Asset Pricing with Bubbles
Lecture 3

Philip Protter, Cornell University

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Review from Lecture 2

- Let S be a semimartingale modeling a risky asset price process (so $S \geq 0$)
- Assume that NFLVR holds
- Let $D = (D_t)_{t \geq 0}$ be its **cumulative cash flow of dividends**
- Assume the spot interest rate $r \equiv 0$

- Let X_τ = the terminal payoff, or liquidation value at time τ
- The wealth of the investor at time t is given by

$$W_t = S_t + \int_0^{\tau \wedge t} dD_u + X_\tau 1_{\{t \geq \tau\}}$$

- We assume there exists a probability measure $Q \sim P$ such that W is a Q-local martingale
- A **trading strategy** is a vector process $(\pi_t, \eta_t)_{t \geq 0}$

- $W_0^\pi = 0$
- The **Value Process** V corresponding to the strategy (π, η) is given by

$$V_t^{\pi,\eta} = \int_0^t \pi_u dW_u$$

- Let $\alpha > 0$. A strategy π is **α-admissible** if $V_t^{\pi,\eta} \geq -\alpha$. The strategy π is **admissible** if it is α-admissible for some $\alpha > 0$
- **The Second Fundamental Theorem of Finance:** A market under $\mathcal{H}$ is **complete** if and only if for every $X \in \mathcal{H}$ there exists a hedging strategy π such that

$$X = \alpha + \int_0^T \pi_s dW_s$$

  This is **equivalent to there being only one equivalent probability measure Q such that W is a Q local martingale**.

The Fundamental Price in a Complete Market Setting

- Since markets are assumed complete, let $Q \sim P$ be the **unique** risk neutral measure
- We define the **fundamental price of the risky asset S, denoted $S^\star$**, to be the future discounted future cash flow one expects to get, conditional on current information
- In mathematics, $S^\star$ is given by

$$S_t^\star = E_Q\{\int_t^T dD_s + X_\tau 1_{\{\tau<\infty\}}|\mathcal{F}_t\}1_{\{t<\tau\}}$$

- The payoff at time $t = \infty$ does not contribute to $S^\star$

Theorem:

$S_t^\star$ is well defined. Moreover,

$$\lim_{t\to\infty} S_t^\star = 0 \text{ a.s.}$$

- Observe that W is a nonnegative Q supermartingale, so $S_t^\star \in L^1(dQ)$, and the result follows the supermartingale convergence theorem, and the facts that $(D_t)_{t\geq 0}$ and X_τ are nonnegative.
- Note that in contrast, we cannot assume that $S_t^\star$ is in $L^1(dP)$
- **Corollary:**

$$W_t^\star = S_t^\star + \int_0^{t\wedge\tau} dD_u + X_\tau 1_{\{\tau\leq t\}}$$

is a uniformly integrable martingale under Q, and

$$W_\infty^\star = \int_0^{\tau} dD_u + X_\tau 1_{\{\tau<\infty\}}$$

Bubbles

- A **bubble** is defined to be a process $\beta = (\beta_t)_{t \geq 0}$ given by

$$\beta_t = S_t - S_t^\star$$

- Note that $\beta_t \geq 0$ for all t, a.s.
- **Theorem:** *If there exists a non trivial bubble (ie, $\beta_t \neq 0$ for some $t > 0$) for the risky asset price process S, then*

1. *If $P(\tau = \infty) > 0$ then β is a local martingale without restrictions (it can even be a uniformly integrable martingale)*
2. *If $P(\tau < \infty) = 1$, and β is unbounded, then β is a local martingale, and it* cannot *be a uniformly integrable martingale*
3. *If τ is bounded, and then β must be a* strict *local martingale*

Theorem (Bubble Decomposition)

The risky asset price admits a unique decomposition

$$S = S^\star + (\beta^1 + \beta^2 + \beta^3)$$

where

1. β^1 *is a càdlàg nonnegative uniformly integrable martingale with* $\lim_{t\to\infty} \beta^1_t = X_\infty$ *a.s.*
2. β^2 *is a càdlàg nonnegative NON uniformly integrable martingale with* $\lim_{t\to\infty} \beta^2_t = 0$ *a.s.*
3. β^3 *is a càdlàg non-negative supermartingale (and strict local martingale) such that* $\lim_{t\to\infty} E\{\beta^3_t\} = 0$ *and* $\lim_{t\to\infty} \beta^3_t = 0$ *a.s.*

