A new approach to LIBOR modeling

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Based on joint work with Martin Keller-Ressel and Josef Teichmann

Istanbul Workshop on Mathematical Finance
Istanbul, Turkey, 18 May 2009

Outline of the talk

Interest rates – Notation

- $B(t, T)$: time-t price of a zero coupon bond for T; $B(T, T) = 1$;
- $L(t, T)$: time-t forward LIBOR for $[T, T + \delta]$;

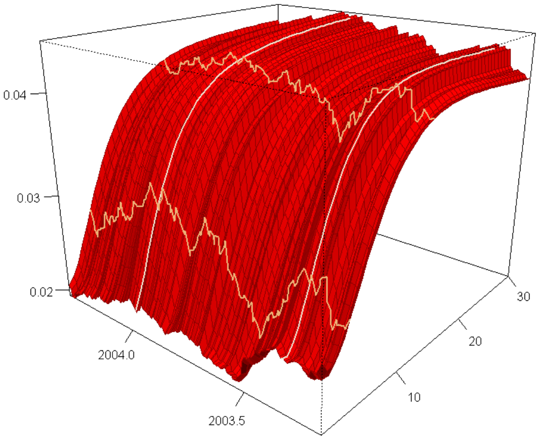
$$L(t, T) = \frac{1}{\delta}\left(\frac{B(t, T)}{B(t, T + \delta)} - 1\right)$$

- $F(t, T, U)$: time-t forward price for T and U; $F(t, T, U) = \frac{B(t,T)}{B(t,U)}$

"Master" relationship

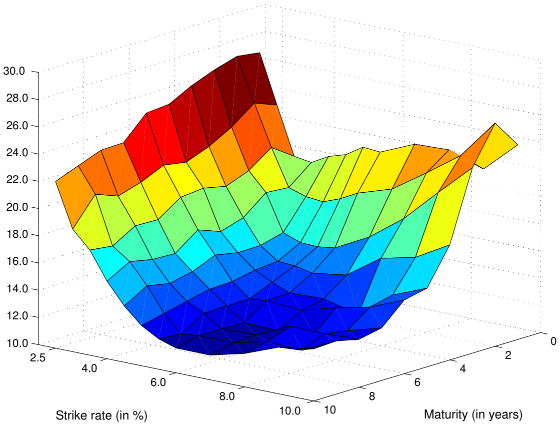
$$F(t, T, T + \delta) = \frac{B(t, T)}{B(t, T + \delta)} = 1 + \delta L(t, T) \tag{1}$$

Interest rates evolution



- Evolution of interest rate term structure, 2003 – 2004 (picture: Th. Steiner)

Calibration problems



1. Implied volatilities are constant neither across strike nor across maturity
2. Variance scales non-linearly over time (see e.g. D. Skovmand)

LIBOR model: Axioms

Economic thought dictates that LIBOR rates should satisfy:

Axiom 1

The LIBOR rate should be non-negative, i.e. $L(t, T) \geq 0$ *for all* t.

Axiom 2

The LIBOR rate process should be a martingale under the corresponding forward measure, i.e. $L(\cdot, T) \in \mathcal{M}(P_{T+\delta})$.

Practical applications require:

Models should be analytically tractable ($\rightsquigarrow$ *fast calibration*).

Models should have rich structural properties ($\rightsquigarrow$ *good calibration*).

- What axioms do the existing models satisfy?

LIBOR models I (Sandmann et al, Brace et al, ..., Eberlein & Özkan)

Ansatz: model the LIBOR rate as the exponential of a semimartingale H:

$$L(t,T_k) = L(0,T_k)\exp\left(\int_0^t b(s,T_k)ds + \int_0^t \lambda(s,T_k)dH_s^{T_{k+1}}\right), \qquad (2)$$

where $b(s,T_k)$ ensures that $L(\cdot,T_k) \in \mathcal{M}(P_{T_{k+1}})$.
H has the $P_{T_{k+1}}$-canonical decomposition

$$H_t^{T_{k+1}} = \int_0^t \sqrt{c_s}dW_s^{T_{k+1}} + \int_0^t \int_{\mathbb{R}} x(\mu^H - \nu^{T_{k+1}})(ds,dx), \qquad (3)$$

where the $P_{T_{k+1}}$-Brownian motion is

$$W_t^{T_{k+1}} = W_t^{T_*} - \int_0^t \left(\sum_{l=k+1}^{N} \frac{\delta_l L(t-,T_l)}{1+\delta_l L(t-,T_l)}\lambda(t,T_l)\right)\sqrt{c_s}ds, \qquad (4)$$

