Optimal portfolio liquidation with execution cost and risk

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Trading and liquidation

- Understanding trade execution strategies:
  - key issue for market practitioners
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- Liquidation of large block orders of shares

▶ Challenging problem due to the following dilemma:

- Quick trading $\rightarrow$ higher costs due to market impact $\leftrightarrow$ depth of the limit order book
  $\Longrightarrow$ Break up a large order into smaller blocks

Liquidation problem: cost/risk tradeoff

- However, more gradual trading over time
$\longrightarrow$ risk of price depreciation in an uncertain environment during the trading horizon

▶ Considerable interest in the literature on such liquidity effects, taking into account permanent and/or temporary price impact:

Bertsimas and Lo (1998), Almgren and Criss (01), Platen and Schweizer (98), Bank and Baum (04), Cetin, Jarrow and Protter (04), Obizhaeva and Wang (05), He and Mamayski (05), Ly Vath, Mnif and P. (07), Schied and Schöneborn (08), Rogers and Singh (08), Cetin, Soner and Touzi (08), etc ....

Discrete vs continuous-time trading

- Discrete-time formulation
  - fixed deterministic times
  - exogenous random times (e.g. associated to buy/sell arrivals)
  - discrete times decided optimally by the investor: impulse control formulation

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- Discrete-time formulation
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  - exogenous random times (e.g. associated to buy/sell arrivals)
  - discrete times decided optimally by the investor: impulse control formulation
    $\longrightarrow$ one usually assumes the existence of a fixed transaction fee paid at each trading
    $\longrightarrow$ this ensures that strategies do not accumulate in time and occur really at discrete points in time, so that the problem is well-posed.

Continuous-time trading

- Continuous-time formulation
  - not realistic in practice
  - but commonly used due to the tractability and powerful theory of stochastic calculus
  - in perfect liquid markets (without transaction costs and market impact), this is often justified by arguing that continuous-time trading is a limit approximation of discrete-time trading when time step goes to zero.

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▶ Under illiquidity cost, it is not clear and suitable how to define the portfolio value of a position in stock shares. And this is a crucial issue given the bank regulation and solvency constraints!

Our illiquidity market modelling

• Continuous-time framework taking into account the main liquidity features and risk/cost tradeoff of portfolio execution:

- bid-ask spread in the limit order book
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- Continuous-time framework taking into account the main liquidity features and risk/cost tradeoff of portfolio execution:
  - bid-ask spread in the limit order book
  - temporary market price impact penalizing rapid execution trades
- However, we do not assume continuous-time trading strategies
  - instead, real trading in discrete-time
  - without assuming ad hoc any any fixed transaction fee, in accordance with practitioner literature

Our illiquidity modelling via impulse control

▶ This is formulated through an impulse control problem including the lag variable tracking the time interval between trades:

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$\longrightarrow$ Combine the advantages of stochastic calculus techniques, and the realistic modelling of liquidity constraints

▶ We study the optimal portfolio execution problem for an investor seeking to liquidate an initial position in stock shares over a finite horizon.

▶ Important result: we show that nearly optimal execution strategies in this modelling lead actually to a finite number of trading times

Outline

Notations and state variables

• Uncertainty and information: $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$, W 1-dim BM, trading interval $[0, T]$.

• Market stock price process: $P = (P_t)$ without permanent price impact, and with BS dynamics

$$dP_t = P_t(bdt + \sigma dW_t).$$

• Amount of money (cash holdings): $X = (X_t)$

• Cumulated number of shares: $Y = (Y_t)$

• Time interval between trades: $\Theta = (\Theta_t)$

$\rightarrow$ Relevant state variables: $(Z, \Theta) = (X, Y, P, \Theta)$.