Examples

- We call the three types of bubbles in the decomposition **bubbles of Type 1, Type 2 and Type 3**
- **Example of a Type 1 bubble:** Let $S_t = 1$, all $t, 0 < t < \infty$, and no dividends. This is an example of **fiat money**
- In this case $\tau = \infty$ a.s., and $X_\infty = 1$, and $D_t \equiv 0$ all $t \geq 0$.
- Therefore

$$S^\star_\infty = E_Q\left(\int_0^{t\wedge\tau} dD_u + X_\tau 1_{\{t\geq\tau\}}|\mathcal{F}_t\right) = 0$$

- Hence

$$\beta_t = S_t - S^\star_t = 1$$

A second example of a Type 1 bubble

- Let $B_t = (B_t^1, B_t^2, B_t^3)$, the three dimensional standard Brownian motion
- $\| B_t \|$ is called the Bessel process;

$$X_t = \frac{1}{\| B_t \|}$$

  is known as the **inverse Bessel process**
- Assume there are no dividends, only the asset price. One can show that

$$\lim_{t \to \infty} X_t = 0$$

  and that X is a local martingale; indeed, X satisfies the SDE

$$dX_t = -X_t^2 dB_t; \quad X_0 = 1$$

  where B is a Brownian motion
- Also $E(X_0) = 1$ and $\lim_{t \to \infty} E(X_t) = 0$

Example of a Type 2 bubble

- Let τ be a stopping time with $P(\tau > t) > 0$, for all $t > 0$, and $P(\tau < \infty) = 1$
- Let

$$\begin{aligned} S_t^\star &= 1_{\{t<\tau\}}, \text{ payoff 1 at time } \tau \\ \beta_t &= \frac{1 - 1_{\{\tau \le t\}}}{P(\tau > t)} \\ S_t &= S_t^\star + \beta_t \end{aligned}$$

- One can show that β is a martingale which is *not uniformly integrable*, and $\beta_\infty = 0$
- So β is a bubble which is **not** uniformly integrable

Example of a Type 3 bubble [A. Cox and D. Hobson, 2005]

- Let T be a fixed (non random) time, and define

$$S_t^\star = 1_{\{[0,T)\}}(t), \quad X_T = 1$$

- Let the bubble be given by

$$\beta_t = \int_0^t \frac{\beta_u}{\sqrt{T-u}} dB_u$$

- Then β is a strict local martingale, with $B_t = 0$; define

$$S_t = S_t^\star + \beta_t$$

Historical example of an option

- Aristotle, in his treatise *Politics; Book 1, Part XI*, writes of Thales of Miletus, a pre-Socratic Greek philosopher and one of the Seven Sages of Greece
- Thales wanted to justify his beliefs in astronomy, which allowed him to predict (correctly, as it turned out) that there would be a bumper olive crop harvest (Source: Walter Schachermayer)
- According to Aristotle, Thales "gave deposits for the use of all the olive-presses in Chios and Miletus, which he hired at a low price because no one bid against him. When the harvest-time came, and many were wanted all at once and of a sudden, he let them out at any rate which he pleased, and made a quantity of money."

An ancient olive press used to make oil

Put-Call Parity in the Presence of Bubbles

- A Call Option has the payoff structure at the maturity time T of $(S_T - K)_+$ and a put $(K - S_T)_+$ and a **forward contract** at strike price K and maturity time T has a payoff at time T of $S_T - K$
- Recall that trivially

$$(S_T - K)_+ - (K - S_T)_+ = S_T - K$$

- Let $C_t(K)$, $P_t(K)$, and $V_t(K)$ be the **market prices** at time t and strike price K with common maturity time T of a call, a put, and a forward
- Let $C_t(K)^\star$, $P_t(K)^\star$, and $V_t(K)^\star$ be the **fundamental prices** at time t and strike price K with common maturity time T of a call, a put, and a forward

- The traditional approach for a complete market for put call parity is to **define** the time t price of (for example) a European call to be

$$E_Q\{(S_t - K)_+ | \mathcal{F}_t\}$$

and then put-call parity follows from the linearity of conditional expectation
- The issue of whether or not market prices agree with the conditional expectation prices is assumed to be true
- With bubbles, the market prices of calls, puts, and forwards need not satisfy put-call parity

