Asset Pricing with Bubbles
Lecture 1

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A Little History

- Stock markets date back to at least 1531 in Antwerp, Belgium
- There are over 150 stock market exchanges world wide, of which the most significant count 103:
- There are 34 in Europe, including 5 in the U.K. and 4 in France
- There are 20 in North America (7 in Canada, 12 in the U.S., and 1 in Mexico)
- There are 5 in the Middle East: (Amman, Beirut, Istanbul, Palestine, and Tel Aviv)
- 3 in Africa; 26 in Asia, Australia, and New Zealand; 15 in Central and South America, Caribbean islands

Why do Stock Markets Exist?

- In the U.S., for example, the railroads needed vast amounts of capital to build their tracks, and created the need for a stock exchange
- The Dow Jones Industrial Average officially began in 1896
- In 1884, 12 years earlier, its predecessor began: **Customer's Afternoon Letter** which contained 11 stocks, 9 of which were railroads
- In 1885, there are 12 railroads and 2 industrials in the Dow Jones letter
- In 1886, there were 10 railroads and 2 industrials in the Dow Jones letter

The Dow Jones Industrial Average in January, 1896

*American Sugar
Chicago, Milwaukee & St. Paul
Chicago, Rock Island & Pacific
Delaware, Lackawanna & Western
Missouri Pacific
Union Pacific
Chicago, Burlington & Quincy
Chicago & North Western
Delaware & Hudson Canal
Louisville & Nashville
Northern Pacific preferred
*Western Union

*Indicates an industrial (not a railroad)

The initial Dow Jones Industrial Average without Railroads (May 26, 1896)

American Cotton Oil
American Tobacco
Distilling & Cattle Feeding
Laclede Gas
North American
U.S. Leather preferred

American Sugar
Chicago Gas
General Electric
National Lead
Tennessee Coal & Iron
U.S. Rubber

North American was replaced by US. Cordage Preferred, and Distilling & Cattle Feeding became American Spirits, in August, 1896

Basic Mathematical Models for Asset Pricing Finance

- Let $S = (S_t)_{0 \leq t \leq T}$ represent the (nonnegative) price process of a risky asset (e.g., the price of a stock, a commodity such as "pork bellies," a currency exchange rate, etc.)
- The present is often thought of as time $t = 0$. One is interested in the unknown price at some future time T, and thus S_T constitutes a "risk."

- **Example:** An American company contracts at time $t = 0$ to deliver machine parts to Germany at time T. Then the unknown price of Euros at time T (in dollars) constitutes a risk for that company.
- In order to reduce this risk, one may use "derivatives": one can purchase — at time $t = 0$ — the right to buy Euros at time T at a price that is fixed at time 0, and which is called the "strike price."
- If the price of Euros is higher at time T, then one exercises this right to buy the Euros, and the risk is removed. This is one example of a derivative, called a **call option**.
- More generally, a **derivative** is any financial security whose value is derived from the price of another asset, financial security, or commodity.

Call and Put Options

- A **call option** with strike price K, and payoff at time T can be represented mathematically as

$$C = (S_T - K)^+$$

  where $x^+ = \max(x, 0)$.

- Analogously, the payoff to a **put option** with strike price K at time T is

$$P = (K - S_T)^+$$

  and this corresponds to the right to *sell* the security at price K at time T.

- Calls and Puts are related, and we have

$$S_T - K = (S_T - K)^+ - (K - S_T)^+.$$

  This relation is known as **put – call parity**.

More complicated simple options

- We can calls and puts as building blocks for more complicated derivatives.
- For example, if

$$V = \max(K, S_T)$$

then

$$V = S_T + (K - S_T)^+ = K + (S_T - K)^+.$$

- More generally, if $f : \mathbb{R}_+ \to \mathbb{R}_+$ is convex then

$$f(x) = f(0) + f'_+(0)x + \int_0^\infty (x - y)^+ \mu(dy) \qquad (1)$$

where $f'_+(x)$ is the right continuous version of the (mathematical) derivative of f, and μ is a positive measure on $\mathbb{R}$ with $\mu = f''$, where the mathematical derivative is in the generalized function sense.

