot \mid \mathcal{F}_t)}{dP(\cdot \mid \mathcal{F}_t)} \mid \mathcal{F}_t\right]$$

In this case: "time-consistent", i.e.

$$\rho_t(-\rho_{t+1}(X)) = \rho_t\left(-\frac{1}{\beta}\log E_P[e^{-\beta X} \mid \mathcal{F}_{t+1}]\right)$$

$$= \frac{1}{\beta} \log E_P[E_P[e^{-\beta X} \mid \mathcal{F}_{t+1}] \mid \mathcal{F}_t]$$

$$= \frac{1}{\beta} \log E_P[e^{-\beta X} \mid \mathcal{F}_t] = \rho_t(X)$$

projective

Time consistency ?
(as t varies)

<u>strong</u> consistency: (looking backward

$$\rho_{t+1}(X) \leq \rho_{t+1}(Y) \quad \text{a.s.}$$

$$\Longrightarrow \quad \rho_t(X) \leq \rho_t(Y)$$

equivalent to

<u>recursiveness</u>

$$\rho_t = \rho_t(-\rho_{t+1})$$

(see references above)

~ projectivity of conditional expectat

~ Bellman principle

Characterizations of

<u>strong time consistency</u>

in terms of acceptance sets and penalty functions:

$$\mathcal{A}_{t,t+s} := \{X \in \mathcal{L}^\infty_{t+s} \mid \rho_t(X) \le 0\}$$

$$\alpha_{t,t+s}(Q) := \operatorname*{ess.sup}_{X \in \mathcal{A}_{t,t+s}} E_Q[-X \mid \mathcal{F}_t]$$

$$\mathcal{Q}_0 := \{Q \approx R \mid \alpha_0(Q) < \infty\}$$

ADEHK (2004)
Cheridito-Delbaen-Kupper (2006)
Bion-Nadal (2006)
Burgert (2005)
F-Penner (2006)

Theorem:

strong time-consistency

$$\iff \quad \mathcal{A}_t = \mathcal{A}_{t,t+s} + \mathcal{A}_{t+s}$$

$$\iff \quad \alpha_t(Q) = \alpha_{t,t+s}(Q) + E_Q[\alpha_{t+s}(Q) \mid \mathcal{F}_t]$$

$\iff$ for $X \in L^\infty$, $Q \in \mathcal{Q}_0$:

$$V_t^Q(X) := \rho_t(X) + \alpha_t(Q), \quad t = 0, 1, \ldots$$

is a <u>Q-supermartingale</u>

Corollary (for $X \equiv 0$)

$$\alpha_t(Q) \quad (t = 0, 1, \ldots)$$

is a <u>Q- supermartingale</u> $\forall Q \in \mathcal{Q}_c$

"learning effect"

— a minimal requirement, equivalent to

<u>"weak"</u> time - consistency

$$\rho_{t+1}(X) \leq 0 \implies \rho_t(X) \leq 0$$

But: <u>Strong</u> time - consistency yields more : explicit <u>Doob decomposition</u>

i.e.

$$\alpha_t(Q) + \underbrace{\sum_{k=0}^{t-1} \alpha_{k,k+1}(Q)}_{\nearrow \text{ predictable}} = \underline{Q\text{- martingale}}$$

$$(*) = \underbrace{E_Q\left[\sum_{k=0}^{\infty} \alpha_{k,k+1}(Q) \,\middle|\, \mathcal{F}_t\right]}_{\text{"fundamental"}} + \underline{\underline{\text{"bubble"}}}$$

Asymptotic Behavior:

- $V_t^Q(X) = \rho_t(X) + \alpha_t(Q) \geq E_Q[-X|\mathcal{F}_t]$
- $\alpha_t(Q) \geq 0$

are both supermartingales bounded from below, hence

<u>convergent</u> Q-a.s.

$\Longrightarrow$

$$\exists \quad \rho_\infty(X) = \lim_{t \uparrow \infty} \rho_t(X)$$

= non-linear analogue of martingale convergence

Asymptotic safety
(precision)

$$:\Longleftrightarrow \qquad \rho_\infty(X) \geqq -X$$

$$(=)$$

may or may not hold
(counter-examples)

Proposition (F., Penner):

$\Longleftrightarrow$ <u>no bubbles</u> in (*)

i.e.

each $\alpha_t(Q)$ $(t=0,1,..)$
is a "Q – Potential"

i.e. <u>no excessive penalization</u> (neglect!) of relevant models

Risk measures on cash-flows:

Cheridito, Delbaen, Kupper (2006, ...)

