Asset Pricing with Bubbles
Lecture 2

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- We have a stochastic process $(S_t)_{t\geq 0}$ modeling the price of a risky asset; S exists on a complete, filtered probability space $(\Omega, \mathcal{F}, P, \mathbb{F})$, where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$
- We discussed that S needs to be a **semimartingale**: a semimartingale is a process S that has a decomposition $S_t = S_0 + M_t + A_t$, where $M_0 = A_0 = 0$ a.s., and M is a local martingale and A is a càdlg̀process with paths of finite variation on compact time intervals; a local martingale M is a process such that there exists an increasing sequence of stopping times $(T_n)_{n\geq 0}$ with $T_0 = 0$ a.s. and $\lim_{n\to\infty} T_n = \infty$ a.s.
- **Folk Theorem:** There are no arbitrage opportunities if and only if there is an equivalent probability measure Q such that under Q the risky asset price process S is a martingale.

- We let L^0_+ denote finite-valued, nonnegative random variables (a.s.). We define

$$\begin{aligned} \mathcal{K} &= \{(H \cdot S)_\infty | H \text{ is admissible}\} \\ \mathcal{K}_\alpha &= \{H \cdot S)_\infty | H \text{ is } \alpha - \text{admissible}\} \end{aligned}$$

- **No Arbitrage (NA):** $\mathcal{K} \cap L^0_+ = \{0\}$
- Intuition: Starting with nothing, the only nonnegative result we can end up with is identically 0; i.e., nothing
- Next we define

$$\begin{aligned} \mathcal{A}_0 &= \mathcal{K} - L^0_+ = \{X = H \cdot S)_\infty | H \text{ is admissible}, f \geq 0, \text{ finite}\} \\ \mathcal{A} &= \mathcal{A}_0 \cap L^\infty = \{|X| \leq k, \text{ some } k : X = (H \cdot S)_\infty - f\} \end{aligned}$$

- **No Free Lunch (NFL) [Kreps]:** $\bar{\mathcal{A}} \cap L^\infty_+ = \{0\}$, where the $\bar{(\cdot)}$ denotes closure in the Mackey topology, that is $\sigma(L^\infty, L^1)$

- **No Free Lunch with Vanishing Risk (NFLVR) [Delbaen-Schachermayer]:** $\bar{\mathcal{A}} \cap L^\infty_+ = \{0\}$, where the closure of $\mathcal{A}$ is in L^∞, that is, the a.s. sup norm, as opposed to the Mackey closure of Kreps and NFL
- **Theorem:** NFLVR is invariant under a change to an equivalent probability measure
- NFLVR has become the accepted definition of no arbitrage; it is considered to be the "gold standard."
- However, we will see when we consider bubbles, that NFLVR is just a bit too weak.
- The idea of No Dominance was introduced by Robert Merton in 1973, but largely forgotten

No Dominance

- Let $P(S)$ be all probabilities equivalent to the underlying probability P such that if $Q \in P(S)$ then S is a $Q\sigma$ martingale. Let

$$\begin{aligned} \mathcal{J} &= \{J \in \mathcal{F}_T | J \text{ is bounded from below and} \\ &\quad \sup_{Q \in P(S)} E_Q(S) < \infty\} \\ \Lambda(J)_t &= \quad \text{is the market price at time } t \text{ of the contingent claim } J \end{aligned}$$

- **Definition:** An element D of $\mathcal{J}$ ***Q*-dominates** another element C of $\mathcal{J}$ if there exists a time $t < T$ such that

$$\begin{aligned} C - \Lambda(C)_t &\leq D - \Lambda(D)_t, \text{ for all } t \geq 0, Q \text{ a.s., and} \\ Q\{C - \Lambda(C)_t &< D - \Lambda(D)_t\} > 0 \text{ for some} t \geq 0 \end{aligned}$$

  We say that the model has **No Dominance (ND) under *P*** if for any contingent claim $C \in P(S)$, **there does not exist** another claim D in $P(S)$ which dominates C

- **Theorem:** If No Dominance holds for one $Q \in P(S)$, then it holds for $Q \in P(S)$
- **Theorem:** If for any $H \in \mathcal{A}$ we have $\Lambda((H \cdot S)_T)_0 = 0$, then No Dominance implies (NA).
- **Theorem:** If for any $H \in \mathcal{A}$ we have $\Lambda((H \cdot S)_T)_0 = 0$ and Λ is lower semicontinuous on L^∞ with the $\| \cdot \|$ norm, then No Dominance implies (NFLVR)