LIBOR models II

and the $P_{T_{k+1}}$-compensator of μ^H is

$$\nu^{T_{k+1}}(ds, dx) = \left(\prod_{l=k+1}^{N} \frac{\delta_l L(t-, T_l)}{1 + \delta_l L(t-, T_l)} \left(e^{\lambda(t, T_l)x} - 1 \right) + 1 \right) \nu^{T_*}(ds, dx).$$

LIBOR models II

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Consequences for continuous semimartingales:

1. caplets can be priced in closed form;
2. swaptions and multi-LIBOR products cannot be priced in closed form;
3. Monte-Carlo pricing is very time consuming $\rightsquigarrow$ coupled high dimensional SDEs!

LIBOR models II

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Consequences for general semimartingales:

1. even caplets cannot be priced in closed form!
2. ditto for Monte-Carlo pricing.

LIBOR models III

The equation for the dynamics yield the following matrix for the "dependence" structure

$$
\begin{array}{cccccccc}
\ddots & L(t, T_{i-1}) & & & & & & \\
\ddots & \vdots & \ddots & \ddots & & & & \\
 & L(t, T_{N-2}) & \dots & \dots & L(t, T_{N-2}) & & & \\
 & L(t, T_{N-1}) & \dots & \dots & L(t, T_{N-1}) & L(t, T_{N-1}) & & \\
\dots & L(t, T_N) & \dots & \dots & L(t, T_N) & L(t, T_N) & L(t, T_N) & \\
\hline
\dots & L(t, T_i) & \dots & \dots & L(t, T_{N-3}) & L(t, T_{N-2}) & L(t, T_{N-1}) & L(t, T_N)
\end{array}
$$

Bottom line: LIBOR rates we wish to simulate.

LIBOR models IV: Remedies

1. **"Frozen drift" approximation**
   - Brace et al, Schlögl, Glassermann et al, ...
   - replace the random terms by their deterministic initial values:

$$\frac{\delta_l L(t-, T_l)}{1 + \delta_l L(t-, T_l)} \approx \frac{\delta_l L(0, T_l)}{1 + \delta_l L(0, T_l)} \tag{5}$$

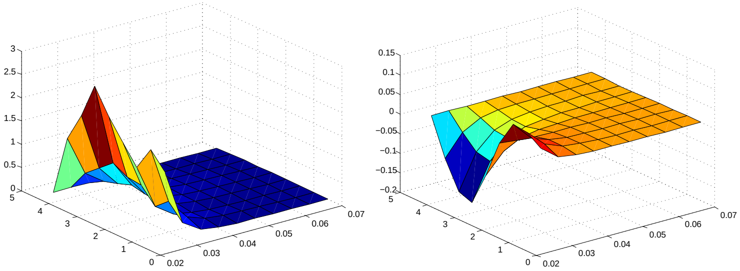
   - **(+)** deterministic characteristics $\rightsquigarrow$ closed form pricing
   - **(−)** "ad hoc" approximation, no error estimates, compounded error ...
2. **Log-normal and/or Monte Carlo methods**
   - best log-normal approximation (e.g. Schoenmakers)
   - interpolations and predictor-corrector MC methods
   - Joshi and Stacey (2008): overview paper

LIBOR models V: Remedies

3. **Strong Taylor approximation**
   - approximate the LIBOR rates in the drift by

$$L(t, T_l) \approx L(0, T_l) + Y(t, T_l)_+ \tag{6}$$

     where Y is the (scaled) exponential transform of H ($Y = \mathcal{L}oge^H$)
   - theoretical foundation, error estimates, simpler equations for MC
   - Siopacha and Teichmann; Hubalek, Papapantoleon & Siopacha



Difference in implied vols between full SDE vs frozen drift and full SDE vs strong Taylor.