N. El Karoui, C. Ravanelli (2008)

B. Acciao, F., I. Penner (2009)

cash flow: C_t $(t = 0, 1, \dots)$

adapted process

$\cong$ a measurable function

$$\bar{C} \in \bar{L}^\infty := L^\infty(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$$

on

$$\bar{\Omega} = \Omega \times \{0, 1, \dots\}$$

$$\bar{\mathcal{F}} = \sigma(\text{adapted processes})$$

$$\bar{P} = P \otimes \mu$$

$$\rho : \mathbb{L}^\infty \longrightarrow \mathbb{R}^1$$

convex risk measure on

<u>cumulated cash flows</u>

$$X_t := \sum_{s=0}^{t} C_s \quad (t=0,1,\ldots)$$

$$\cong \bar{X} \in \bar{\mathbb{L}}^\infty$$

<u>cash - invariant</u>:

$$\rho(\bar{X} + \bar{m}) = \rho(\bar{X}) - m$$

$$\bar{m} = m I_{\{0,1,\ldots\}}$$

$$\Downarrow$$

$$\rho(\bar{X}) = \sup_{\bar{Q} \ll \bar{P}} \left(E_{\bar{Q}}[-\bar{X}] - \alpha(\bar{Q}) \right)$$

$$= \; ?$$

$\bar{Q} \ll \bar{P} = P \otimes \mu$

on optional σ-field $\bar{\mathcal{F}}$

$\Longrightarrow$

$\bar{Q} = Q \otimes \gamma$

via multiplicative decomposition (Itô - Watanabe in discrete time)

$Q \ll_{loc} P$

γ optional random measure on $\{0, 1, \dots\}$

resp. $D_t := \sum_{s \geq t} \gamma_s \searrow$, predictable, $D_0 = 1$

Thus:

$$\rho(\bar{X}) \quad \text{resp.} \quad \rho(\bar{C})$$

$$= \sup_{Q \ll_{loc} P} \; \sup_{(D) \text{ predictable}} \left(E_Q\left[\sum_{t=0}^{\infty} D_t (-C_t)\right] - \alpha_t(Q, D) \right)$$

"combines model uncertainty with (subjective) discounting preferences" translated into

<u>monetary valuation</u> $\varphi = -\rho$;

<u>dynamic</u> version:

$$\varphi_t(C_{t+1}, C_{t+2}, \dots)$$

$$= \underset{\substack{Q \ll_{loc} P \\ D \downarrow \text{ predictable}}}{\text{ess. inf}} \left(E_Q\left[\sum_{s=t+1}^{\infty} \frac{D_s}{D_t} C_s \right] + \alpha_t(Q, D) \right)$$

<u>Time consistency</u> $\iff$ $\forall Q, D$, $\alpha_t := \alpha_t(Q, D)$

$$D_t \alpha_t = D_t \alpha_{t,t+1} + E_Q[D_{t+1} \alpha_{t+1} \mid \mathcal{F}_t]$$

<u>Q-supermartingale</u>, = "potential" + <u>"bubble"</u>

$$\Longleftrightarrow \quad \forall \bar{C}, \quad \forall Q, D :$$

$$\alpha_t := \alpha_t(Q, D)$$

$$\varphi_t := \varphi_t(C_{t+1}, C_{t+2}, \ldots)$$

$$\underbrace{D_t \varphi_t}_{\text{"proxy" for future cash-flow}} + \sum_{s=0}^{t} D_s C_s - D_t \alpha_t$$

$$= \underline{Q\text{-submartingale}}$$

$$\leq E_Q\left[\sum_{s=0}^{\infty} D_s C_s \,\middle|\, \mathcal{F}_t\right]$$

hence

<u>convergent</u> Q-a.s.

Thus:

$$\exists \lim_{t \uparrow \infty} D_t \varphi_t \quad Q\text{-a.s.}$$

Asymptotic safety (precision)

$:\Longleftrightarrow \quad \varphi_t := \varphi_t(C_{t+1}, C_{t+2} \dots), \quad \alpha_t := \alpha_t(Q, D)$

$$\underbrace{D_t \varphi_t}_{\text{"proxy" for future cash flow}} + \sum_{s=0}^{t} D_s C_s \longrightarrow \underset{(=)}{\leq} \sum_{s=0}^{\infty} D_s C_s$$

Q-a.s. and in $L^1(Q)$ $\quad \forall Q, D: \alpha_0(Q, D) < \infty$

$\Longleftrightarrow \quad D_t \alpha_t$ is Q-potential,

i.e.

$\nexists$ bubbles in the penalization

i.e. no excessive neglect of relevant models !

Classical setting: $\alpha_{t,t+1}(Q) \equiv 0$

i.e.

$\alpha_t \equiv 0$ or martingale $\neq 0$ = "bubble"