Robust utility maximization problem from terminal wealth and consumption : BSDE approach

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Plan de l'exposé

PLAN

- Bordigoni G., M. A., Schweizer, M. : A Stochastic control approach to a robust utility maximization problem. *Stochastic Analysis and Applications. Proceedings of the Second Abel Symposium, Oslo, 2005*, *Springer*, *125-151 (2007)*.
- Jeanblanc, M., M. A., Ngoupeyou, A. : Robust utility maximization from terminal wealth and consumption in a discontinuous filtration. forthcoming paper.

PROBLEM

We present a problem of *utility maximization* under *model uncertainty* :

$$\sup_{\pi} \inf_{Q} \mathbf{U}(\pi, Q), \tag{1}$$

where

- π runs through a set of strategies (portfolios, investment decisions, ...)
- Q runs through a set of models $\mathcal{Q}$.

ONE KNOWN MODEL CASE

- If we have a one known model P : in this case, $\mathcal{Q} = \{P\}$ for P a given reference probability measure and $\mathbf{U}(\pi, P)$ has the form of a P-expected utility from terminal wealth and/or consumption, namely

$$\mathbf{U}(\pi, P) = \mathbb{E}\big(U(X_T^\pi)\big)$$

  where
- X^π is the wealth process

  and
- U is some utility function.

REFERENCES : DUAL APPROACH

- Schachermayer (2001) (one single model)
- A. Schied (2007), Schied and Wu (2005)
- H. Föllmer and A. Gundel, A. Gundel (2005)
- others missing references . . . (sorry !)

REFERENCES : BSDE APPROACH

- El Karoui, Quenez and Peng (2001) : Dynamic maximum principle (one single model)
- Lazrak-Quenez (2003), Quenez (2004), $\mathcal{Q} \neq \{\mathbb{P}\}$ but one keep $\mathbf{U}(\pi, \mathbb{Q})$ as an expected utility
- Duffie and Epstein (1992), Duffie and Skiadas (1994), Skiadas (2003), Schroder & Skiadas (1999, 2003, 2005) : Stochastic Differential Utility and BSDE.
- Hansen & Sargent : they discuss the problem of robust utility maximization when model uncertainty is penalized by a relative entropy term.
- They study the problem in Markovian settings and use mainly formal manipulations of Hamilton-Jacobi-Bellman (HJB) equations to provide the optimal investment behaviour in these situations.

PLAN

1. INTRODUCTION

2. THE MINIMIZATION PROBLEM

3. COMPARISON THEOREM AND REGULARITIES FOR THE BSDE

4. MAXIMIZATION OVER CONSUMPTION AND TERMINAL WEALTH

Let us given :

- Final horizon : $T < \infty$
- $(\Omega, \mathcal{F}, \mathbb{F}, P)$ a filtered probability space where $\mathbb{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$ is a filtration satisfying the usual conditions of right-continuity and $\mathbb{P}$-completness.
- Possible scenarios given by

$$\mathcal{Q} := \{Q \ \text{ probability measure on } \ \Omega \text{ such that } \ Q \ll P \ \text{ on } \ \mathcal{F}_T\}$$

- the density process of $Q \in \mathcal{Q}$ is the càdlàg P-martingale

$$Z_t^Q \quad = \quad \frac{dQ}{dP}\Big|_{\mathcal{F}_t} = \mathbb{E}_P\Big[\frac{dQ}{dP} \,\Big|\, \mathcal{F}_t\Big]$$

- we may identify Z^Q with Q.
- Discounting process : $S_t^\delta := \exp(-\int_0^t \delta_s \, ds)$ with a discount rate process $\delta = \{\delta_t\}_{0 \leq t \leq T}$.