Example of Put-Call Parity Failing

- Let $B^i, 1 \leq i \leq 5$ be five iid standard Brownian motions
- Define

$$\begin{aligned} M_t^1 &= \exp(B_t^1 - t/2) \\ M_t^i &= 1 + \int_0^t \frac{M_s^i}{\sqrt{T-s}} dB_s^i, \quad 2 \leq i \leq 5 \end{aligned}$$

- Consider a market with finite time horizon T
- It is complete, given M^i, $1 \leq i \leq 5$
- M^1 is a uniformly integrable martingale, and the rest are strict local martingales on $[0, T]$
- Let

$$\begin{aligned} S_t^\star &= \sup_{s \leq t} M_t^1; \quad S_t = S_t^\star + M_t^2; \quad C(K)_t = C^\star(K)_t + M_t^3 \\ P(K)_t &= P^\star(K)_t + M_t^4; \quad V(K)_t = V^\star(K)_t + M_t^5 \end{aligned}$$

- All the traded securities in the this example have bubbles
- Let δ_t^C, δ_t^P, and δ_t^F be the bubbles parts of the market prices for the Call, Put, and Forward.
- Under special conditions only (the absence of bubbles) do we have **market price put-call parity:**

$$\begin{aligned} C_t(K) - P_t(K) &= F_t(K) \text{ if and only if } \delta_t^F = \delta_t^C - \delta_t^P \\ C_t(K) - P_t(K) &= S_t - K \text{ if and only if } \delta_t^S = \delta_t^C - \delta_t^P \end{aligned}$$

Implications for Models in the Black-Scholes Paradigm

- To take advantage of these bubbles based on the convergence at time T, one needs only to short sell at least one asset
- Such a strategy, however, is not admissible due to possible unbounded losses
- By the Black-Scholes paradigm we mean a continuous risky asset price process under the now standard NFLVR structure
- The important consequence is that in the presence of bubbles, the **Black-Scholes formula need not hold**
- This is because the time t market price of a call option, $C_t(K)$, can differ from the price $E_Q\{(S_t - K)_+\}$

Consequence for Black-Scholes Paradigm Models

- Implied volatility from the B-S formula need not equal historical volatility; indeed, if there is a bubble, implied volatility should exceed historical volatility
- However, if one assumes No Dominance, then the usual understanding of the Black-Scholes model applies
- Another issue is **Merton's No Early Exercise Theorem**
- This theorem states that while an American call option with strike price K and maturity time T has the *a priori* impression of presenting more flexibility in the exercise of the option, in reality the optimal strategy is to exercise it at maturity T. Therefore the fair prices of an American call option and that of a European call option are the same
- The proof of Merton's theorem uses Jensen's inequality and assumes the risky asset risk neutral price process is a martingale

Under NFLVR and continuous complete markets, Merton's No Early Exercise Theorem need not hold

- We give an example where No Early Exercise fails to hold
- Let $B_t = (B_t^1, B_t^2, B_t^3)$ be a standard Brownian motion with $B_0 = (1, 0, 0)$
- Recall that the **inverse Bessel process** is

$$X_t = \frac{1}{\parallel B_t \parallel}$$

  which is a strict local martingale
- If X models a risky asset price process, then the price process is a bubble
- $(X_t)_{t \geq 0}$ is a uniformly integrable collection, $E(X_0) = 1$, and $\lim_{t \to \infty} X_t = 0$ a.s. and in L^1

- If X is a risk neutral (Q) martingale, then by Jensen's inequality, $t \mapsto E_Q\{(X_t - K)_+\}$ is monotone increasing
- For the inverse Bessel process, with **Soumik Pal**, we have shown that the prices of European calls decrease as a function of time to expiration
- That is, for S the inverse Bessel process, the function

$$T \mapsto E\{(S_T - K)_+\}$$

  is monotone decreasing if $K \leq \frac{1}{2}$, and otherwise it is initially increasing and then strictly decreasing for

$$T \geq \left(K \log \frac{2K+1}{2K-1}\right)^{-1}.$$

- A similar results holds for all continuous strict local martingales with asymptotic behavior similar to that of the inverse Bessel process
- This result is intuitive in the presence of bubbles, since in a bubble, the best strategy is to get in and out early, and not to wait a long time to liquidate your positions

Bubble Decomposition

Theorem [Bubble Decomposition]:

$$S_t = S_t^\star + \beta_t = S_t^\star + \left(\beta_t^1 + \beta_t^2 + \beta_t^3\right),$$

is a unique decomposition such that

- $\beta^1 \geq 0$ is a uniformly integrable martingale with $\lim_{t\to\infty} \beta_t^1 = X_\infty$ a.s.
- $\beta^2 \geq 0$, is **not** a uniformly integrable martingale, but of course is a local martingale and is possibly a martingale, and $\lim_{t\to\infty} \beta_t^2 = 0$ a.s.
- $\beta^3 \geq 0$ is a **strict** local martingale such that $\lim_{t\to\infty} \beta_t^3 = 0$ a.s. and in L^1

Why Does Short Selling Not Correct for Bubbles?