Forward price model I (Eberlein & Özkan, Kluge)

Ansatz: model the forward price as the exponential of a semimartingale H:

$$F(t, T_k) = F(0, T_k) \exp\left(\int_0^t b(s, T_k) ds + \int_0^t \lambda(s, T_k) dH_s^{T_{k+1}} \right), \qquad (7)$$

where $b(s, T_k)$ ensures that $F(\cdot, T_k) = 1 + \delta L(\cdot, T_k) \in \mathcal{M}(P_{T_{k+1}})$.
H has the $P_{T_{k+1}}$-canonical decomposition

$$H_t^{T_{k+1}} = \int_0^t \sqrt{c_s} dW_s^{T_{k+1}} + \int_0^t \int_{\mathbb{R}} x(\mu^H - \nu^{T_{k+1}})(ds, dx), \qquad (8)$$

where the $P_{T_{k+1}}$-Brownian motion is

$$W_t^{T_{k+1}} = W_t^{T_*} - \int_0^t \left(\sum_{l=k+1}^{N} \lambda(t, T_l) \right) \sqrt{c_s} ds, \qquad (9)$$

Forward price model II

and the $P_{T_{k+1}}$-compensator of μ^H is

$$\nu^{T_{k+1}}(ds, dx) = \exp\left(x \sum_{l=k+1}^{N} \lambda(t, T_l)\right) \nu^{T_*}(ds, dx).$$

Consequences:

1. the model structure is preserved;
2. caps, swaptions and multi-LIBOR products priced in closed form.

So, what is wrong?

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So, what is wrong?

Negative LIBOR rates can occur!

Aim: *design a model where the model structure is preserved and LIBOR rates are positive.*

Tool: *Affine processes on* $\mathbb{R}^d_{\geqslant 0}$.

Affine processes I

Let $X = (X_t)_{0\leq t\leq T}$ be a conservative, time-homogeneous, stochastically continuous Markov process taking values in $D = \mathbb{R}^d_{\geqslant 0}$; and $(P_x)_{x\in D}$ a family of probability measures on $(\Omega, \mathcal{F})$, such that $X_0 = x$, P_x-a.s. for every $x \in D$. Setting

$$\mathcal{I}_T := \left\{ u \in \mathbb{R}^d : E_x\left[e^{\langle u, X_T\rangle} \right] < \infty, \text{ for all } x \in D \right\}, \tag{10}$$

we assume that

(i) $0 \in \mathcal{I}_T^\circ$;

(ii) the conditional moment generating function of X_t under P_x has exponentially-affine dependence on x; i.e. there exist functions $\phi_t(u) : [0, T] \times \mathcal{I}_T \to \mathbb{R}$ and $\psi_t(u) : [0, T] \times \mathcal{I}_T \to \mathbb{R}^d$ such that

$$E_x\left[\exp\langle u, X_t\rangle \right] = \exp\left(\phi_t(u) + \langle \psi_t(u), x\rangle \right), \tag{11}$$

for all $(t, u, x) \in [0, T] \times \mathcal{I}_T \times D$.

Affine processes II

The process X is a regular affine process in the spirit of Duffie, Filipović & Schachermayer (2003).
Using Theorem 3.18 in Keller-Ressel (2008)

$$F(u) := \frac{\partial}{\partial t}\Big|_{t=0+}\phi_t(u) \qquad \text{and} \qquad R(u) := \frac{\partial}{\partial t}\Big|_{t=0+}\psi_t(u) \tag{12}$$

exist for all $u \in \mathcal{I}_T$ and are continuous in u. Moreover, F and R satisfy Lévy–Khintchine-type equations:

$$F(u) = \langle b, u\rangle + \int_D \left(e^{\langle \xi, u\rangle} - 1\rangle\right) m(d\xi) \tag{13}$$

and

$$R_i(u) = \langle \beta_i, u\rangle + \left\langle \frac{\alpha_i}{2}u, u\right\rangle + \int_D \left(e^{\langle \xi, u\rangle} - 1 - \langle u, h^i(\xi)\rangle\right)\mu_i(d\xi), \tag{14}$$

where $(b, m, \alpha_i, \beta_i, \mu_i)_{1\le i\le d}$ are admissible parameters.

Affine processes III

The time-homogeneous Markov property of X implies:

$$E_x\Big[\exp\langle u, X_{t+s}\rangle \Big| \mathcal{F}_s\Big] = \exp\big(\phi_t(u) + \langle \psi_t(u), X_s\rangle\big), \tag{15}$$

for all $0 \leq t+s \leq T$ and $u \in \mathcal{I}_T$.