- Let $\mathcal{U}^{\delta}_{t,T}(Q)$ be a quantity given by

$$\mathcal{U}^{\delta}_{t,T}(Q) \quad = \quad \alpha \int_t^T \frac{S^{\delta}_s}{S^{\delta}_t} U_s \, ds + \alpha' \frac{S^{\delta}_T}{S^{\delta}_t} \overline{U}_T$$

- where $U = (U_t)_{t \in [0,T]}$ is a utility rate process which comes from consumption and $\overline{U}_T$ is the terminal utility at time T which corresponds to final wealth.
- α, α' are some constant which can be used to obtain special cases.
- Let $\mathcal{R}^{\delta}_{t,T}(Q)$ be a penalty term

$$\mathcal{R}^{\delta}_{t,T}(Q) \quad = \quad \int_t^T \delta_s \frac{S^{\delta}_s}{S^{\delta}_t} \log \frac{Z^Q_s}{Z^Q_t} \, ds + \frac{S^{\delta}_T}{S^{\delta}_t} \log \frac{Z^Q_T}{Z^Q_t}.$$

for $Q \ll P$ on $\mathcal{F}_T$.

COST FUNCTIONAL

- We consider the cost functional

$$c(\omega, Q) := \mathcal{U}^{\delta}_{0,T}(Q) + \beta \mathcal{R}^{\delta}_{0,T}(Q) \ .$$

  with $\beta > 0$ is a constant which determines the strength of this penalty term.
- Our first goal is to

$$\text{minimize the functional} \quad Q \longmapsto \Gamma(Q) := \mathbb{E}_Q\big[c(.,Q)\big]$$

  over a suitable class of probability measures $Q \ll P$ on $\mathcal{F}_T$.

Relative entropy

- Under the reference probability P the cost functional $\Gamma(Q)$ can be written :

$$\Gamma(Q) = \mathbb{E}_P \left[Z_T^Q \Big(\alpha \int_0^T S_s^\delta U_s \, ds + \alpha' S_T^\delta \overline{U}_T \Big) \right] + \beta \mathbb{E}_P \left[\int_0^T \delta_s S_s^\delta Z_s^\delta \log Z_s^Q ds + S_T^\delta Z_T^Q \log Z_T^Q \right].$$

- The second term is a discounted relative entropy with both an entropy rate as well a terminal entropy :

$$H(Q|P) := \begin{cases} \mathbb{E}_Q \left[\log Z_T^Q \right], & \text{if } Q \ll P \text{ on } \mathcal{F}_T \\ +\infty, & \text{if not} \end{cases}$$

FUNCTIONAL SPACES AND HYPOTHESES

- D_0^{exp} is the space of progressively measurable processes $y = (y_t)$ such that

$$\mathbb{E}_P\Big[\exp\left(\gamma \operatorname{ess\,sup}_{0\le t\le T}|y_t|\right)\Big] < \infty, \quad \text{for all } \gamma > 0\,.$$

- D_1^{exp} is the space of progressively measurable processes $y = (y_t)$ such that

$$\mathbb{E}_P\Big[\exp\left(\gamma \int_0^T |y_s|\,ds\right)\Big] < \infty \quad \text{for all } \gamma > 0\,.$$

- **Assumption (A)** : $0 \le \delta \le \|\delta\|_\infty < \infty$, $U \in D_1^{exp}$ and $\mathbb{E}_P\left[\exp\left(\gamma|\overline{U}_T|\right)\right] < \infty$, for all $\gamma > 0$.

THE CASE : $\delta = 0$

- The spacial case $\delta = 0$ corresponds to the cost functional

$$\Gamma(Q) = \mathbb{E}_Q\left[\mathcal{U}^0_{0,T}\right] + \beta H(Q|P) = \beta H(Q|P_{\mathcal{U}}) - \beta \log \mathbb{E}_P\left[\exp\left(-\frac{1}{\beta}\mathcal{U}^0_{0,T}\right)\right.$$

where $P_{\mathcal{U}} \approx P$ and $\frac{dP_{\mathcal{U}}}{dP} = c \exp\left(-\frac{1}{\beta}\mathcal{U}^0_{0,T}\right)$.

- Csiszar (1997) have proved the existence and uniqueness of the optimal measure $Q^* \approx P_{\mathcal{U}}$ which minimize the relative entropy $H(Q|P_{\mathcal{U}})$.