A General Framework

- Let S be a semimartingale modeling a risky asset price process (so $S \geq 0$)
- Assume that NFLVR holds
- Let $D = (D_t)(t \geq 0$ be its **cumulative cash flow of dividends**
- Assume the spot interest rate $r \equiv 0$
- We assume that the risky asset has a (finite valued) maturity, or lifetime, of the risky asset

- Let X_τ = the terminal payoff, or liquidation value at time τ
- Assume that $X_\tau \geq 0$ and $D_t \geq 0$
- $\Delta D_t = D_t - D_{t-}$, and if $\Delta D_{t_0} > 0$, for some t_0, then S_{t_0} denotes the price **ex dividend**
- **Ex dividend** refers to the trading of shares when a declared dividend belongs to the seller, rather than to the buyer.

The Wealth Process

- The wealth of the investor at time t is given by

$$W_t = S_t + \int_0^{\tau \wedge t} dD_u + X_\tau 1_{\{t \geq \tau\}}$$

- We assume there exists a probability measure $Q \sim P$ such that W is a Q-local martingale
- Note that since $W \geq 0$ always, we do not have need of σ martingales

Trading Strategies

- A **trading strategy** is a vector process $(\pi_t, \eta_t)_{t\geq 0}$
- $(\pi_t)_{t\geq 0}$ is the trading strategy for the risky asset
- $(\eta_t)_{t\geq 0}$ is the (risk-free) money market trading strategy
- $W_t^\pi = \pi_t S_t + \eta_t$ is the **Wealth process corresponding to the strategy** (π, η)

Self-Financing Trading Strategies

- $W_0^\pi = 0$
- The **Value Process** V corresponding to the strategy (π, η) is given by

$$V_t^{\pi,\eta} = \int_0^t \pi_u dW_u$$

- Let $\alpha > 0$. A self-financing strategy π is **α-admissible** if $V_t^{\pi,\eta} \geq -\alpha$. The strategy π is **admissible** if it is α-admissible for some $\alpha > 0$

The Case of Complete Markets

- A market is **complete** for a class of contingent claims $\mathcal{H}$ if every claim $X \in \mathcal{H}$ can be perfectly hedged
- How to interpret this statement in mathematics?
- First we must discount for the time value of money; usually we work in a finite horizon, ie, a time interval $[0, T]$
- We can take, for example, $\mathcal{H}$ to be all **bounded** $X \in \mathcal{F}_T$; or all random variables X in $\mathcal{F}_T$ such that X/R_T is bounded, where $R_T = 1 + \int_0^T R_s r_s ds$, and r is the spot interest rate process

- **The Second Fundamental Theorem of Finance:** A market under $\mathcal{H}$ is **complete** if and only if every $X \in \mathcal{H}$ can be perfectly hedged. That is, for any $X \in \mathcal{H}$ there exists a hedging strategy π such that

$$X = \alpha + \int_0^T \pi_s dW_s$$

This is **equivalent to there being only one equivalent probability measure Q such that W is a Q local martingale**.

- The unique equivalent measure Q that turns W into a local martingale is called **the risk neutral measure**
- Recall that we are operating under the assumption of NFLVR, so we know that at least one such Q exists
- The price of such a claim X is now intuitively clear: if $X \in L^1_Q(\mathcal{F}_T)$ and $\int_0^T \pi_s dW_s$ is a martingale, then the price should be $E_Q(X) = \alpha$

The Problem of Unique Prices

- A **suicide strategy** is a strategy σ such that $W_0^\sigma = 1$ but $W_T^\sigma = 0$
- Suicide strategies can lead to the non-uniqueness of prices
- For example let Y be a contingent claim that suppose θ is a strategy such that
$$Y = c + W_T^\theta$$
- The fair price of Y should be $E_Q(Y) = c$. But if one adds the suicide strategy σ, one gets
$$Y = c + 1 + W_T^{\theta+\sigma}$$
which by the same reasoning should have price $c + 1$.