Lemma (Flow property)

The functions ϕ and ψ satisfy the **semi-flow equations***:*

$$\begin{aligned} \phi_{t+s}(u) &= \phi_t(u) + \phi_s(\psi_t(u)) \\ \psi_{t+s}(u) &= \psi_s(\psi_t(u)) \end{aligned} \tag{16}$$

with initial condition

$$\phi_0(u) = 0 \quad \text{and} \quad \psi_0(u) = u, \tag{17}$$

for all suitable $0 \leq t+s \leq T$ and $u \in \mathcal{I}_T$.

Affine processes IV

1. **Affine processes on $\mathbb{R}$:** the admissibility conditions yield

$$F(u) = bu + \frac{a}{2}u^2 + \int_{\mathbb{R}} \left(e^{zu} - 1 - uh(z)\right) m(dz)$$
$$R(u) = \beta u,$$

for $a \in \mathbb{R}_{\geqslant 0}$ and $b, \beta \in \mathbb{R}$.

- Every affine process on $\mathbb{R}$ is an Ornstein–Uhlenbeck (OU) process.

2. **Affine processes on $\mathbb{R}_{\geqslant 0}$:** the admissibility conditions yield

$$F(u) = bu + \int_{D} \left(e^{zu} - 1\right) m(dz)$$
$$R(u) = \beta u + \frac{\alpha}{2}u^2 + \int_{D} \left(e^{zu} - 1 - uh(z)\right) \mu(dz),$$

for $b, \alpha \in \mathbb{R}_{\geqslant 0}$ and $\beta \in \mathbb{R}$.

- There exist affine process on $\mathbb{R}_{\geqslant 0}$ which are not OU process, e.g. CIR.

Affine LIBOR model: martingales $\geqslant 1$

Idea:

1. insert an affine process in its moment generating function with inverted time; the resulting process is a martingale;
2. if the affine process is positive, the martingale is greater than one.

Theorem

The process $M^u = (M_t^u)_{0 \le t \le T}$ defined by

$$M_t^u = \exp\left(\phi_{T-t}(u) + \langle \psi_{T-t}(u), X_t \rangle\right), \tag{18}$$

is a martingale. Moreover, if $u \in \mathcal{I}_T \cap \mathbb{R}^d_{\geqslant 0}$ then $M_t \ge 1$ a.s. for all $t \in [0, T]$, for any $X_0 \in \mathbb{R}^d_{\geqslant 0}$.

Affine LIBOR model: martingales $\geqslant 1$

Proof.

Using (17) and (15), we have that:

$$\begin{aligned} E_x\left[M_T^u|\mathcal{F}_t\right] &= E_x\left[\exp\langle u, X_T\rangle|\mathcal{F}_t\right] \\ &= \exp\left(\phi_{T-t}(u) + \langle\psi_{T-t}(u), X_t\rangle\right) = M_t^u. \end{aligned}$$

Regarding $M_t^u \geq 1$ for all $t \in [0, T]$: note that if $u \in \mathcal{I}_T \cap \mathbb{R}^d_{\geqslant 0}$, then

$$M_t^u = E_x\left[\exp\langle u, X_T\rangle\middle|\mathcal{F}_t\right] \geq 1. \tag{19}$$

□

Affine LIBOR model: martingales $\geqslant 1$

Example (Lévy process)

Consider a Lévy subordinator, then

$$\begin{aligned} M_t^u &= \exp\left(\phi_{T-t}(u) + \langle \psi_{T-t}(u), X_t \rangle\right) \\ &= \exp\left((T-t)\kappa(u) + u \cdot X_t\right) \geqslant 1 \\ &= \exp(T\kappa(u)) \exp\left(u \cdot X_t - t\kappa(u)\right) \in \mathcal{M}, \end{aligned} \tag{20}$$

which is a martingale $\geqslant 1$ for $u \in \mathbb{R}^d_{\geqslant 0}$.

