Asset Pricing with Bubbles
Lecture 4

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- At the end of Lecture 3 we considered the problem of defining a fundamental price of a risky asset when the market is incomplete, and there are an infinite number of risk neutral measures.
- The current methods share two properties:
    1. In one sense or another they can be considered to be *ad hoc*
    2. Once a risk neutral measure is chosen, it remains fixed in the model for all time (which is often a finite horizon model, and time is modeled by an interval $[0, T]$)

- Instead of these approaches, one can **let the market choose the risk neutral measure**
- *En passant*, the market is often wrong (otherwise we would not have bubbles!), but the issue is not "truth," as it is, for example, in physics
- **Work with Jean Jacod**
- Related recent results are by **M. Schweizer and J. Wissel**, and also by **R. Carmona and S. Nadtochiy**

How Do We Let the Market Choose the Risk Neutral Pricing Measure?

- Begin with the risky asset price process S and a (possibly random) savings rate;
- Next assume there are market-given price processes for a large number of (European style) derivatives of the form $g(S_T)$, where T varies;
- Find the collection $\mathcal{Q}$ of risk neutral measures that make both the price process and all of the derivative price processes local martingales

- **If there are enough derivative prices, the cardinality of $\mathcal{Q}$ might be one;**
- **Alternatively the cardinality of $\mathcal{Q}$ might be zero or ∞** (compatibility issues are serious here).
- More specifically, we consider a contingent claim of the form $g(S_T)$ where g is a positive, convex function; g is fixed throughout (eg, $g(x) = (x - K)_+$)
- Next, with g fixed, let $P(T)_t$ denote the price of the claim $g(X_T)$ at time T, and fix t and consider $P(T)_t$ as a function of T (ie, $T \mapsto P(T)_t$)

- Under a quasi-left continuous assumption on the filtration, $T \to P(T)_t$ is continuous, and we assume further that it is **absolutely continuous** in the following sense:

$$P(T)_t = g(S_t) + \int_t^T f(t,s)ds, \text{ for } t \le T \text{ and with } f(t,s) \in \mathcal{F}_t$$

- We also assume f is an Itô process and hence a semimartingale, with some regularity on its decomposition

- Next we look at the collection of all probability measures equivalent to the historical probability P, which make S a local martingale, call this collection $\mathcal{M}_{\text{loc}}$
- For t fixed and a collection of times $T \in \mathcal{J}$, we obtain a condition under which the values of $(P(T)_t)_{\{T \in \mathcal{J}\}}$ determine that $\mathcal{M}_{\text{loc}}$ is either empty, an infinite collection, **or a singleton**
- If $\mathcal{M}_{\text{loc}} = \emptyset$ then we have arbitrage, which is uninteresting for our purposes; also when it is infinite is uninteresting.
- But when the index family $\mathcal{J}$ has infinite cardinality, we can often obtain that $\mathcal{M}_{\text{loc}}$ is a singleton
- This means that **the market has chosen a unique risk neutral measure**

Regime Change

- Recall that we have an infinite number of possible risk neutral measures
- Let us assume that market has chosen a unique one determined by (an infinite number of) option prices
- *A fortiori* the option prices must be internally consistent as well; else we would have no risk neutral measure matching, due to arbitrage opportunities (Delbaen-Schachermayer)

- Given the nature of the market over time, it is unreasonable to assume that it stays with that risk neutral measure for all time
- When the market changes from one risk neutral choice to a different one, we call it a **regime change**
- **Caution:** This is a controversial idea, and a new one

The Fundamental Price for Incomplete Markets

- What follows is based on work with **Robert Jarrow and Kazuhiro Shimbo**
- Recall that for the case of complete markets with a finite horizon T, with risk neutral measure Q, and for $t < T$ the **fundamental price** of the risky asset is defined to be:

$$S_t^\star = E_Q\{\int_t^T dD_u + X_T | \mathcal{F}_t\}$$

- In incomplete markets, if one Q is chosen by the market for all time (**a "static market"**), the definition is analogous.
- In an incomplete market with an infinite horizon, we assume there exists a countable sequence of stopping times $0 = T_0 < T_1 < T_2 < \dots$ increasing to ∞ a.s. which represents change times from one risk neutral probability to another
- The stochastic interval $[T_i, T_{i+1})$ consists of the i^{th} regime

- In an incomplete market with an infinite horizon and regime change, the **fundamental price** of the risky asset with end time τ for the asset, $t < \tau$, and for regime i at time t, is defined to be:

$$S_t^\star = E_{Q^i}\{\int_t^\tau dD_u + X_\tau 1_{\{\tau<\infty\}} | \mathcal{F}_t\}$$

  where Q^i is the risk neutral measure chosen by the market
- Note that $X_\tau 1_{\{\tau=\infty\}}$ is not included