Trading strategies

- **Trading strategies:** impulse control $\alpha = (\tau_n, \zeta_n)_{n \geq 0}$:
  - $0 \leq \ldots \leq \tau_n \leq \tau_{n+1} \leq \ldots T$: stopping times representing the intervention times of the investor
  - ζ_n $\mathcal{F}_{\tau_n}$-measurable real-valued random variable: number of stocks traded at time τ_n

    $\longrightarrow$ Dynamics of Y:

$$Y_t = Y_{\tau_n}, \;\; \tau_n \leq t < \tau_{n+1}, \qquad Y_{\tau_{n+1}} = Y_{\tau_{n+1}^-} + \zeta_{n+1}, \;\; n \geq 0.$$

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- Lag variable: $\Theta_t = \inf\{t - \tau_n, \; \tau_n \leq t\}$, evolves according to

$$\Theta_t \; = \; t - \tau_n, \;\; \tau_n \leq t < \tau_{n+1}, \qquad \Theta_{\tau_{n+1}} \; = \; 0, \;\; n \geq 0.$$

Cost of illiquidity

If the current market price is p, and the time lag from the last order is θ, then the price the investor get for an order of size e is:

$$Q(e, p, \theta) \quad = \quad pf\Big(\frac{e}{\theta}\Big),$$

where f is a temporary price impact function from $\mathbb{R}$ into $(0, \infty)$, with the convention $0/0 = 0$ in $f(e/\theta)$, satisfying:

- **(H1f)** $f(0) = 1$, and f is nondecreasing,
- **(H2f)** (i) $f(-\infty) = 0$, and (ii) $f(\infty) = \infty$,
- **(H3f)** $\kappa_b := f(0^-) < 1$ and $\kappa_a := f(0^+) > 1$.

Cost of illiquidity and bid-ask spread

• A usual form, suggested by empirical studies, see Lillo, Farmer and Mantagna (03), Potters and Bouchaud (03), Almgren, Thum, Hauptmann and Li (05), is:

$$f(\eta) = e^{\lambda|\eta|^\beta \mathrm{sgn}(\eta)}\Big(\kappa_a \mathbf{1}_{\eta>0} + \mathbf{1}_{\eta=0} + \kappa_b \mathbf{1}_{\eta<0}\Big),$$

where $0 < \kappa_b < 1 < \kappa_a$, $\kappa_a - \kappa_b$ is the bid-ask spread parameter, $\lambda > 0$ is the temporary price impact factor, and $\beta > 0$ is the price impact exponent.

Cash holdings

- Assuming zero interest rate, bank account is constant between two trading times;

$$X_t \;=\; X_{\tau_n}, \quad \tau_n \leq t < \tau_{n+1}, \quad n \geq 0.$$

- When a trading $(\tau_{n+1}, \zeta_{n+1})$ occurs, this results in a variation of cash holdings by:

$$\begin{aligned} X_{\tau_{n+1}} &= X_{\tau_{n+1}^-} - \zeta_{n+1} Q(\zeta_{n+1}, P_{\tau_{n+1}}, \Theta_{\tau_{n+1}^-}) \\ &= X_{\tau_{n+1}^-} - \zeta_{n+1} P_{\tau_{n+1}} f\Big(\frac{\zeta_{n+1}}{\tau_{n+1} - \tau_n}\Big), \quad n \geq 0. \end{aligned}$$

Remarks

- We **do not assume fixed transaction fee** to be paid at each trading

- We can then not exclude a priori trading strategies with immediate trading times, i.e. $\Theta_{\tau_{n+1}^-} = \tau_{n+1} - \tau_n = 0$.

- However, under condition **(H2f)**, an immediate sale does not increase the cash holdings, i.e. $X_{\tau_{n+1}} = X_{\tau_{n+1}^-} = X_{\tau_n}$, while an immediate purchase leads to a bankruptcy, i.e. $X_{\tau_{n+1}} = -\infty$.

Liquidation value and solvency constraint

• **No-short sale constraint**:

$$Y_t \;\geq\; 0, \quad \forall t.$$

• Nonnegative liquidation value (portfolio value by a single block trade):

$$L(X_t, Y_t, P_t, \Theta_t) \;:=\; X_t + Y_t P_t f\Big(\frac{-Y_t}{\Theta_t}\Big) \;\geq\; 0, \quad \forall t.$$