Eliminating suicide strategies

- The problem arises because a local martingale Z need not have constant expectation, and can even have $Z_T = 0$ at a finite time T
- Harrison and Pliska eliminate this possibility be restricting themselves to martingales, at the cost of generality
- One can also eliminate suicide strategies with Merton's No Dominance assumption, and then allow local martingales (recall that No Dominance implies NFLVR)

NFLVR does not imply ND

- Consider two risky assets maturing at time τ with payoff X_τ and Y_τ, respectively. Suppose $X_\tau \geq Y_\tau$ a.s. Then:

$$
\begin{aligned}
X_t^\star &= E_Q\{X_\tau | \mathcal{F}_t\} 1_{\{t<\tau\}} \\
&\geq E_Q\{Y_\tau | \mathcal{F}_t\} 1_{\{t<\tau\}} = Y_t^\star
\end{aligned}
$$

- Let β be a nonnegative local martingale such that $\beta_\tau = 0$ a.s., and $\beta_{t_0} > X_{t_0}^\star - Y_{t_0}^\star$ for some t_0
- Such a non-trivial β exists, and it is unbounded, of necessity (if it were bounded, it would be a nonnegative [true] martingale with terminal value 0, and hence identically 0)

- Suppose next the two risky asset prices are

$$\begin{aligned} X_t &= X_t^\star \\ Y_t &= \beta_t + Y_t^\star \end{aligned}$$

- No Dominance is violated, because

$$\begin{aligned} Y_{t_0} &> X_{t_0} \text{ for some } t_0, \\ X_\tau &\geq Y_\tau \end{aligned}$$

- But NFLVR is **not** violated. **Why?**

- Suppose we have a strategy designed to take advantage of this mis-pricing: Sell Y and buy X, say at time t_0, and then hold it to maturity. This give a gain of $\beta_{t_0} > 0$ at time t_0, and since $X_\tau - Y_\tau \geq 0$, with no other cash flows, we are safe to keep our gain β_{t_0}; this creates an apparent arbitrage
- However, to do this, if $t_0 < u \leq \tau$, then the value of the trading strategy is

$$-Y_u + X_u = \beta_u + (X_u^\star - Y_u^\star)$$

- Since β is of necessity unbounded, the admissibility condition eliminate this strategy of apparent arbitrage; doubling all over again!

What is the difference between S being a martingale under the risk neutral measure, and a local martingale?

- It turns out that this nuance can be used to give an explanation of financial bubbles
- But first, let us describe the popular conception of a financial bubble

US Stock Prices 1929 (Donaldson & Kamstra [1996])

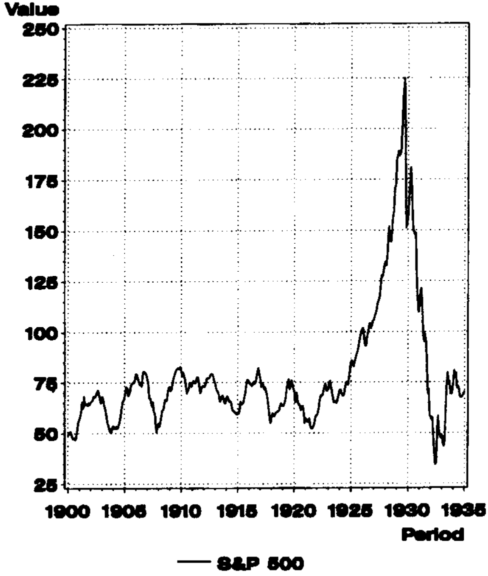


Figure 1

Important Recent Bubbles

- Minor crashes in the 1960s and 1980s
- Junk bond financing led to the major crash of 1987
- Japanese housing bubble circa 1970 to 1989
- The "dot com" crash, from March 11th, 2000 to October 9th, 2002. Led by speculation due to the promise of the internet; The Nasdaq Composite lost 78% of its value as it fell from 5046.86 to 1114.11.
- Current US housing bubble and subprime mortgages

NASDAQ Index 1998-2000 (Brunnermeir & Nagel)