Thus if f is convex, and if

$$V = f(S_T)$$

is our financial derivative, then V is **a portfolio** consisting of a continuum of European call options:

$$V = f(0) + f'_+(0)S_T + \int_0^\infty (S_T - K)^+ \mu(dK).$$

Other kinds of derivatives

- We can also have *path dependent derivatives*.

$$\begin{aligned} V &= F(S)_T \\ &= F(S_t; 0 \le t \le T) \end{aligned}$$

  which are functionals of the paths of S.

- For example if S has càdlàg paths (càdlàg is a French acronym for "right continuous with left limits") then $F\colon D \to \mathbb{R}_+$, where D is the space of functions $f\colon [0, T] \to \mathbb{R}_+$ which are right continuous with left limits.

The time value of money

- Inflation makes money worth less as time goes on
- Deflations makes it worth more
- Evaluating a claim that pays off $\$D$ at time T, when current time is zero, can be done in time T dollars, or in time 0 dollars; if we use time T dollars for the payoff, but time 0 dollars for the evaluation, we must discount the payoff by the rate of inflation (deflation)
- Suppose we have $\$D$ at time 0, and invest it in a bank which pays interest rate r for one time unit (eg, one year). After one year, we have $\$(D + rD)$.

- If we are paid interest every 3 months , or $1/4$ year, and leave the interest in the bank, we have $\$D + Dr/4$ after the first quarter, $\$D(1 + r/4)^2$ after the second, and $\$D(1 + r/4)^4$ after one year.
- If we compound n times in one year and leave the money in the bank, we have $\$D(1 + r/n)^n$
- Taking limits $\lim_{n\to\infty} \$D(1 + r/n)^n = \$De^r$; for t time units analogously the limit $= \$De^{rt}$, which solves the ODE

$$\frac{dR_t}{R_t} = r; \quad R_0 = D$$

- In general if r is a stochastic process $(r_t)_{t\geq 0}$, then

$$R_t = D + \int_0^t r_s R_s ds \quad \Rightarrow \quad R_t = De^{\int_0^t r_s ds}.$$

A simple Portfolio

- A simple portfolio has a varying quantity of shares of a stock, plus a varying amount of money in a liquid, risk-free money account.
- The value of a portfolio, V, depends on the trading strategy a for stocks, and b for the money account
- A **trading strategy** is a vector of stochastic processes (a, b)
- Following a strategy (a, b) gives a dynamic portfolio value process:

$$V_t(a, b) = a_t S_t + b_t R_t.$$

- A trading strategy (a, b) is called **self-financing** if

$$a_t S_t + b_t R_t = a_0 S_0 + b_0 R_0 + \int_0^t a_s dS_s + \int_0^t b_s dR_s$$

Comments on self-financing

$$a_t S_t + b_t R_t = a_0 S_0 + b_0 R_0 + \int_0^t a_s dS_s + \int_0^t b_s dR_s \qquad (2)$$

- Intuitively Self-financing means that we do not consume money for other purposes, or add new money; we will soon give a heuristic justification of equation (2)
- S is taken, by assumption, to have sample paths which are right continuous and have left limits (càdlàg), and R is continuous; hence the right side of (2) is at least càdlàg
- This creates implicit restrictions on the illusory arbitrariness of the choice of a (predictable) and b (right continuous)
- If $r \equiv 0$ then $(R_t)_{t \geq 0} \equiv 1$, hence $dR_t = 0$ and (2) becomes

$$a_t S_t + b_t = a_0 S_0 + b_0 + \int_0^t a_s dS_s \qquad (3)$$

- This means once we have chosen strategy a, then b is determined

Heuristic justification of self-financing

- Suppose a, b, S are all three semimartingales, and that $R \equiv 1$. Then we have:

$$(a_{t+dt} - a_t)S_{t+dt} = -(b_{t+dt} - b_t) \tag{4}$$

which says that the change in stock holdings creates a corresponding change in the money account.