The Evaluation Measure

- We can piece all of these measures Q^i together to get one measure $Q^\star$
- $Q^\star$ **need not be a risk neutral measure**; (if it were, then in effect we would not have regime change)
- Some people find $Q^\star$ not being a risk neutral measure, although it is equivalent, to be troubling
- We call $Q^\star$ the **evaluation measure**, and write it $Q^{t\star}$ to denote that it changes with the time t

- The Fundamental Price can be written compactly:

$$\begin{aligned} S_t^\star &= \sum_i 1_{[T_i, T_{i+1})}(t) E_{Q^i}\{\int_t^\tau dD_u + X_\tau 1_{\{\tau<\infty\}} | \mathcal{F}_t\} \\ \text{or using } Q^\star &= Q^{t\star} \\ S_t^\star &= E_{Q^{t\star}}\{\int_t^\tau dD_u + X_\tau 1_{\{\tau<\infty\}} | \mathcal{F}_t\} \end{aligned}$$

Bubble Birth

- Recall the definition of a bubble:
  Defintion[Bubble]: A bubble in a static market for an asset with price process S is defined to be:

$$\beta_t = S_t - S_t^\star \geq 0, \quad t \geq 0$$

- **A bubble in a dynamic market for $t < \tau$ in regime i** is:

$$\beta_t = S_t - E_{Q^{t\star}}\{\int_t^\tau dD_u + X_\tau 1_{\{\tau<\infty\}}|\mathcal{F}_t\} \geq 0$$

- Since we are in regime i, we have in this case $Q^{t\star} = Q^i$.
- If there are no bubbles for regime $i-1$, a change to a new risk neutral measure at change time T_i might in effect create a bubble; if it does, we call this **bubble birth**

Derivatives

- Recall we touched on European Calls and Puts in Lecture 3
- For a European call with payoff $(S_T - K)_+$ at time T, we denote its market price at time $t < T$ as $C_t(K)$
- Analogously for a put with payoff $(K - S_T)_+$ at time T, its time t market price is $P_t(K)$
- A forward with payoff $S_T - K$ has market price at time t denoted $V_t(K)$
- Their fundamental prices are given by

$$
\begin{aligned}
C_t^\star(K) &= E_{Q^{t\star}}\{(S_T - K)_+\} \\
P_t^\star(K) &= E_{Q^{t\star}}\{(K - S_T)_+\} \\
V_t^\star(K) &= E_{Q^{t\star}}\{(S_T - K)\}
\end{aligned}
$$

- **Theorem [Put-Call Parity]:** We have put-call parity for fundamental prices

$$C_t^\star(K) - P_t^\star(K) = V_t^\star(K)$$

- The proof follows from the linearity of the conditional expectation with respect to $Q^{t\star}$
- **Theorem:** $P_t(K) = P_t^\star(K)$
- The proof follows from the fact that it is bounded, plus the hypothesis of No Dominance
- In contrast, if there is a bubble in the risky asset price S, then it is captured by the call option market price process $C_t(K)$

- **Theorem:** for $K > 0$ we have

$$C_t(K) - C_t^\star(K) = S_t - E_{Q^{t\star}}\{S_T|\mathcal{F}_t\}$$

- The proof of this theorem follows from the observation regarding bubbles of forwards:

$$\begin{aligned} V_t^\star(K) &= E_{Q^{t\star}}\{S_T - K|\mathcal{F}_t\} \\ &= E_{Q^{t\star}}\{S_T|\mathcal{F}_t\} - K \\ &= S_t^\star - K \le S_t - K = V_t^f(K) \end{aligned}$$

which means that a **forward has a type 3 bubble** of size

$$\begin{aligned} \delta_t^3 &= V_t^f(K) - V_t^{f\star}(K) = (S_t - K) - (S_t^\star - K) \\ &= S_t - S_t^\star \end{aligned}$$

- Put-call parity and the fact that $P_t(K) = P_t^\star(K)$ imply

$$C_t(K) - C_t^\star(K) = S_t - E_{Q^{t\star}}\{S_T|\mathcal{F}_t\}$$

- Note that with calls we are dealing with Type 3 bubbles, since they exist on the compact time interval $[0, T]$