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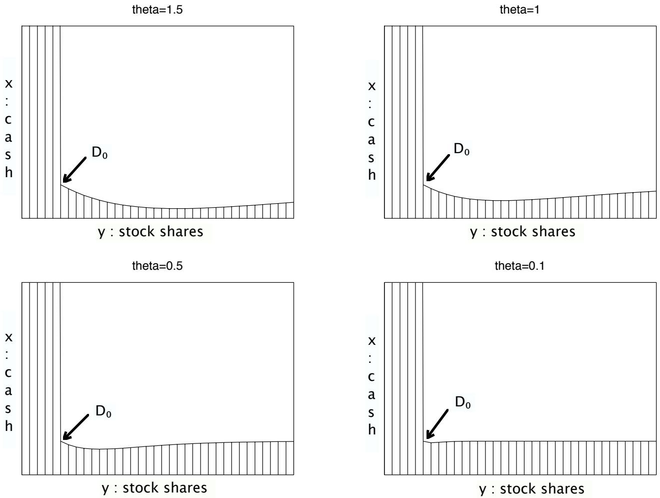
▶ Liquidation solvency region

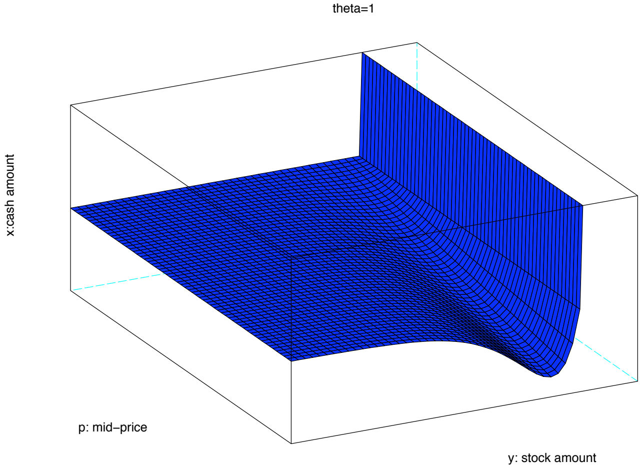
$$\mathcal{S} \;=\; \Big\{(z,\theta) = (x,y,p,\theta) \in \mathbb{R}\times\mathbb{R}_+\times\mathbb{R}_+^*\times[0,T] : \\ y > 0 \;\text{ and }\; L(z,\theta) \;:=\; x + ypf\Big(\frac{-y}{\theta}\Big) > 0\Big\},$$

and $\bar{\mathcal{S}} = \mathcal{S} \cup \partial\mathcal{S}$.

Graph of $\mathcal{S}$ in the plan (x, y)



Graph of $\mathcal{S}$ in 3-dim (x, y, p) for fixed θ



Admissible trading strategies

Given $(t, z, \theta) \in [0, T] \times \bar{\mathcal{S}}$, we say that the impulse control strategy $\alpha = (\tau_n, \zeta_n)_n$ is *admissible*, denoted by $\alpha \in \mathcal{A}(t, z, \theta)$, if the associated state process (Z, Θ), starting from (z, θ) at t stays in $\bar{\mathcal{S}}$ for all $t \leq s \leq T$.

Remark:
The impulse control strategy consisting in liquidating immediately all stock shares, and then doing no more trading, is admissible:
$\rightarrow \mathcal{A}(t, z, \theta) \neq \emptyset$.

Portfolio liquidation problem

- Utility function $U : \mathbb{R}_+ \rightarrow \mathbb{R}$, nondecreasing, concave, with $U(0) = 0$, and s.t. there exists $K \geq 0$ and $\gamma \in [0,1)$:

$$0 \leq U(x) \leq Kx^\gamma, \quad \forall x \in \mathbb{R}_+.$$

▶ **Value function**:

$$v(t,z,\theta) = \sup_{\alpha \in \mathcal{A}_\ell(t,z,\theta)} \mathbb{E}\big[U(X_T)\big], \quad (t,z,\theta) \in [0,T] \times \bar{\mathcal{S}},$$

where $\mathcal{A}_\ell(t,z,\theta) = \big\{\alpha \in \mathcal{A}(t,z,\theta) : Y_T = 0\big\}$.