- Equation (4) becomes

$$\begin{aligned}(a_{t+dt} - a_t)(S_{t+dt} - S_t) + (a_{t+dt} - a_t)S_t &= -(b_{t+dt} - b_t) \\ \approx d[a, S]_t + S_{t-}da_t &= -b_t \end{aligned} \tag{5}$$

- By Integration by parts, (5) becomes

$$\begin{aligned} a_t S_t &= a_0 S_0 + b_0 + \int_0^t a_s dS_s + \int_0^t S_{s-} da_s + [a, S]_t \\ &\Rightarrow d(a_t S_t) - a_{t-} dS_t = -db_t \\ &\equiv a_t S_t + b_t = a_0 S_0 + b_0 + \int_0^t a_s dS_s. \end{aligned} \tag{6}$$

What is Arbitrage?

- In language: Arbitrage is the chance, no matter how small, to make a profit without taking any risk

- Definition
  A model is **arbitrage free on** $[0, 1]$ if there does not exist a self-financing strategy (a, b) such that

$$V_0(a, b) = 0, \quad V_T(a, b) \geq 0, \quad P(V_T(a, b) > 0) > 0. \qquad (7)$$

- We want to convert this idea into useful mathematics
- **Folk Theorem:** There is **no arbitrage** if and only if there exist a new probability Q, equivalent to P (ie, same sets of probability zero, written $Q \sim P$), such that S is a martingale.
- The above folk theorem is based on it being true in simple cases (eg, finite probability space Ω [J.M. Harrison & S.R. Pliska])

Martingales, local martingales, and sigma martingales

- We assume given a complete, filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$
- A stochastic process M is a **martingale** if $E(|M_t|) < \infty$, and for $s \leq t$, $E(M_t | \mathcal{F}_s) = M_s$ a.s.
- Martingales are insufficient; for example:
    - If X is a submartingale, we want a decomposition of $X = M + A$, where M is a martingale and A is an increasing, predictably measurable process. This is not true in general, instead we need the concept of *local martingale*.
    - If N is a martingale, we would like the stochastic integral $\int_0^t H_s dN_s$ to be a martingale, too. This is not true in general, but instead (if X has continuous paths) it is a *local martingale*.
    - In general, if N is a martingale, then the stochastic integral $\int_0^t H_s dN_s$ is a σ *martingale*

Definitions

- A stochastic process X with $X_0 = 0$ is a **local martingale** if there exists a sequence of stopping times $(T_n)_{n\geq 1}$ with $\lim_{n\to\infty} T_n = \infty$ a.s., such that $X_{t\wedge T_n}$ is a martingale for every $n \geq 1$
- A stochastic process X with $X_0 = 0$ is a σ **martingale** if there exists a martingale M and a predictable process H such that $X_t = \int_0^t H_s dM_s$ for all $t \geq 0$
- **Note: Martingales $\subset$ Local Martingales $\subset$ σ Martingales**
- If X is a nonnegative (or just bounded from below) σ martingale, then it is a local martingale. So $X \geq 0 \Rightarrow$ **Local Martingales = σ Martingales**
- Stochastic integration is closed for σ martingales
- For continuous processes, stochastic integration is closed for local martingales

One way local martingales can arise

- Let X be the unique weak solution of the stochastic differential equation

$$dX_t = \sigma(X_t)dB_t, \quad X_0 = 1$$

where B is a standard Brownian motion

- (Blei-Engelbert)If there exists an $\alpha \in (0,1)$ such that

$$\int_\alpha^\infty \frac{1}{\sigma(y)^2} dy < \infty$$

then X is a local martingale, and not a martingale. We call such a process a **strict local martingale**

The Canonical Example of a Local Martingale

- Let $B_t = (B_t^1, B_t^2, B_t^3)$ be standard 3 dimensional Brownian motion, with $B_0 = (1, 0, 0)$.
- Let $u : \mathbb{R}^3 \backslash \{(0,0,0)\} \to \mathbb{R}_+$ be given by

$$u(x) = \frac{1}{\parallel x \parallel}$$

- $X_t = u(B_t)$ is a positive real valued local martingale, with $E(X_0) = 1$; X is called the **inverse Bessel process**
- X is not a martingale, because one can show that

$$\lim_{t \to \infty} E(X_t) = 0$$

  and therefore it is not constant
- The inverse Bessel process satisfies the SDE

$$dX_t = -(X_t)^2 dB_t, \quad X_0 = x_0 > 0$$

The Canonical Example of a σ martingale

- τ is an exponential r.v. with parameter $\lambda = 1$
- U is independent of τ and $P(U = 1) = P(U = -1) = \frac{1}{2}$
- $X_t = U1_{\{t \geq \tau\}}$; then X is a martingale
- Let $H_s = \frac{1}{s}$ for $s > 0$, and let $M_t = \int_0^t H_s dX_s$
- Note that M has unbounded positive and negative jumps
- $E(|M_\nu|) = \infty$ for every stopping time ν with $P(\nu > 0) > 0$, so M is not a martingale, and not a local martingale, **but M is in fact a σ martingale**.