Affine LIBOR model: Ansatz

Consider a discrete tenor structure $0 = T_0 < T_1 < T_2 < \cdots < T_N$; discounted bond prices must satisfy:

$$\frac{B(\cdot, T_k)}{B(\cdot, T_N)} \in \mathcal{M}(P_{T_N}), \qquad \text{for all } k \in \{1, \dots, N-1\}. \tag{21}$$

Ansatz

We model quotients of bond prices using the martingales M:

$$\frac{B(t, T_1)}{B(t, T_N)} = M_t^{u_1} \tag{22}$$

$$\vdots$$

$$\frac{B(t, T_{N-1})}{B(t, T_N)} = M_t^{u_{N-1}}, \tag{23}$$

with initial conditions: $\frac{B(0,T_k)}{B(0,T_N)} = M_0^{u_k}$, *for all* $k \in \{1, \dots, N-1\}$.

Affine LIBOR model: initial values

Proposition

Let $L(0,T_1),\dots,L(0,T_N)$ be a tenor structure of non-negative initial LIBOR rates; let X be an affine process starting at the canonical value $\mathbf{1}$.

1. *If $\gamma_X := \sup_{u\in\mathcal{I}_T\cap\mathbb{R}^d_{>0}} E_{\mathbf{1}}\left[e^{\langle u, X_T\rangle}\right] > \frac{B(0,T_1)}{B(0,T_N)}$, then there exists a decreasing sequence $u_1 \geq u_2 \geq \cdots \geq u_N = 0$ in $\mathcal{I}_T \cap \mathbb{R}^d_{\geqslant 0}$, such that*

$$M_0^{u_k} = \frac{B(0,T_k)}{B(0,T_N)}, \qquad \text{for all } k \in \{1,\dots,N\}. \tag{24}$$

   In particular, if $\gamma_X = \infty$, then the affine LIBOR model can fit any term structure of non-negative initial LIBOR rates.

2. *If X is one-dimensional, the sequence $(u_k)_{k\in\{1,\dots,N\}}$ is unique.*
3. *If all initial LIBOR rates are positive, the sequence $(u_k)_{k\in\{1,\dots,N\}}$ is strictly decreasing.*

Affine LIBOR model: forward prices

Forward prices have the following form

$$\frac{B(t,T_k)}{B(t,T_{k+1})} = \frac{B(t,T_k)}{B(t,T_N)}\frac{B(t,T_N)}{B(t,T_{k+1})} = \frac{M_t^{u_k}}{M_t^{u_{k+1}}}$$
$$= \exp\Big(\phi_{T_N-t}(u_k) - \phi_{T_N-t}(u_{k+1})$$
$$+ \big\langle \psi_{T_N-t}(u_k) - \psi_{T_N-t}(u_{k+1}), X_t \big\rangle\Big). \quad (25)$$

Now, $\phi_t(\cdot)$ and $\psi_t(\cdot)$ are order-preserving, i.e.

$$u \le v \;\Rightarrow\; \phi_t(u) \le \phi_t(v) \;\text{ and }\; \psi_t(u) \le \psi_t(v).$$

Consequently: positive initial LIBOR rate yields positive LIBOR rates for all times.

Affine LIBOR model: forward measures

Forward measures are related via:

$$\frac{dP_{T_k}}{dP_{T_{k+1}}}\Big|_{\mathcal{F}_t} = \frac{F(t, T_k, T_{k+1})}{F(0, T_k, T_{k+1})} = \frac{B(0, T_{k+1})}{B(0, T_k)} \times \frac{M_t^{u_k}}{M_t^{u_{k+1}}} \qquad (26)$$

or equivalently:

$$\frac{dP_{T_{k+1}}}{dP_{T_N}}\Big|_{\mathcal{F}_t} = \frac{B(0, T_N)}{B(0, T_{k+1})} \times \frac{B(t, T_{k+1})}{B(t, T_N)} = \frac{B(0, T_N)}{B(0, T_{k+1})} \times M_t^{u_{k+1}}. \qquad (27)$$

Hence, we can easily see that

$$\frac{B(\cdot, T_k)}{B(\cdot, T_{k+1})} = \frac{M^{u_k}}{M^{u_{k+1}}} \in \mathcal{M}(P_{T_{k+1}}), \qquad \text{for all } k \in \{1, \ldots, N-1\}. \qquad (28)$$