Testing for Bubbles

- Recall that we have put-call parity even in the presence of bubbles
- This is in contrast to the views expressed in the literature
- For example, Battalio and Schultz find no violations of put call parity during the internet stock price bubble, and argue that this evidence is inconsistent with an internet stock price bubble
- However a "correct" model for the price operator $E_Q\left(\cdot\middle|\,\mathcal{F}_t\right)$, when applied to both call and put options on the same spot commodity, would give differential results in the presence of price bubbles
- Puts would be priced correctly, but calls would not. If type 3 bubbles exist, this difference would be observable
- Also, the mispricings would be independent of the moneyness of the options, but dependent on the time to maturity

Forwards and Futures on Durable Commodities

- With commodities, it is good to include the time value of money; let
$$R_t = \exp\left(\int_0^t r_u du\right)$$
- We consider commodities whose liquidation dates exceed the maturity of the contract (eg, gold, oil, or a stock index)
- Therefore we can assume $T < \tau$
- Let Φ_m consist of linear combinations of random variables generated by admissible and self-financing strategies involving the risky asset and the money market account, as well as all static strategies involving forwards and futures, and European calls and puts on the risky asset

- We assume given a unique **market price operator**

$$\Lambda_t : \Phi_m(t) \to L^0(\Omega, \mathcal{F}_t, P)$$

  which at time t gives a market price $\Lambda_t(\Phi)$
- We have (by our assumptions) that

$$\Lambda_t(R_T) = R_t \text{ and } \Lambda_t(W_T) = S_t$$

- We assume linearity of Λ, and of course No Dominance, and we treat only the static case (ie, no regime change, no bubble birth)

- Let Q denote the risk neutral probability chosen by the market
- The price of a zero coupon bond of \$1, maturing at time T, is

$$p(t,T) = \Lambda_t(1_{\{T\}}) = \Lambda_t^{\star}(1_{\{T\}}) = E_Q\left(\frac{1_{\{T\}}}{R_T}|\mathcal{F}_t\right)R_t$$

- Since we are working on a compact time interval $[0,T]$, we can have bubbles only of type 3
- Let

$$\begin{aligned}\hat{S}_T &= S_T 1_{\{T<\tau\}} + R_T \frac{X_T}{R_T} 1_{\{\tau \le T\}} \\ \mathrm{div}_{t,T} &= \Lambda_t\left(R_T \int_t^{T\wedge\tau} \frac{1}{R_u} dD_u\right)\end{aligned}$$

- These represent respectively the payoff to the risky asset at time T (less the cash flows prior to time T), and the market price of the cash flow stream between times t and T
- Linearity of the market price operator gives

$$S_t = \Lambda_t(\hat{S}_T + \mathrm{div}_{t,T})$$

- We let as usual

$$S_t = S_t^{\star} + (\beta_t^1 + \beta_t^2 + \beta_t^3)$$

Forward Prices and Bubbles

- A **forward contract** written on a risky asset price S obligates the owner (the long) to purchase the risky asset on the delivery date T for a predetermined price, called the **forward price**
- **Theorem [Forward Price]:**

$$V_{t,T}\, p(t, T) = S_t - \text{div}_{t,T}$$

- **Theorem [Forward Price Bubbles]:**

$$V_{t,T}\, p(t, T) = S_t^\star - \text{div}_{t,T} + \beta_t, \text{ where } \beta_t = S_t - S_t^\star$$

- Note that since we are only working up to time T, we have Type 3 bubbles only

Futures Prices

- Futures prices are more complicated than are forwards
- Futures contracts are superficially similar to forward contracts
- A **futures contract** is written on the risky asset price S with a fixed maturity time T. It represents the purchase of the risky asset at time T via a prearranged payment procedure.
- The prearranged payment procedure is called **marking-to-market**.
- Marking-to-market obligates the purchaser (long position) to accept a continuous cash flow stream equal to the continuous changes in the futures prices for this contract.

- The time t **futures prices**, denoted $F_{t,T}$, are set (by market convention) such that newly issued futures contracts (at time t) on the same risky asset with the same maturity date T, have **zero market value**.
- Hence, futures contracts (by construction) have zero market value at all times, and a continuous cash flow stream equal to $dF_{t,T}$.
- At maturity, the last futures price must equal the asset's price $F_{T,T} = S_T$