Semimartingales and arbitrage

- Suppose S has continuous paths and is a semimartingale with decomposition $S_t = S_0 + M_t + A_t$, with $M_0 = A_0 = 0$, and $Q \sim P$; take
$$Z_t = E_P(\frac{dQ}{dP}|\mathcal{F}_t)$$
which is a martingale
- By Girsanov's theorem the decomposition of S under Q is given by
$$S_t = (M_t - \int_0^t \frac{1}{Z_{s-}} d[Z,M]_s) + (A_t + \int_0^t \frac{1}{Z_{s-}} d[Z,M]_s);$$

- Therefore if Z can be chosen so that

$$A_t = -\int_0^t \frac{1}{Z_{s-}} d[Z, M]_s, \tag{8}$$

we have that S is a Q-local martingale.

- By the Kunita-Watanabe inequality, from (8) we have

$$d[Z, M]_t \ll \begin{cases} d[Z, Z]_t \\ d[M, M]_t \end{cases}$$

- Recall (8):

$$A_t = -\int_0^t \frac{1}{Z_{s-}} d[Z, M]_s,$$

therefore we must have that

$$dA_t \ll d[M, M]_t$$

in order for M to be martingale or local martingale under Q.

- This is not always the case; for example by Tanaka's formula, if

$$\begin{aligned} S_t &= 1 + |B_t| = 1 + \int_0^t \mathrm{sign}(B_s) dB_s + L_t^0 \\ &= 1 + \beta_t + L^0(B)_t, \end{aligned} \tag{9}$$

then $d[\beta, \beta]_t = dt$, but $dL_t^0 \not\ll dt$.

- Therefore $\nexists\, Q \sim P$ for (9) such that S is a Q (local) martingale

What if S is continuous and not a semimartingale?

- If there exists a $Q \sim P$ such that S is a local martingale under Q, then let $Y_t = E_Q(\frac{dP}{dQ}|\mathcal{F}_t)$
- By Girsanov,

$$S_t = (S_t + \int_0^t \frac{1}{Y_s} d[Y,S]_s) - \int_0^t \frac{1}{Y_s} d[Y,S]_s$$

  is a P decomposition of S. Therefore S is a P semimartingale, a contradiction
- Thus a **necessary condition** for $Q \sim P$ is that S be a P semimartingale

Why are (local) martingales so important?

- Martingales model fair gambling games
- A price process which is a model under the *risk neutral* measure should have constant expectation
- Martingales have the property that $t \mapsto E(M_t)$ is constant
- **Theorem** A stochastic process X is a martingale if and only if $E(M_\tau) = E(M_0)$ for every bounded stopping time τ.
- Thus, M has constant expectation not just for fixed times, but for stopping times as well.

The First Fundamental Theorem of Asset Pricing

- **First Version: J. M. Harrison and S. R. Pliska, circa 1979** showed that a *finite* probability space $(\Omega, \mathcal{F}, (S_n)_{n=0,1,2,\dots}, P)$ has **No Arbitrage** if and only if there exists another probability measure $Q \sim P$ such that S is a martingale
- **Second Version: David Kreps, circa 1981** realized that No Arbitrage was not a strong enough condition to guarantee such a result in a more general case. He created a new condition and called it **No Free Lunch**
- Ignoring admissibility conditions for now, Kreps said that S admits a **Free Lunch** on $[0, T]$ if there exists a function $f \in L^\infty_+(\Omega, \mathcal{F}, P)$ such that $P(f > 0) > 0$, and a *net* $(f_\alpha)_{\alpha \in I} = (g_\alpha - h_\alpha)_{\alpha \in I}$, with $h_\alpha \geq 0$ and $g_\alpha = \int_0^T H_s^\alpha dS_s$, for *admissible* H^α. And also $f_\alpha \to f$ in the Mackey topology on L^∞ induced by L^1
- The Mackey topology is often written as $\sigma(L^\infty, L^1)$, which means that for a sequence $(X_n)_{n\geq 1} \in L^\infty$, then $X_n \to X$, if for any $Y \in L^1$, $E(X_n Y) \to E(XY)$.