- I. Csiszár : *I*-divergence geometry of probability distributions and minimization problems. *Annals of Probability* **3**, p. 146-158 (1975).

Class of probability measure

- Due to the assumption on δ, a simple estimation gives

$$\mathbb{E}_P\left[S_T^\delta Z_T^Q \log Z_T^Q\right] \geq -e^{-1} + e^{-\|\delta\|_\infty} H(Q|P)\,.$$

- Hence the second term in $\Gamma(Q)$ explodes unless $H(Q|P) < \infty$.
- This explains why we only consider measures Q in $\mathcal{Q}_f$:= the space of all probability measures Q on $(\Omega, \mathcal{F})$ with $Q \ll P$ on $\mathcal{F}_T$, $Q = P$ on $\mathcal{F}_0$ and $H(Q|P) < \infty$.
- $\mathcal{Q}_f^e := \{Q \in \mathcal{Q}_f \mid Q \approx P \text{ on } \mathcal{F}_T\}$.

- We have the following result :

Theorem (Bordigoni G., M. A., Schweizer, M.)

(i) There exits a unique $Q^* \in \mathcal{Q}_f$ *which minimizes* $Q \mapsto \Gamma(Q)$ *aver all* $Q \in \mathcal{Q}_f$.

(ii) The optimal measure Q^* *is equivalent to* P.

We embed the minimization of $\Gamma(Q)$ in a stochastic control problem :

- The minimal conditional cost

$$J(\tau, Q) := Q - \text{ess}\inf_{Q' \in \mathcal{D}(Q,\tau)} \Gamma(\tau, Q')$$

  with$\Gamma(\tau, Q) := \mathbb{E}_Q\left[c(\cdot, Q) \,|\, \mathcal{F}_\tau\right]$,

- $\mathcal{D}(Q, \tau) = \{Z^{Q'} \,|\, Q' \in \mathcal{Q}_f \text{ et } Q' = Q \text{ sur } \mathcal{F}_\tau\}$ and $\tau \in \mathcal{S}$.

- So, we can write our optimization problem as

$$\inf_{Q \in \mathcal{Q}_f} \Gamma(Q) = \inf_{Q \in \mathcal{Q}_f} \mathbb{E}_Q\left[c(\cdot, Q)\right] = \mathbb{E}_P\left[J(0, Q)\right].$$

- We obtain the following martingale optimality principle from stochastic control :

We have obtained by following El Karoui (1981) :

PROPOSITION (BORDIGONI G., M. A., SCHWEIZER, M.)

1. *The family* $\{J(\tau, Q) \,|\, \tau \in \mathcal{S}, Q \in \mathcal{Q}_f\}$ *is a submartingale system ;*
2. $\tilde{Q} \in \mathcal{Q}_f$ *is optimal if and only if* $\{J(\tau, \tilde{Q}) \,|\, \tau \in \mathcal{S}\}$ *is a* $\tilde{Q}$*-martingale system ;*
3. *For each* $Q \in \mathcal{Q}_f$*, there exists an adapted RCLL process* $J^Q = (J_t^Q)_{0 \le t \le T}$ *which is a right closed* Q*-submartingale such that*

$$J_\tau^Q = J(\tau, Q)$$

- We define for all $Q' \in \mathcal{Q}_f^e$ and $\tau \in \mathcal{S}$:

$$\tilde{V}(\tau, Q') := \mathbb{E}_{Q'}\left[\mathcal{U}^{\delta}_{\tau,T} \mid \mathcal{F}_\tau\right] + \beta \mathbb{E}_{Q'}\left[\mathcal{R}^{\delta}_{\tau,T}(Q') \mid \mathcal{F}_\tau\right]$$

- The value of the control problem started at time τ instead of 0 is :

$$V(\tau, Q) := Q - \text{ess inf}_{Q' \in \mathcal{D}(Q,\tau)} \tilde{V}(\tau, Q')$$