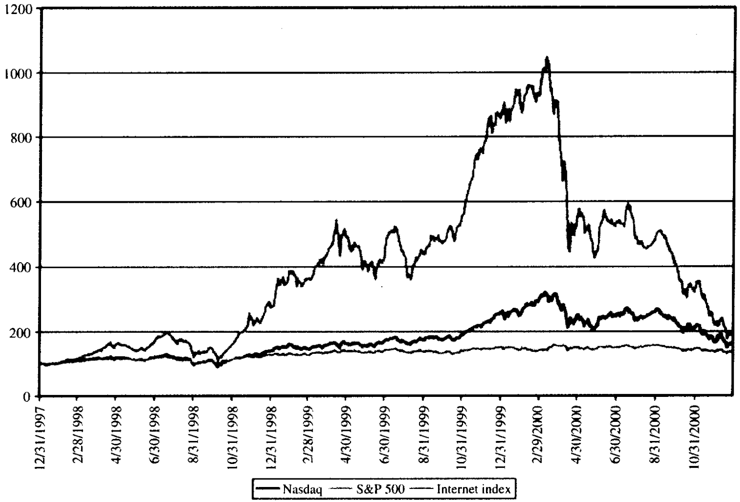
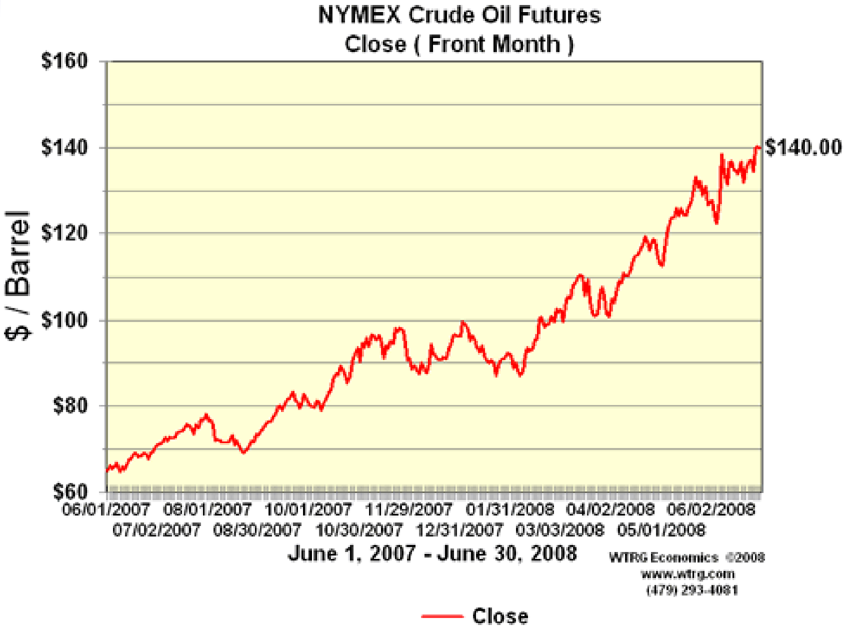


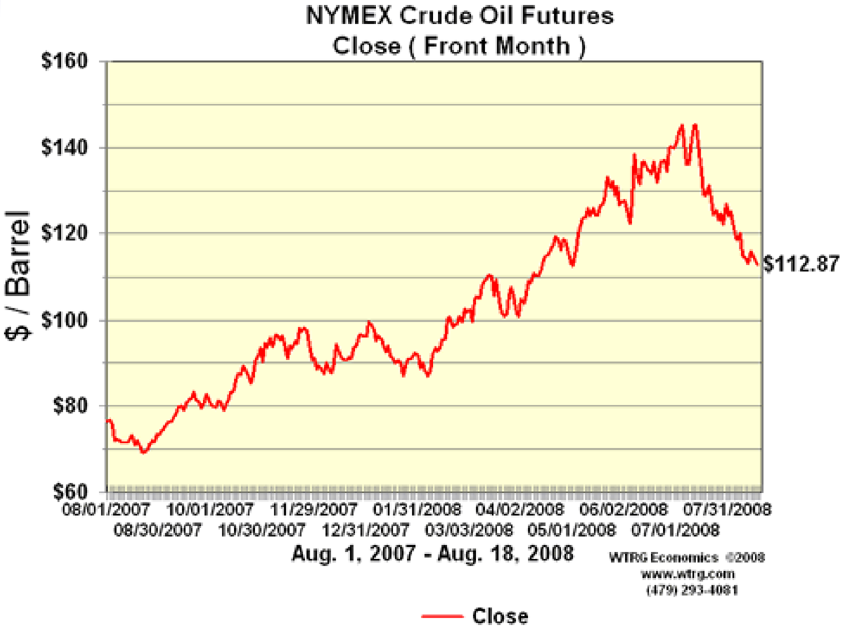
Figure 1. Returns on equally weighted Internet index, S&P 500 and Nasdaq composite. Comparison of index levels of the equally weighted Internet index, the S&P 500 index, and the Nasdaq composite index for the period 1/1/1998–12/31/2000. All three indexes are scaled to be 100 on 12/31/1997.