Affine LIBOR model: dynamics under forward measures

The moment generating function of X_t under any forward measure is

$$\begin{aligned} E_{P_{T_{k+1}}}\left[e^{vX_t}\right] &= M_0^{u_{k+1}} E_{P_{T_N}}[M_t^{u_{k+1}} e^{vX_t}] \\ &= \exp\Big(\phi_t\big(\psi_{T_N-t}(u_{k+1}) + v\big) - \phi_t(\psi_{T_N-t}(u_{k+1})) \\ &\qquad + \big\langle \psi_t\big(\psi_{T_N-t}(u_{k+1}) + v\big) - \psi_t(\psi_{T_N-t}(u_{k+1})), x\big\rangle\Big). \end{aligned} \tag{29}$$

Denote by $\frac{M_t^{u_k}}{M_t^{u_{k+1}}} = e^{A_k + B_k \cdot X_t}$; the moment generating function is

$$\begin{aligned} E_{P_{T_{k+1}}}\left[e^{v(A_k + B_k \cdot X_t)}\right] &= \frac{B(0, T_N)}{B(0, T_{k+1})} \\ &\quad \times \exp\Big(v\phi_{T_N-t}(u_k) + (1-v)\phi_{T_N-t}(u_{k+1}) \\ &\qquad + \phi_t\big(v\psi_{T_N-t}(u_k) + (1-v)\psi_{T_N-t}(u_{k+1})\big) \\ &\qquad + \big\langle \psi_t\big(v\psi_{T_N-t}(u_k) + (1-v)\psi_{T_N-t}(u_{k+1})\big), x\big\rangle\Big). \end{aligned} \tag{30}$$

Affine LIBOR model: caplet pricing

We can re-write the payoff of a caplet as follows (here $\mathcal{K} := 1 + \delta K$):

$$\begin{aligned}\delta(L(T_k, T_k) - K)^+ &= (1 + \delta L(T_k, T_k) - 1 + \delta K)^+ \\ &= \left(\frac{M_{T_k}^{u_k}}{M_{T_k}^{u_{k+1}}} - \mathcal{K}\right)^+ \\ &= \left(e^{A_k + B_k \cdot X_{T_k}} - \mathcal{K}\right)^+. \qquad (31)\end{aligned}$$

Then we can price caplets by Fourier-transform methods:

$$\begin{aligned}\mathbb{C}(T_k, K) &= B(0, T_{k+1}) E_{P_{T_{k+1}}}\left[\delta(L(T_k, T_k) - K)^+\right] \\ &= \frac{\mathcal{K} B(0, T_{k+1})}{2\pi} \int_{\mathbb{R}} \mathcal{K}^{iv-R} \frac{\Lambda_{A_k + B_k \cdot X_{T_k}}(R - iv)}{(R - iv)(R - 1 - iv)} dv \qquad (32)\end{aligned}$$

where $\Lambda_{A_k + B_k \cdot X_{T_k}}$ is given by (30).

CIR martingales

The Cox-Ingersoll-Ross (CIR) process is given by

$$dX_t = -\lambda\,(X_t - \theta)\,dt + 2\eta\sqrt{X_t}dW_t, \quad X_0 = x \in \mathbb{R}_{\geqslant 0}, \tag{33}$$

where $\lambda, \theta, \eta \in \mathbb{R}_{\geqslant 0}$. This is an affine process on $\mathbb{R}_{\geqslant 0}$, with

$$E_x\left[e^{uX_t}\right] = \exp\Big(\phi_t(u) + x \cdot \psi_t(u)\Big), \tag{34}$$

where

$$\phi_t(u) = -\frac{\lambda\theta}{2\eta}\log\left(1 - 2\eta b(t)u\right) \quad \text{and} \quad \psi_t(u) = \frac{a(t)u}{1 - 2\eta b(t)u}, \tag{35}$$

with

$$b(t) = \begin{cases} t, & \text{if } \lambda = 0 \\ \frac{1-e^{-\lambda t}}{\lambda}, & \text{if } \lambda \neq 0 \end{cases}, \qquad \text{and} \qquad a(t) = e^{-\lambda t}.$$

CIR martingales: closed-form formula I

Definition

A random variable Y has location-scale extended non-central chi-square distribution, $Y \sim \mathrm{LSNC}{-}\chi^2(\mu, \sigma, \nu, \alpha)$, if $\frac{Y-\mu}{\sigma} \sim \mathrm{NC}{-}\chi^2(\nu, \alpha)$