- By using the Bayes formula and the definition of $\mathcal{R}^{\delta}_{\tau,T}(Q')$, one sees that each $\tilde{V}(\tau, Q')$ depends only on the values of $Z^{Q'}$ on $]\tau, T[$ and therefore not on Q, since $Q' \in \mathcal{D}(Q, \tau)$ only says that $Z^{Q'} = Z^Q$ on $[0, \tau]$.
- So we can equally well take the ess inf under $P \approx Q$ and over all $Q' \in \mathcal{Q}_f$ and $V(\tau) \equiv V(\tau, Q')$ and one proves that V is P-special semimartingale with canonical decomposition

$$V = V_0 + M^V + A^V$$

SEMIMARTINGALE BSDE

- We need precise information on the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \leq T}$.
- Let first consider the following quadratic semimartingale BSDE with :

DEFINITION

A solution of the BSDE is a pair of processes (Y, M) such that Y is a P-semimartingale and M is a locally square-integrable locally martingale with $M_0 = 0$ such that :

$$\begin{cases} -dY_t = (\alpha U_t - \delta_t Y_t)dt - \dfrac{1}{2\beta} d < M >_t \; - dM_t \\ Y_T = \alpha' \overline{U}_T \end{cases}$$

- *Note that Y is then automatically P-special, and that if M is continuous, so is Y.*

A^V AND M^V : THE CONTINUOUS FILTRATION CASE

THEOREM (BORDIGONI G., M. A., SCHWEIZER, M.)

Assume that $\mathbb{F}$ is continuous. Then the couple (V, M^V) is the unique solution in $D_0^{\exp} \times \mathcal{M}_{0,loc}(P)$ of the BSDE

$$\begin{cases} -dY_t = (\alpha U_t - \delta_t Y_t)dt - \dfrac{1}{2\beta} d < M >_t \; - dM_t \\ Y_T = \alpha' U'_T \end{cases}$$

- *Moreover, $\mathcal{E}\left(-\frac{1}{\beta} M^V\right) = Z^{Q^*}$ is a $P-$martingale such that it's supremum belongs to $L^1(P)$ where Q^* is the optimal probability.*

Recursive relation

Lemma

Let (Y, M) be a solution of BSDE with M continuous. Assume that $Y \in D_0^{\exp}$ or $\mathcal{E}\left(-\frac{1}{\beta}M\right)$ is P−martingale.
For any pair of stopping times $\sigma \leq \tau$, then we have the recursive relation

$$Y_\sigma = -\beta \log E_P\Big[\exp\left(\frac{1}{\beta}\int_\sigma^\tau (\delta_s Y_s - \alpha U_s)\, ds - \frac{1}{\beta} Y_\tau\right) \Big| \mathcal{F}_\sigma\Big]$$

- As a consequence one gets the uniqueness result for the semimartingale BSDE.

$\delta = 0$: THE ENTROPIC SOLUTION

- In the case of $\delta = 0$, $\sigma = t$ et $\tau = T$, we get from the recursive relation the explicit solution, which corresponds to the entropic process (also entropic risk measure) :

$$Y_t = -\beta \log E_P\Big[\exp\left(-\frac{\alpha}{\beta} \int_t^T U_s \, ds - \frac{1}{\beta} Y_T \right) \Big| \mathcal{F}_t \Big]$$

BSDE : Brownian filtration

Remark

- *If $\mathbb{F} = \mathbb{F}^W$, for a given Brownian mtotion, then the semimartingale BSDE takes the standard form of quadratique BSDE :*

$$\begin{cases} -dY_t = \left(\rho_t + \delta_t Y_t - \frac{1}{2\beta}|Z_t|^2\right))dt \; - \; Z_t \cdot dW_t \\ Y_T = \xi \end{cases}$$

- *Kobylanski (2000), Lepletier et San Martin (1998), El Karoui and Hamadène (2003), Briand and Hu (2005).*