Current US Housing Price Trend (Center for Responsible Lending)

Oil Futures (WTRG Economics)



Oil Futures (WTRG Economics)



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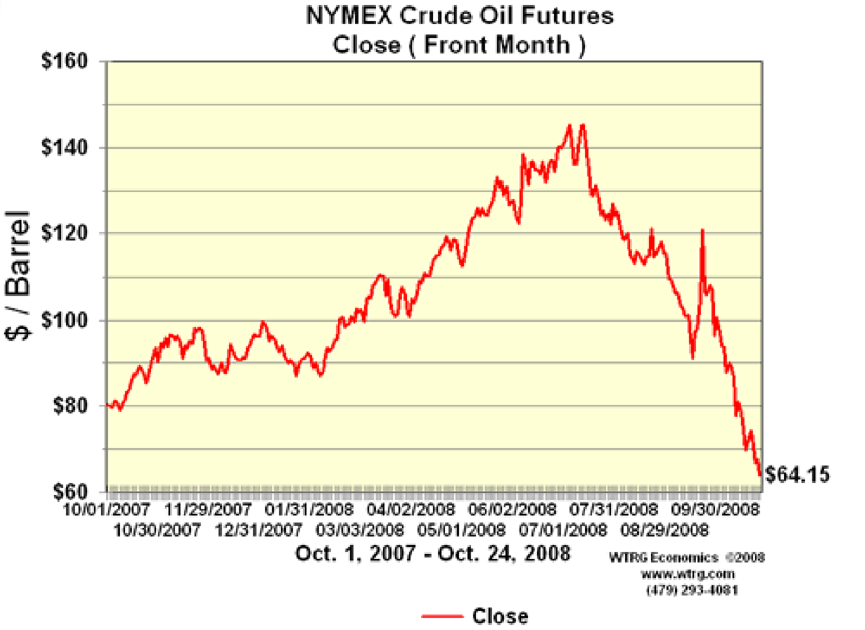

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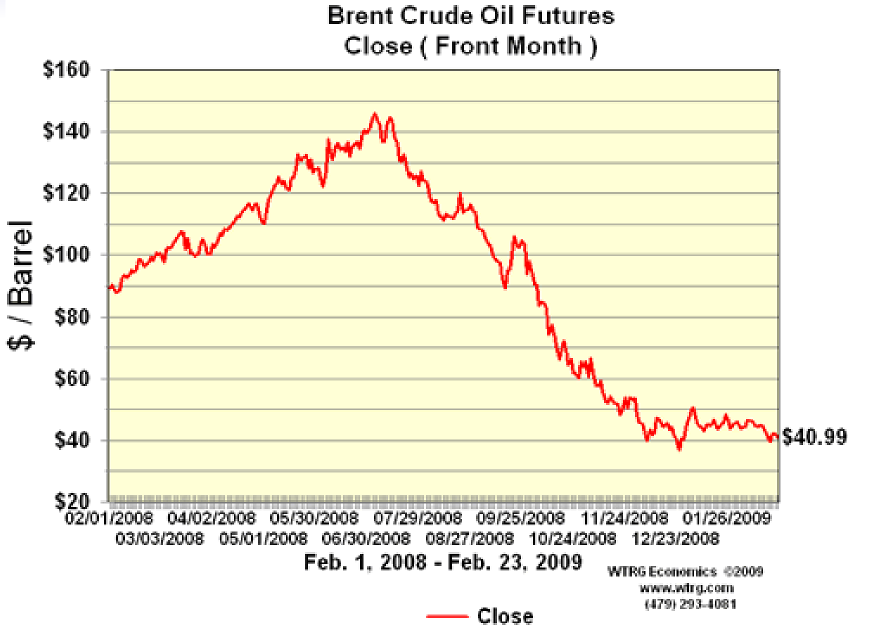

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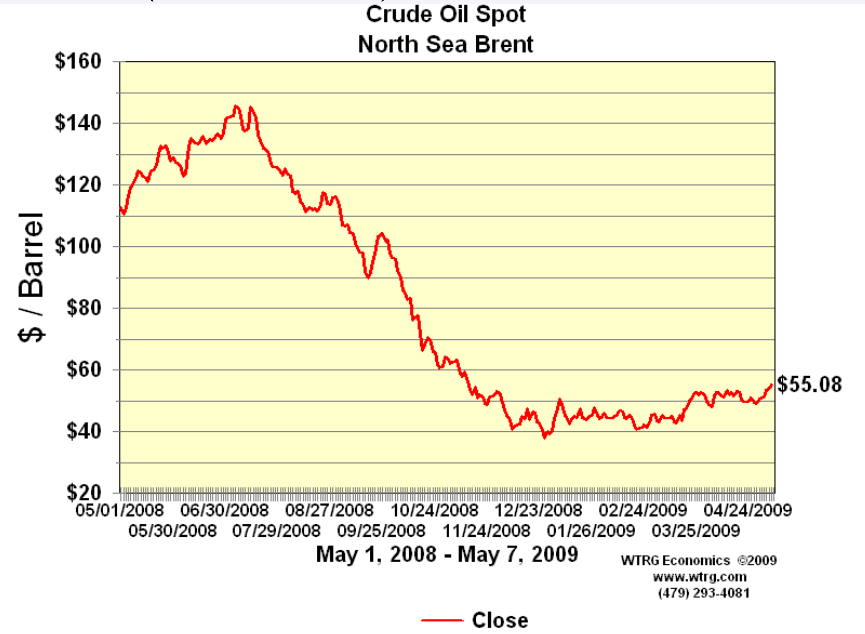