Economic intuition of No Free Lunch

- Often we think of f as being of the form $f = \int_0^T H_s dS_s$
- Kreps saw that f *could not* in general be restricted to this form for an admissible process H. (If it were, one could follow this trading strategy H and replicated f, and have classical arbitrage [starting with 0 and ending with $f \geq 0$])
- But suppose f can be approximated by $(f_\alpha)_{\alpha \in I}$ in a suitable topology
- Let $(h_\alpha)_{\alpha \in I}$ be the "errors" in the approximation, representing "money thrown away."
- No Free Lunch does not allow arbitrage, but it does allow arbitrage to exist in the limit

Kreps' Theorem

Theorem (Kreps, 1981) A *bounded* process $S = (S_t)_{0 \leq t \leq T}$ admits NFL if and only if there exists $Q \sim P$ such that S is a martingale under Q.

This creates three immediate questions:

1. Can we replace $[0, T]$ with $[0, \infty)$?
2. What if S is not bounded?
3. What does convergence in **nets** mean vis à vis an economics interpretation?

The Four Fundamental Papers that Clarified the Issues Surrounding the First Fundamental Theorem

1. Harrison, J.M, Kreps, D.M. (1979) Martingales and Arbitrage in Multiperiod Securities Markets, *Journal of Economic Theory* **20**, 381-408
2. Harrison, J.M, Pliska, S.R. (1981) Martingales and Stochastic Integrals in the Theory of Continuous Trading, *Stochastic Processes and their Applications* **11**, 215-260
3. Kreps, D.M. (1981) Arbitrage and Equilibrium in Economics with infinitely many Commodities, *Journal of Mathematical Economics* **8**, 15-35
4. Harrison, J.M, Pliska, S.R. (1983) A stochastic calculus model of continuous trading: Complete markets *Stochastic Processes and their Application* **11**, 313-316

13 years later: Delbaen and Schachermayer

- **Delbaen and Schachermayer, 1994:** Convergence with nets is replaced with convergence of sequences; S bounded is replaced with S *locally* bounded, and M a martingale is replaced with M a local martingale
- **Delbaen and Schachermayer, 1998:** The general case is treated, where S can be càdlàg, and does not have to be locally bounded, and M is replaced with a σ martingale.
- Before we discuss these results, we need the concept of an *admissible* trading strategy

The Doubling Strategy

- Bet \$1 at even money
- Stop betting if you win and collect \$1 net winnings; otherwise bet again, waging \$2
- Stop if you win; you have now lost \$1 and won \$2, for a profit of \$1; otherwise bet again, waging \$4
- In general: stop whenever you win, otherwise bet again, doubling your last bet; your net winnings will be \$1
- The probability is 1 that you will eventually win \$1, so this is an arbitrage strategy, known as **the doubling strategy**

Problems with the Doubling Strategy

- Need to make an unlimited number of bets (time constraints)
- Need "no fees" to make such bets (transaction costs)
- Need to have a counterparty (liquidity)
- But the above are *practical* problems; a theoretical problem is the need for infinite resources
- We can eliminate the doubling strategy with an admissibility condition

Admissibility

Definition: *Let S be a semimartingale, $\alpha > 0$. A predictable process H is* α-**admissible** *if $H_0 = 0$, and $\int_0^t H_s dS_s \geq -\alpha$, for all $t \geq 0$.*

H is **admissible** *if there exists an $\alpha > 0$ such that H is α-admissible.*

Note:

- We are implicity assuming That if H is admissible it is predictably measurable and is in the space of S-integrable processes
- This condition of admissibility is intrinsically asymmetric: H can increase without bound, but is strictly limited in how much it can be negative