Remark: remove the terminal liquidation constraint

Define the terminal liquidation utility by:

$$U_L(z,\theta) \quad = \quad U(L(z,\theta)), \quad (z,\theta) \in \bar{\mathcal{S}}.$$

Then, the value function is written equivalently in

$$v(t,z,\theta) \quad = \quad \sup_{\alpha \in \mathcal{A}(t,z,\theta)} \mathbb{E}\big[U_L(Z_T,\Theta_T)\big], \quad (t,z,\theta) \in [0,T] \times \bar{\mathcal{S}}.$$

Remark: continuous-time trading version

- Trading strategy in terms of instantaneous trading rate $(\eta)_t$:

$$\begin{aligned} dY_t &= \eta_t dt, \\ dX_t &= -\eta_t P_t f(\eta_t) dt. \end{aligned}$$

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- We may define the portfolio value in absence of liquidity cost:

$$X_t + Y_t P_t,$$

But how to define the liquidation value under illiquidity cost in continuous-time!

Well posedness of the problem

Property 0.
The value function is bounded by the Merton bound:
For all $(t, z = (x, y, p), \theta) \in [0, T] \times \bar{\mathcal{S}}$, we have

$$\begin{aligned} v(t, z, \theta) \leq v_0(t, z) &:= \mathbb{E}[U(x + yP_T^{t,p})] \\ &\leq Ke^{\rho(T-t)}(x + yp)^{\gamma} \end{aligned}$$

where $\rho \geq \frac{\gamma}{1-\gamma} \frac{b^2}{2\sigma^2}$.

Finiteness of the total amount traded

Property 1.
Under the existence of a bid-ask spread, the total number of shares and amount in absolute value associated to an admissible trading strategy is finite:

For any $\alpha = (\tau_n, \zeta_n)_n \in \mathcal{A}(t, z, \theta)$, we have

$$\sum_n |\zeta_n| \; < \; \infty, \quad \text{and} \quad \sum_n |\zeta_n| P_{\tau_n} f\Big(\frac{\zeta_n}{\Theta_{\tau_n^-}}\Big) \; < \; \infty, \quad a.s.$$

Nearly optimal strategies $\longrightarrow$ finite number of trading times

Property 2.

$$v(t,z,\theta) = \sup_{\alpha \in \mathcal{A}_\ell^b(t,z,\theta)} \mathbb{E}[U(X_T)], \quad (t,z,\theta) \in [0,T] \times \bar{\mathcal{S}},$$

where

$$\begin{aligned} \mathcal{A}_\ell^b(t,z,\theta) &= \Big\{ \alpha = (\tau_n,\zeta_n)_n \in \mathcal{A}_\ell(t,z,\theta) : \\ & \qquad N_T(\alpha) := \sum_n \mathbf{1}_{\tau_n \leq T} < \infty \ a.s. \\ & \qquad \text{and} \ \ \tau_n < \tau_{n+1} \ a.s., \ \ 0 \leq n \leq N_T(\alpha) - 1 \Big\}. \end{aligned}$$

Quasi-Variational dynamic programming equation

The QVI associated to the optimal portfolio liquidation problem is:

$$\min \Big[-\frac{\partial v}{\partial t} - \frac{\partial v}{\partial \theta} - \mathcal{L}v \ , \ v - \mathcal{H}v \Big] \;=\; 0, \quad \text{in } \ [0,T) \times \bar{\mathcal{S}}, \tag{1}$$

together with the relaxed terminal condition:

$$\min \big[v - U_L \ , \ v - \mathcal{H}v \big] \;=\; 0, \quad \text{in } \ \{T\} \times \bar{\mathcal{S}}, \tag{2}$$

dividing the time-space liquidation solvency region into:

- *A no-trade region*

$$\mathbf{NT} \;=\; \big\{ (t,z,\theta) \in [0,T] \times \bar{\mathcal{S}} : v > \mathcal{H}v \big\}$$

- *An impulse trading region*

$$\mathbf{IT} \;=\; \big\{ (t,z,\theta) \in [0,T] \times \bar{\mathcal{S}} : v = \mathcal{H}v \big\}$$

Local and nonlocal operators of the QVI

• $\mathcal{L}$ is the second order local operator associated to the **no-trading strategy**:

$$\mathcal{L}v = bp\frac{\partial v}{\partial p} + \frac{1}{2}\sigma^2 p^2 \frac{\partial^2 v}{\partial p^2}$$

• $\mathcal{H}$ is the nonlocal operator associated to the jumps of (Z, Θ) for an **impulse trading**:

$$\mathcal{H}v(t,x,y,p,\theta) = \sup_{e \in \mathcal{C}(z,\theta)} v(t, x - epf(e/\theta), y+e, p, 0)$$

and $\mathcal{C}(z,\theta)$ is the **admissible transaction set**:

$$\mathcal{C}(z,\theta) = \{e \in \mathbb{R} : (x - epf(e/\theta), y+e, p, 0) \in \bar{\mathcal{S}}\}.$$

Viscosity properties for the value function

Theorem.
The value function v is a *constrained* viscosity solution to (1)-(2).