Existence proof : main steps

1. We use the martingale optimality principle to show that (V, M^V) is solution of the BSDE. For each $Q \in \mathcal{Q}_f^e$, we have $Z^Q = \mathcal{E}(L^Q)$ for some continuous local P-martingale L^Q null at 0, and we have
$$d(\log Z^Q) = dL^Q - \frac{1}{2} d\langle L^Q \rangle.$$
The semimartingale decomposition of J^Q + Girsanov theorem + optimality imply that :
$$A^V = \int (\delta_t V_t - \alpha U_t)dt - \text{ess} \inf_{Q \in Q_f^e} \left(\langle M^V, L^Q \rangle + \frac{\beta}{2} \langle L^Q \rangle \right).$$
2. We show that
$$\text{ess} \inf_{Q \in Q_f^e} \left(\langle M^V, L^Q \rangle + \frac{\beta}{2} \langle L^Q \rangle \right) = -\frac{1}{2\beta} \langle M^V \rangle$$
that is the ess inf is attained for $L^{Q^*} = -\frac{1}{\beta} M^V$.
3. $\mathcal{E}\left(\frac{1}{\beta} M^V\right)$ is a true P-martingale.

- For any $i = 1, \dots, d$, we note $H_t^i = \mathbf{1}_{\{\tau_i \le t\}}$ the jump process associated with τ_i, where τ_i is the $\mathbb{F}$-stopping time representing the default time of the firm i. We assume that $P(\tau_i = \tau_j) = 0, \forall i \neq j$.
- Let also $N^i, i = 1, \dots, d$ be given by $N_t^i := H_t^i - \int_0^t \lambda_s^i ds$ assumed to be $\mathbb{F}$-martingales for a non-negative processes λ_i. Obviously, the process λ^i is null after the default time τ_i, and these stopping times are totally inaccessible.
- Any special semimartingale Y admits a canonical decomposition $Y = Y_0 + A + Y^c + Y^d$ where A is a predictable finite variation process, Y^c is a continuous martingale and Y^d is a discontinuous martingale. In our case, there exists predictable processes y and $\widehat{Y}^i$ such that

$$dY_t^c = y_t dW_t, \; dY_t^d = \sum_{i=1}^{d} \widehat{Y}_t^i dN_t^i.$$

- Let first consider the following quadratic semimartingale BSDE with jumps :

DEFINITION

A solution of the BSDE is a triple of processes $(Y, M^{Y,c}, \widehat{Y})$ *such that* Y *is a* P*-semimartingale,* M *is a locally square-integrable locally martingale with* $M_0 = 0$ *and* $\widehat{Y} = (\widehat{Y}^1, \cdots, \widehat{Y}^d)$ *a* $\mathbb{R}^d$*-valued predictable locally bounded process such that :*

$$\begin{cases} dY_t = [\sum_{i=1}^{d} g(\widehat{Y}_t^i)\lambda_t^i - \alpha U_t + \delta_t Y_t]dt + \frac{1}{2}d\langle M^{Y,c}\rangle_t + dM_t^{Y,c} + \sum_{i=1}^{d} \widehat{Y}_t^i dN_t^i \\ Y_T = \bar{\alpha}\bar{U}_T \end{cases} \qquad (2)$$

where $g(x) = e^{-x} + x - 1$.

THEOREM (JEANBLANC, M., M. A., NGOUPEYOU A.)

There exists a unique triple of process $(Y, M^{Y,c}, \widehat{Y}) \in D_0^{\exp} \times \mathcal{M}_{0,loc}(P) \times \left(D_0^{\exp}\right)^{\otimes d}$ *solution of the semartingale BSDE with jumps. Furthermore, the optimal measure* Q^* *solution of our minimization problem is given :*

$$dZ_t^{Q^*} = Z_{t-}^{Q^*} dL_t^{Q^*}, \quad Z_0^{Q^*} = 1$$

where

$$dL_t^{Q^*} = -dM_t^{Y,c} + \sum_{i=1}^{d} \left(e^{-\widehat{Y}_t^i} - 1\right) dN_t^i.$$

Plan

Comparison for semimartingale BSDE

Theorem (Jeanblanc, M., M. A., Ngoupeyou A.)