The Kreps-Delbaen-Schachermayer Theory

- We work on the semi-infinite time interval $[0, \infty]$, on a filtered complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$.
- We further assume we have a risky asset price process $S = (S_t)_{t \geq 0}$ and that the spot interest rate $r = 0$
- A **Contingent Claim** is simply an $\mathcal{F}_T$ measurable random variable; examples are $C = (S_T - K)_+$, which is a **call at strike price** K **and maturity time** T; another example is $P = (K - S_T) +$ which is a **put** We let $H \cdot S$ denote the stochastic integral process $(\int_0^t H_s dS_s)_{t \geq 0}$
- Note that a call C is unbounded if S is unbounded, but a put P is always bounded, irrespective of the behavior of S

Definitions

- We let L^0_+ denote finite-valued, nonnegative random variables (a.s.). We define

$$\begin{aligned} \mathcal{K} &= \{(H \cdot S)_\infty | H \text{ is admissible}\} \\ \mathcal{K}_\alpha &= \{H \cdot S)_\infty | H \text{ is } \alpha - \text{admissible}\} \end{aligned}$$

- **No Arbitrage (NA):** $\mathcal{K} \cap L^0_+ = \{0\}$
- Intuition: Starting with nothing, the only nonnegative result we can end up with is identically 0; i.e., nothing
- Next we define

$$\begin{aligned} \mathcal{A}_0 &= \mathcal{K} - L^0_+ = \{X = H \cdot S)_\infty | H \text{ is admissible}, f \geq 0, \text{ finite}\} \\ \mathcal{A} &= \mathcal{A}_0 \cap L^\infty = \{|X| \leq k, \text{ some } k : X = (H \cdot S)_\infty - f\} \end{aligned}$$

- **No Free Lunch (NFL) [Kreps]:** $\bar{\mathcal{A}}^M \cap L^\infty_+ = \{0\}$, where the $\bar{(\cdot)}^M$ denotes closure in the Mackey topology $\sigma(L^1, L^\infty)$

- **No Free Lunch with Vanishing Risk (NFLVR) [Delbaen-Schachermayer]:** $\bar{\mathcal{A}} \cap L^\infty_+ = \{0\}$, where the closure of $\mathcal{A}$ is in L^∞, that is, the a.s. sup norm, as opposed to the Mackey closure of Kreps and NFL
- **Theorem:** NFLVR is invariant under a change to an equivalent probability measure
- NFLVR has become the accepted definition of no arbitrage; it is considered to be the "gold standard."
- However, we will see when we consider bubbles, that NFLVR is just a bit too weak.
- The idea of No Dominance was introduced by Robert Merton in 1973, but largely forgotten

No Dominance

- Let $P(S)$ be all probabilities equivalent to the underlying probability P such that if $Q \in P(S)$ then S is a $Q\sigma$ martingale. Let

$$\begin{aligned}
\mathcal{J} &= \{J \in \mathcal{F}_T | J \text{ is bounded from below and} \\
&\quad \sup_{Q \in P(S)} E_Q(S) < \infty\} \\
\Lambda(J)_t &= \{\text{the market price at time } t \text{ of the contingent claim } J\}
\end{aligned}$$

- **Definition:** An element D of $\mathcal{J}$ **Q-dominates** another element C of $\mathcal{J}$ if there exists a time $t < T$ such that

$$\begin{aligned}
C - \Lambda(C)_t &\leq D - \Lambda(D)_t, \text{ for all } t \geq 0, Q \text{ a.s., and} \\
Q\{C - \Lambda(C)_t &< D - \Lambda(D)_t\} > 0 \text{ for some } t \geq 0
\end{aligned}$$

- We say that the model has **No Dominance (ND) under P** if for any contingent claim $C \in P(S)$, **there does not exist** another claim D in $P(S)$ which dominates C
- **Theorem:** If No Dominance holds for one $Q \in P(S)$, then it holds for $Q \in P(S)$
- **Theorem:** If for any $H \in \mathcal{A}$ we have $\Lambda((H \cdot S)_T)_0 = 0$, then No Dominance implies (NA).
- **Theorem:** If for any $H \in \mathcal{A}$ we have $\Lambda((H \cdot S)_T)_0 = 0$ and Λ is lower semicontinuous on L^∞ with the $\| \cdot \|$ norm, then No Dominance implies (NFLVR)

End of Lecture 1