Then we have that

$$X_t \overset{P_{T_N}}{\sim} \mathrm{LSNC}{-}\chi^2\left(0, \eta b(t), \frac{\lambda\theta}{\eta}, \frac{xa(t)}{\eta b(t)}\right) ,$$

and

$$X_t \overset{P_{T_{k+1}}}{\sim} \mathrm{LSNC}{-}\chi^2\left(0, \frac{\eta b(t)}{\zeta(t, T_N)}, \frac{\lambda\theta}{\eta}, \frac{xa(t)}{\eta b(t)\zeta(t, T_N)}\right),$$

hence

$$\log\left(\frac{B(t, T_k)}{B(t, T_{k+1})}\right) \overset{P_{T_{k+1}}}{\sim} \mathrm{LSNC}{-}\chi^2\left(A_k, \frac{B_k \eta b(t)}{\zeta(t, T_N)}, \frac{\lambda\theta}{\eta}, \frac{xa(t)}{\eta b(t)\zeta(t, T_N)}\right).$$

CIR martingales: closed-form formula II

Then, denoting by $M = \log\left(\frac{B(T_k,T_k)}{B(T_k,T_{k+1})}\right)$ the log-forward rate, we arrive at:

$$\begin{aligned}
\mathbb{C}(T_k, K) &= B(0, T_{k+1})\, E_{P_{T_{k+1}}}\left[\left(e^M - \mathcal{K}\right)^+\right] \\
&= B(0, T_{k+1})\left\{E_{P_{T_{k+1}}}\left[e^M 1_{\{M \geq \log\mathcal{K}\}}\right] - \mathcal{K}\, P_{T_{k+1}}[M \geq \log\mathcal{K}]\right\} \\
&= B(0, T_k)\cdot \overline{\chi}^2_{\nu,\alpha_1}\left(\frac{\log\mathcal{K} - A_k}{\sigma_1}\right) - \mathcal{K}^\star\cdot\overline{\chi}^2_{\nu,\alpha_2}\left(\frac{\log\mathcal{K} - A_k}{\sigma_2}\right),
\end{aligned} \tag{36}$$

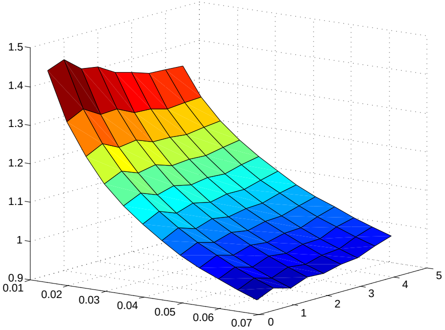
where $\mathcal{K}^\star = \mathcal{K}\cdot B(0, T_{k+1})$ and $\overline{\chi}^2_{\nu,\alpha}(x) = 1 - \chi^2_{\nu,\alpha}(x)$, with $\chi^2_{\nu,\alpha}(x)$ the non-central chi-square distribution function,

$$\nu = \frac{\lambda\theta}{\eta}, \qquad \sigma_{1,2} = \frac{B_k\eta b(T_k)}{\zeta_{1,2}}, \qquad \alpha_{1,2} = \frac{xa(T_k)}{\eta b(T_k)\zeta_{1,2}},$$

and

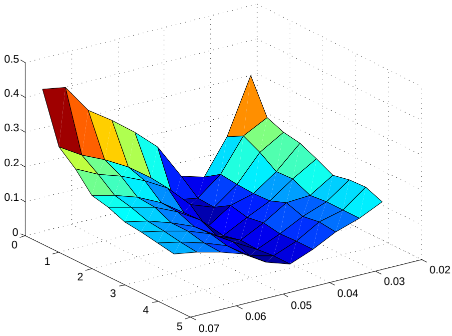
$$\zeta_1 = 1 - 2\eta b(T_k)\psi_{T_N - T_k}(u_k), \qquad \zeta_2 = 1 - 2\eta b(T_k)\psi_{T_N - T_k}(u_{k+1}).$$

CIR martingales: volatility surface



Example of an implied volatility surface for the CIR martingales.

Γ-OU martingales: volatility surface



Example of an implied volatility surface for the Γ-OU martingales.

Summary and Outlook

1. We have presented a LIBOR model that
   - is very simple (Axiom 0 !), and yet . . .
   - captures all the important features . . .
   - especially positivity and analytical tractability.
2. Future work:
   - thorough empirical analysis
   - extensions: multiple currencies, default risk
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Thank you for your attention!