Assume that for $k = 1, 2$, $(Y^k, M^{Y^k,c}, \widehat{Y}^k)$ *is solution of the BSDE associated to* $(\widetilde{U}^k, \bar{U}^k)$. *Then one have*

$$Y_t^1 - Y_t^2 \leq \mathbb{E}^{\mathbb{Q}^{*,2}} \left[\int_t^T \alpha \frac{S_s^\delta}{S_t^\delta} \left(\widetilde{U}_s^1 - \widetilde{U}_s^2 \right) ds + \bar{\alpha} \frac{S_T^\delta}{S_t^\delta} \left(\bar{U}_T^1 - \bar{U}_T^2 \right) \Big| \mathcal{F}_t \right]$$

where $Q^{*,2}$ *the probability measure equivalent to* P *given by*

$$\frac{dZ_t^{Q^{*,2}}}{Z_{t-}^{Q^{*,2}}} = -dM_t^{Y^2,c} + \sum_{i=1}^{d} \left(e^{-\widehat{Y}_t^{i,2}} - 1 \right) dN_t^i.$$

In particular, if $\widetilde{U}^1 \leq \widetilde{U}^2$ *and* $\bar{U}_T^1 \leq \bar{U}_T^2$, *one obtains*

$$Y_t^1 \leq Y_t^2, \quad dP \otimes dt\text{-a.e.}$$

Concavity property for the semimartingale BSDE

Theorem

Let define the map $F : D_1^{\exp} \times D_0^{\exp} \longrightarrow D_0^{\exp}$ *such that for all* $(\widetilde{U}, \bar{U}) \in D_1^{\exp} \times D_0^{\exp}$*, we have*

$$F(\widetilde{U}, \bar{U}) = V$$

where $(V, M^{V,c}, \hat{V})$ *is the solution of BSDE associated to* $(\widetilde{U}, \bar{U})$*. Then* F *is concave ,namely,*

$$F\left(\theta\widetilde{U}^1 + (1-\theta)\widetilde{U}^2, \theta\bar{U}^1_T + (1-\theta)\bar{U}^2_T\right) \geq \theta F(\widetilde{U}^1, \bar{U}^1_T) + (1-\theta)F(\widetilde{U}^2, \bar{U}^2_T).$$

Plan

- The wealth process associated to the corresponding self-financing strategy is :

$$dX_t^{x,\pi,c} = (r_t X_t + \pi_t(\mu_t - r_t.\mathbf{1}) - c_t)\,dt + \pi_t \sigma_t dM_t$$

  where M is the $d+1$-dimensional martingale $M = (N^1, \dots, N^d, W)$.

- The budget constraints reads

$$\mathbb{E}^{\widetilde{P}}\left(\int_0^T c_t dt + X_T^{x,\pi,c}\right) \leq x$$

  where $\widetilde{P}$ is the unique martingale measure.
- Moreover, the strategy is called *feasible* if the constraint of nonnegative wealth holds :

$$X_t^{x,\pi,c} \geq 0 \qquad t \in [0, T]$$

  and this condition holds if the terminal wealth is non negative.

- We assume now that $\widetilde{U}_s = U(c_s)$ and $\overline{U}_s = \overline{U}(X_T)$.
- The main goal is to show there exists an unique pair of strategy that maximize the second part of the optimization problem :

$$\begin{cases} \sup_{\pi,c} V_0^{x,\pi,c} \\ \text{s.t } \mathbb{E}^{\widetilde{P}}\left(\int_0^T c_t dt + X_T^{x,\pi,c}\right) \leq x \end{cases}$$

where V_0 is the initial value process of the problem such that $(V, M^V, M^{V,\cdot})$ is the solution of the BSDE.