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Remark for uniqueness and comparison principle.

• A first key argument for proving a comparison principle in obstacle problem is to produce a strict viscosity supersolution.

• However, in our model, this is not possible! and the reason is the absence of a fixed cost in the impulse transaction operator $\mathcal{H}$.

A model with fixed transaction cost

- We consider a small variation of the original model by adding a fixed transaction fee $\varepsilon > 0$ at each trading:

$$X^{\varepsilon}_{\tau_{n+1}} = X^{\varepsilon}_{\tau^-_{n+1}} - \zeta_{n+1} P_{\tau_{n+1}} f\left(\frac{\zeta_{n+1}}{\Theta_{\tau^-_{n+1}}}\right) - \varepsilon.$$

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▶ Modified liquidation function:

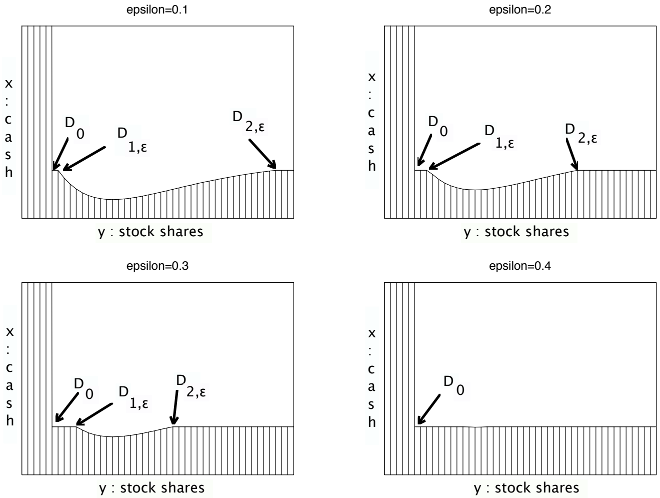
$$L_{\varepsilon}(x, y, p, \theta) \quad = \quad \max[x, L(x, y, p, \theta) - \varepsilon].$$

and solvency region:

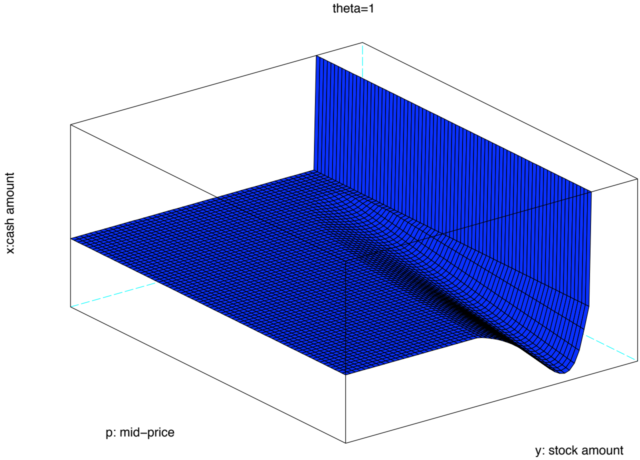
$$\begin{aligned} \mathcal{S}_{\varepsilon} \quad = \quad & \Big\{(z, \theta) = (x, y, p, \theta) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^*_+ \times [0, T] : \\ & y > 0 \ \text{ and } \ L_{\varepsilon}(z, \theta) > 0\Big\}, \end{aligned}$$

$\bar{\mathcal{S}}_{\varepsilon} = \mathcal{S}_{\varepsilon} \cup \partial \mathcal{S}_{\varepsilon}$.