- Two reasons are proposed in the literature:
- The first is **structural limitations:** This is the limited ability and/or expensive cost to borrow an asset for short sales (eg, Duffie, Gârleanu, and Pederson [2002])
- As regards the first, in markets where short selling does not exist (especially the third world), there do not seem to be more bubbles
- The second is the risk the short seller takes that the price will continue to go up (the danger of trying to predict a bubble)
- In mathematics this translates into **admissibility violations**

Two Problems with Complete Markets and Bubbles

- What is nice is that the risk neutral measure Q is unique, and we therefore have a unique fundamental price
- An undesirable property is the impossibility of **bubble birth:** A nonnegative local martingale cannot spring up after being zero; once a nonnegative local martingale reaches zero, it sticks at zero forever after
- The biggest problem is that while bubbles make sense in complete markets under NFLVR, **bubbles do not exist under No Dominance**. This is serious, because we will see later we need No Dominance to establish fundamental put-call parity
- **Theorem:** Under No Dominance, Type 2 and Type 3 bubbles do not exist in a complete market (with NFLVR)

Proof that Bubbles Do Not Exist in Complete Markets under ND

- **Theorem:** Under No Dominance, Type 2 and Type 3 bubbles do not exist in a complete market (with NFLVR)
- **Proof:** For Type 2 and Type 3 bubbles, $\beta_\infty = 0$. Let W be the wealth process corresponding to the risky asset price process S
- There exist hedging processes π^1 and π^2 such that

$$\begin{aligned} W_t^\star &= W_0^\star + \int_0^t \pi_u^1 dW_u \\ \beta_t &= \beta_0 + \int_0^t \pi_u^2 dW_u \end{aligned}$$

- Let η^1 and η^2 make π^1 and π^2 self-financing, so that both π^1 and π^2 are admissible

- We have two ways to generate W
- The first way is buy and hold
- The second way is to follow π^1, obtaining $W^\star$
- The cost of the first position is $W_0 \geq W_0^\star$, with $W_0 > W_0^\star$ if there is a non-trivial bubble
- That means that π^1 dominates the buy and hold strategy, which violates No Dominance; so β cannot exist.

- We conclude: **bubbles exist in a complete market under NFLVR** [Lowenstein and Willard, Cox and Hobson], but cannot be born after time $t = 0$, create a Black-Scholes paradox, and violate put-call parity.
- **Bubbles do not exist in a complete market under No Dominance**, which is stronger than NFLVR
- The non existence of bubbles under ND solves the Black-Scholes paradigm paradox, for example
- **What happens in incomplete markets?** In incomplete markets under No Dominance the argument showing bubbles do not exist, no longer applies

Do Bubbles Exist in Incomplete Markets?

- To discuss bubbles in incomplete markets, we need to decide what we mean by a **fundamental price**, since there is an infinite choice of risk neutral measures
- There are five basic methods to choose such a measure
- The first is **Utility Indifference Pricing:** Risk Neutral prices span an interval on the real line, and choosing the right price depends on the utility function of preferences of the agent selling the contingent claim

- **The Egocentric Method:** Simply choose one arbitrarily
- **The Convenience Method:** Choose a risk neutral measure that gives the price process mathematically nice properties: for example, makes it a Markov process, or even a Lévy process
- **The Canonical Method:** Find a reasonable criterion (eg, minimal variance of the error, minimal distance to the historical measure in a distance one chooses, minimal entropy) and let it determine the risk neutral measure
- **The Ostrich Method:** Prove results under a risk neutral measure already chosen; that is, you do not specify it, but pretend someone else has done so already
- All of these methods assume that, once chosen, the risk neutral measure is fixed and and never changes
- In our next lecture, we will discuss a different method to choose the risk neutral measure, and allow it to change from one choice to another, and study bubbles in incomplete markets

Ben Bernanke and the Federal Reserve

End of Lecture 3
Thank you for your attention