Unconstrainted optimization problem

Theorem

There exists a constant $\nu^* > 0$ *such that :*

$$u(x) = \sup_{(c,\psi)} \left\{ V_0^{(c,\psi)} + \nu^* \left(x - X^{(c,\psi)} \right) \right\}$$

and if the maximum is attained in the above constraint problem by (c^*, ψ^*) *then it is attained in the unconstraint problem by* (c^*, ψ^*) *with* $X^{(c,\psi)} = x$*. Conversely if there exists* $\nu^0 > 0$ *and* (c^0, ψ^0) *such that the maximum is attained in*

$$\sup_{(c,\psi)} \left\{ V_0^{(c,\psi)} + \nu^0 \left(x - X_0^{(c,\psi)} \right) \right\}$$

with $X_0^{(c,\psi)} = x$*, then the maximum is attained in our constraint problem by* (c^0, ψ^0)

- We now study for a fixed $\nu > 0$ the following optimization problem :

$$\sup_{(c,\psi)} L(c,\psi) \tag{3}$$

where the functional L is given by $L(c,\psi) = V_0^{(c,\psi)} - \nu X_0^{(c,\psi)}$

PROPOSITION (JEANBLANC, M., M. A., NGOUPEYOU A.)

The optimal consumption plan (c^0, ψ^0) *which solves* (3) *satisfies the following equations :*

$$U'(c_t^0) = \frac{Z_t^{\widetilde{P}}}{Z_t^{Q^*}} \frac{\nu}{\alpha S_t^\delta} \qquad \bar{U}'(\psi^0) = \frac{Z_T^{\widetilde{P}}}{Z_T^{Q^*}} \frac{\nu}{\bar{\alpha} S_T^\delta} \ a.s \tag{4}$$

where Q^* *is the model measure associated to the optimal consumption* (c^0, ψ^0).

- Let consider the optimal consumption plan (c^0, ψ^0) which solve (3) and another consumption plan (c, ψ). Consider $\epsilon \in (0,1)$ then :

$$L(c^0 + \epsilon(c - c^0), \psi^0 + \epsilon(c - c^0)) \leq L(c^0, \psi^0)$$

Then

$$\frac{1}{\epsilon}\left[V_0^{(c^0+\epsilon(c-c^0),\psi^0+\epsilon(\psi-\psi^0))} - V_0^{(c^0,\psi^0)}\right] - \nu\frac{1}{\epsilon}\left[X_0^{(c^0+\epsilon(c-c^0),\psi^0+\epsilon(\psi-\psi^0)} - X_0^{(c^0,\psi^0)}\right] \leq 0$$

Because $\left(X_t^{(c,\psi)} + \int_0^t c_s ds\right)_{t\geq 0}$ is a $\widetilde{P}$ martinagle we obtain :

$$\frac{1}{\epsilon}\left[X_t^{(c^0+\epsilon(c-c^0),\psi^0+\epsilon(\psi-\psi^0)} - X_t^{(c^0,\psi^0)}\right] = \mathbb{E}^{\widetilde{P}}\left[\int_t^T (c_s - c_s^0)ds + (\psi - \psi^0)\Big|\mathcal{F}_t\right]$$

- Then the wealth process is right differential in 0 with respect to ϵ we define

$$\partial_\epsilon X_t^{(c^0,\psi^0)} = \lim_{\epsilon \longrightarrow 0} \frac{1}{\epsilon}(X_t^{(c^0+\epsilon(c-c^0),\psi^0+\epsilon(c-c^0))} - X_t^{(c^0,\psi^0)})$$

- We take $\lim_{\epsilon \longrightarrow 0}$ above, we obtain :

$$\partial_\epsilon V_0^{(c^0,\psi^0)} - \nu \partial_\epsilon X_0^{(c^0,\psi^0)} \leq 0$$

where $(\partial_\epsilon V^{(c^0,\psi^0)})_{t \geq 0}$ exists and it is given explicitly :