We can also see the bubble in the **entire world's stock markets** bursting. The next slide gives the results from the Morgan Stanley Capital International All Country World Index.

Global Stock Markets (Greed and Fear, Oct 9, 2008)

Our Framework for Modeling Bubbles

- We assume NFLVR
- We have a risky asset price process S with $S \geq 0$, and the spot rate $r_t \equiv 0$, so $R_t \equiv 1$
- We have a dividend process D, a wealth process W, and a liquidation vale X_τ
- We are in a complete market framework, with a unique risk neutral measure Q
- a fair price to pay for the risky asset at time t is the conditional expectation of the future cash flows (taken under the risk neutral measure)

The Fundamental Price

In complete markets with a finite horizon T, we use the risk neutral measure Q, and for $t < T$ the **fundamental price** of the risky asset is defined to be:

$$S_t^\star = E_Q\{\int_t^T dD_u + X_T|\mathcal{F}_t\}$$

Definition (Bubble)

A bubble in a static market for an asset with price process S is defined to be:

$$\beta = S - S^\star$$

Static Markets

Theorem (Three types of bubbles)

1. *β is a local martingale (which could be a uniformly martingale) if $P(\tau = \infty) > 0$;*
2. *β is a local martingale but not a uniformly integrable martingale, if it is unbounded, but with $P(\tau < \infty) = 1$;*
3. *β is a strict Q local martingale, if τ is a bounded stopping time.*

- Type 1 is akin to fiat money
- Type 2 is tested in the empirical literature
- Type 3 is essentially "new." Type 3 are the most interesting!

- Fiat money (Type 1 bubbles) has no intrinsic value, and become almost worthless
- German hyper inflation of the 1920s
- Zimbabwean hyperinflation of the current day

100 million German marks, 1923

100 trillion Zimbabwe dollars, 2009

Hyperinflation in Zimbabwe

In 1980, 1ZWR=US $1

Month	ZWR per USD
Sept 2008	$1,000$
Oct 2008	$90,000$
Nov 2008	$1,200,000$
Mid Dec 2008	$60,000,000$
End Dec 2008	$2,000,000,000$
Mid Jan 2009	$1,000,000,000,000$
2 Feb 2009	$300,000,000,000,000$

Zimbabwean Currency Follow-up

HARARE, Zimbabwe, March 20 (UPI) – Zimbabwean Finance Minister Tendai Biti told members of Parliament the country's currency was essentially dead.

"The death of the (Zimbabwean) dollar is a reality we have to live with," he said during a 2009 budget presentation. "Since October 2008, our national currency has become moribund."

Along with his remarks, Biti announced "the removal of all foreign currency surrender requirements," New Ziana reported Friday

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What this means is that Zimbabwe is currently using hard currencies from other lands (eg, The Rand, Euro, and US Dollar); this is expected to continue for at least one year

Theorem (Bubble Decomposition)

The risky asset price admits a unique decomposition

$$S = S^{\star} + (\beta^1 + \beta^2 + \beta^3)$$

where

1. β^1 *is a càdlàg nonnegative uniformly integrable martingale with* $\lim_{t\to\infty} \beta^1_t = X_\infty$ *a.s.*
2. β^2 *is a càdlàg nonnegative NON uniformly integrable martingale with* $\lim_{t\to\infty} \beta^2_t = 0$ *a.s.*
3. β^3 *is a càdlàg non-negative supermartingale (and strict local martingale) such that* $\lim_{t\to\infty} E\{\beta^3_t\} = 0$ *and* $\lim_{t\to\infty} \beta^3_t = 0$ *a.s.*

Ben Bernanke and the Federal Reserve

End of Lecture 2