Graph of $\mathcal{S}_\varepsilon$ in the plan (x, y)



Graph of $\mathcal{S}$ in 3-dim (x, y, p) for fixed θ



Optimal portfolio liquidation in the approximating problem

- *Admissible trading strategies*: given $(t, z, \theta) \in [0, T] \times \bar{\mathcal{S}}_\varepsilon$, we say that the impulse control strategy $\alpha = (\tau_n, \zeta_n)_n$ is admissible, denoted by $\alpha \in \mathcal{A}^\varepsilon(t, z, \theta)$, if the associated state process (Z^ε, Θ), starting from (z, θ) at t stays in $\bar{\mathcal{S}}_\varepsilon$ for all $t \leq s \leq T$.

Remark. The set $\mathcal{A}^\varepsilon(t, z, \theta)$ is nonempty.

- *Value function*:

$$v_\varepsilon(t, z, \theta) \quad = \quad \sup_{\alpha \in \mathcal{A}^\varepsilon(t,z,\theta)} \mathbb{E}\Big[U_{L_\varepsilon}(Z^\varepsilon_T, \Theta_T)\Big], \quad (t, z, \theta) \in [0, T] \times \bar{\mathcal{S}}_\varepsilon.$$

Remark

For $\varepsilon = 0$, $v_0 = v$.

Convergence of the approximation problem

Proposition.
The sequence $(v_\varepsilon)_\varepsilon$ is nonincreasing, and converges pointwise on $[0,T] \times (\bar{\mathcal{S}} \setminus \partial_L \mathcal{S})$ towards v as ε goes to zero, where

$$\partial_L \mathcal{S} \quad = \quad \big\{ (z,\theta) \in \bar{\mathcal{S}} : L(z,\theta) = 0 \big\}.$$

Dynamic programming equation

The QVI associated to the approximating problem is

$$\min\Big[-\frac{\partial v}{\partial t}-\frac{\partial v}{\partial \theta}-\mathcal{L}v\ ,\ v-\mathcal{H}_\varepsilon v\Big] \;=\; 0, \quad \text{in } [0,T)\times\bar{\mathcal{S}}_\varepsilon, \tag{3}$$

$$\min\Big[v-U_{L_\varepsilon}\ ,\ v-\mathcal{H}_\varepsilon v\Big] \;=\; 0, \quad \text{in } \{T\}\times\bar{\mathcal{S}}_\varepsilon, \tag{4}$$

where

$$\mathcal{H}_\varepsilon v(t,x,y,p,\theta) \;=\; \sup_{e\in\mathcal{C}_\varepsilon(z,\theta)} v(t,x-epf(e/\theta)-\varepsilon,y+e,p,0)$$

and $\mathcal{C}_\varepsilon(z,\theta)$ is the admissible transaction set:

$$\mathcal{C}_\varepsilon(z,\theta) \;=\; \Big\{e\in\mathbb{R} : (x-epf(e/\theta)-\varepsilon,y+e,p,0)\in\bar{\mathcal{S}}_\varepsilon\Big\}.$$

Viscosity characterization

Theorem.
For any $\varepsilon > 0$, the value function v_ε is the *unique constrained viscosity solution* to (3)-(4), satisfying the growth condition:

$$|v_\varepsilon(t,z,\theta)| \quad \leq \quad K(1+(x+yp)^\gamma), \quad (t,z,\theta) \in [0,T] \times \bar{\mathcal{S}}_\varepsilon,$$

for some $K > 0$, and the boundary condition on the corner line D_0 of $\bar{\mathcal{S}}_\varepsilon$:

$$\lim_{(t',z',\theta') \to (t,z,\theta)} v_\varepsilon(t',z',\theta') \quad = \quad U(0), \quad (t, z = (0,0,p), \theta) \in [0,T] \times D_0.$$

Remark.
With respect to usual uniqueness and comparison results, there are some technical difficulties coming from the nonregularity of the solvency boundary (corners), and so we have to specify here the boundary data on D_0, which forms a right angle of $\bar{\mathcal{S}}_\varepsilon$.

Concluding remarks

• We propose a continuous-time model of illiquidity market with bid-ask spread and temporary price impact penalizing speedy trades

- Suitable for defining liquidation value under illiquidity cost
- Discrete nature of trading times is justified by the presence of illiquidity cost

• The value function of the optimal portfolio liquidation problem is the limit of value functions characterized as unique constrained viscosity solutions of an approximation of the dynamic programming equation

- Convergence result useful for numerical purpose