The main steps of the Proof III

$$
\begin{cases}
d\partial_\epsilon V_t = \left(\delta_t \partial_\epsilon V_t - U'(c_t^1)(c_t^2 - c_t^1)\right) dt + d\langle \partial_\epsilon M^{V^1,c}, M^{V^1,c}\rangle_t \\
+ d\partial_\epsilon M_t^{V^1,c} - \sum_{i=1}^{d} \partial_\epsilon \hat{V}_t^i \left(e^{-\hat{V}^{1,i}} - 1\right) \lambda_t^i dt \\
+ \sum_{i=1}^{d} \partial_\epsilon \hat{V}^{1,i} dN_t^i. \\
\partial_\epsilon V_T = \bar{U}'(X_T^1)(X_T^2 - X_T^1)
\end{cases}
$$

THE MAIN STEPS OF THE PROOF IV

- Consider the optimal density $(Z^{Q_t^{*,1}})_{t\geq 0}$ where its dynamics is given by

$$\frac{dZ_t^{Q^{*,1}}}{Z_{t^-}^{Q^{*,1}}} = -dM^{V,c} + \sum_{i=1}^{d} \left(e^{-\widehat{Y}^{1,i}} - 1 \right) dN_t^i$$

then :

$$\partial_\epsilon V_t = \mathbb{E}^{Q^{*,1}} \Big[\frac{S_T^\delta}{S_t^\delta} \bar{U}'(X_T^1)(X_T^2 - X_T^1) + \int_t^T \frac{S_s^\delta}{S_t^\delta} U'(c_s^1)(c_s^2 - c_s^1) ds \Big| \mathcal{F}_t \Big].$$

THE MAIN STEPS OF THE PROOF V

- From the last result and the explicitly expression of $(\partial_\epsilon X_t^{(c^0,\psi^0)})_{t\geq 0}$ we get :

$$
\begin{aligned}
&\partial_\epsilon V_0^{(c^0,\psi^0)} - \nu \partial_\epsilon X_0^{(c^0,\psi^0)} \\
&= \mathbb{E}^P\big[S_T^\delta Z_T^{Q^*}\bar{\alpha}\bar{U}'(\psi^0)(\psi-\psi^0) + \int_0^T S_s^\delta Z_s^{Q^*}\alpha U'(c_s^0)(c_s-c_s^0)ds\big] \\
&-\nu\mathbb{E}^P\big[Z^{\widetilde{P}}(\psi-\psi^0) + \int_0^T Z_s^{\widetilde{P}}(c_s-c_s^0)ds\big]
\end{aligned}
\tag{5}
$$

- Using the equality above we get :

$$
\begin{aligned}
&\mathbb{E}^P\big[\big(S_T^\delta Z_T^{Q^*}\bar{\alpha}\bar{U}'(\psi^0) - \nu Z^{\widetilde{P}}\big)(\psi-\psi^0) \\
&+\int_0^T \big(S_s^\delta Z_s^{Q^*}\alpha U'(c_s^0) - \nu Z_s^{\widetilde{P}}\big)(c_s-c_s^0)ds\big] \leq 0
\end{aligned}
\tag{6}
$$

The main steps of the Proof VI

- Let define the set $A := \{(Z^{Q^*}\bar{\alpha}\bar{U}'(\psi^0) - \nu Z^{\widetilde{P}})(\psi - \psi^0) > 0\}$ taking $c = c^0$ and $\psi = \psi^0 + \mathbf{1_A}$ then using (6) $P(A) = 0$ and we get :

$$(Z^{Q^*}\bar{\alpha}\bar{U}'(\psi^0) - \nu Z^{\widetilde{P}}) \leq 0 \qquad a.s$$

- Let define for each $\epsilon > 0$

$$B := \{(Z^{Q^*}\bar{\alpha}\bar{U}'(\psi^0) - \nu Z^{\widetilde{P}})(\psi - \psi^0) < 0, \psi^0 > \epsilon\}$$

- because $\{\psi^0 > 0\}$ due to Inada assumption, we can define $\psi = \psi^0 - \mathbf{1_B}$ then due to (6) $P(B) = 0$ and we get

$$(Z^{Q^*}\bar{\alpha}\bar{U}'(\psi^0) - \nu Z^{\widetilde{P}}) \geq 0 \qquad a.s$$

We find the optimal consumption with similar arguments.