- The wealth process of a portfolio long one futures contract at time T is

$$R_T \int_0^T \frac{1}{R_u} dF_{u,T} \in \Phi_m(0).$$

- We do **not** *a priori* require futures prices $(F_{t,T})_{t\geq 0}$ to be non-negative
- The **Futures price process** $(F_{t,T})_{t\geq 0}$ is any càdlàg semimartingale such that

$$\Lambda_t \left(R_T \int_t^T \frac{1}{R_u} dF_{u,T} \right) = 0 \textit{ for all } t \in [0, T], \text{ and}$$

$$F_{T,T} = S_T$$

- This definition of the Futures price process is a definition which depends on the processes themselves, and not (in the case of an incomplete market, where there are an infinite number of risk neutral measures) on the choice of a risk neutral measure
- This is similar to the definition in the book of Karatzas and Shreve (*Methods of Mathematical Finance*)
- What we are **not** giving is the classical definition of the futures price, as in (eg) Duffie's or Shreve's books, where futures prices are defined to be martingales

- Indeed, under $r = 0$, in Duffie and Shreve Futures are modeled by expressions of the form $E\{S_T|\mathcal{F}_t\}$, which of course is automatically a martingale
- Thus in the classical case, bubbles are excluded by fiat
- This idea that Futures processes are always martingales under the risk neutral measure is reflected in the literature. For example, K. Miltersen and Eduardo Schwarz wrote in a paper published in 1998 in the *Journal of Financial and Quantitative Analysis* that "...since we know that futures prices are martingales under an equivalent martingale measure"

Why are Futures so Important?

- With a call option or a put option, or a forward, you have the risk that the counter party may not be able to honor the contract at the maturity time
- AIG would not have able to honor its contracts, but the American government-by-Goldman Sachs came to the rescue
- A smaller firm, ACA, which insured CDS contracts, went bankrupt
- Perhaps due to a general lack of trust in the financial markets, the Futures market is very large: according to a BIS report of 2007, the total exchange traded derivatives (which are mostly futures) in Dec 2006 was **70.5 trillion** US dollars, and in Dec 2008 was around 58 trillion US dollars (notional)
- Margins in the US are from 5 to 20 percent, typically

- Futures prices can have their own bubbles that are unrelated to any bubble in the underlying asset's price
- A futures price bubbles **can be positive or negative**
- This is in contrast to bubbles in the underlying asset's price process
- **Theorem [Futures Price Bubbles]:** Let γ_t be a local Q martingale with $\gamma_T = 0$. Then,

$$F_{t,T} = E_Q\left(S_T \middle| \mathcal{F}_t \right) + \gamma_t$$

is a futures price process

- **Corollary:** Let $E_Q\left([F_{\cdot,T}, F_{\cdot,T}]_t^{\frac{1}{2}}\right) < \infty$ *for all* $0 \leq t \leq T$. A futures contract has no bubbles if and only if

$$F_{t,T} = E_Q\left(S_T \middle| \mathcal{F}_t\right)$$

- In addition to its own bubble γ, a futures price can also inherit Type 1 and Type 2 bubbles from the asset price process
- It does not inherit a Type 3 bubble because the futures prices is a bet on the market price of the risky asset S_T at time T

- **Theorem [Futures Price Bubbles]:**

$$
\begin{aligned}
F_{t,T} \quad = \quad & E_Q\left(R_T \middle| \mathcal{F}_t \right) + \gamma_t + (S_t^* - \mathrm{div}_{t,T}) + cov_Q\left(\frac{S_T}{R_T}, R_T \middle| \mathcal{F}_t \right) \\
& + \beta_t - \left[\beta_t^3 - E_Q\left(\frac{\beta_T^3}{R_T} \middle| \mathcal{F}_t \right) R_t - \delta_t \left(R_T \int_t^T \frac{dD_u}{R_u} \right) \right]
\end{aligned}
$$

Forward versus Futures Prices

- It is known since at least 1981 (Jarrow and Oldfield; Cox, Ingersoll, and Ross) that forwards and futures prices are equal under deterministic interest rate, but not equal in general under stochastic interest rates
- **Theorem [Deterministic Interest Rates]:**

$$F_{t,T} = V_{t,T} \text{ for all } t$$

- Our proof uses No Dominance
- The case of stochastic interest rates is more complicated

- **Theorem [Stochastic Interest Rates]:**

$$
\begin{aligned}
V_{t,T} \quad = \quad & F_{t,T} + \ \mathrm{cov}_Q\left(S_T, \frac{1}{R_T} \middle| \mathcal{F}_t \right) \frac{R_t}{p(t,T)} - \gamma_t \\
& + \frac{\beta_t^3}{p(t,T)} - E_Q\left(\frac{\beta_T^3}{A_T} \middle| \mathcal{F}_t \right) \frac{R_t}{p(t,T)} - \delta_t \left(R_T \int_t^T \frac{dD_u}{R_u} \right)
\end{aligned}
$$

End of Lecture 4 and All of the Lectures

Thank You for Your Patience