# Hedging Survivor Bonds with Mortality-linked Securities

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# Securitization of longevity risk

- Tontine bonds – one of the debt management schemes which ultimatively led to the creation of the Bank of England. A fixed yearly coupon was paid as long as a designated individual named by the investor was still alive.
- BNP Paribas longevity bond – coupon payments were supposed to be a nominal times the survival probability (at the time of the coupon payment) of a male from an English/Welsh reference population who was 65 years old in 2003. In the end it was not launched due to a lack of interest because of certain design, pricing, and institutional issues.
- CAT-type mortality bonds, where coupon payments or even the principal is at risk if mortality exceeds catastrophic levels.
- Mortality swaps written on some prespecified longevity index.

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- Let $(\mathcal{G}_t) = (\mathcal{H}_t) \vee (\mathcal{F}_t)$. We assume that $(\mathcal{F}_t)$ is immersed into $(\mathcal{G}_t)$, that is, $W$ stays a Brownian motion in the larger filtration $(\mathcal{G}_t)$.

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- Let $(\mathcal{G}_t) = (\mathcal{H}_t) \vee (\mathcal{F}_t)$. We assume that $(\mathcal{F}_t)$ is immersed into $(\mathcal{G}_t)$, that is, $W$ stays a Brownian motion in the larger filtration $(\mathcal{G}_t)$.
- The survival probability process $G_t$ associated to $\tau$ is supposed to fulfill

$$G_t := P\left(\tau > t \middle| \mathcal{F}_t\right) = \exp\left(-\int_0^t \mu_s \, ds\right) := \exp\left(-\Gamma_t\right)$$

such that $\mu_t$ can be interpreted as stochastic mortality intensity.

- One example would be the stochastic Gompertz-Makeham model,

$$\mu_t = Z_1(t) + Z_2(t)\, e^{\gamma t}, \qquad \gamma > 0,$$

where the stochastic factors $Z_1$, $Z_2$ follow e.g. the law of Cox-Ingersoll-Ross processes.

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- The survival probability $G_t$ could be inferred at time $T$ from some publicly accessible longevity index which should be specified in the bond contract.

- The combined position in survivor as well as mortality bonds resembles the mortality swap introduced by *Dahl, Melchior & Møller (2008)*, which generates a payment stream of

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- However, the rate $1 - H_t$ gives only a constant contribution up to time $\tau$; a potential rate change can only happen at $\tau$ and therefore would not affect our portfolio which does not generate any payments after $\tau \wedge T$.
- The (undiscounted) market value process associated with the mortality bond is given by the conditional expectation

$$V_t = kE_Q\Big[\int_0^{\tau\wedge T} G_s\, ds \Big| \mathcal{G}_t\Big],$$

assuming that $Q$ is some pricing measure. Our goal is now to hedge the risk exposure from having sold the survivor bond $(1 - H_T)$ by trading dynamically in the mortality bond with value process $V$.

# Dynamic hedging with mortality bonds

- *Föllmer & Sondermann (1986)* approach (mean-variance hedging under $Q$):

$$\min_{c,\vartheta} E_Q\left[\left((1-H_T)-c-\int_0^T \vartheta_t\, dV_t\right)^2\right],$$

where we minimize over all constants $c$ and admissible strategies $\vartheta$. Here admissibility means that the resulting integral process $\int \vartheta\, dV$ is a square-integrable $Q$-martingale.

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- It results that the fair price in this framework is given by $c = E_Q\left[1-H_T\right]$, and the optimal hedging strategy $\vartheta^{opt}$ can be found via the Kunita-Watanabe decomposition

$$E_Q\left[(1-H_T)\middle|\,\mathcal{G}_t\right] = c + \int_{0+}^t \vartheta_s^{opt}\, dV_s + R_t,$$

where $R$ is a $Q$-martingale strongly orthogonal to $V$ (i.e. $VR$ is a local $Q$-martingale).

- This decomposition can be found by simple algebra once we have established representations of the $Q$-martingales $E_Q\left[(1-H_T)\,|\,\mathcal{G}_{\cdot}\right]$ and $V$ in terms of stochastic integrals with respect to the Brownian motion $W$ and the fundamental counting process martingale $M$ associated with $H$ where

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- **Notation.** From now on, $\mu_t$ and associated quantities are understood with respect to the chosen martingale measure $Q$, in particular $M$ is a $Q$-martingale. We also denote, by a slight abuse of notation, by $W$ the $Q$-Brownian motion which is the Girsanov transform of our original $P$-Brownian motion. Moreover, in the sequel all expectations are with respect to $Q$ as well without that we make this explicit in the notation.

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- We define the $(P, (\mathcal{F}_t))$-martingale $U$ and the predictable process $\widetilde{\xi}$ by

$$U_t := E\left[ e^{-\Gamma_s} Y \middle| \mathcal{F}_t \right] = E\left[ e^{-\Gamma_s} Y \right] + \int_0^t \widetilde{\xi}_u \, dW_u,$$

where the second equality follows from the martingale representation theorem with respect to a Brownian filtration.

- Let $X = (1 - H_s)\, Y$, $0 \leq s \leq T$ be a simple claim. Then

$$X = E\left[e^{-\Gamma_s} Y\right] + \int_0^{\tau \wedge T} \xi_t \, dW_t + \int_{0+}^{\tau \wedge T} \zeta_t \, dM_t$$

where $\xi_t = \widetilde{\xi}_t L_{t-}$ and $\zeta_t = U_t L_{t-} \mathbb{I}_{[0,s]}(t)$.

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- A general representation result by *Kunita (1979)* can be obtained by approximating $\mathcal{G}_T$-measurable random variables in $L^2(P)$ by simple claims, and by using that the spaces of stochastic integrals of admissible integrands wrt. to $W$, resp. $M$, are closed in $L^2(P)$. The result then follows by a localization argument and states that each $(\mathcal{G}_t)$-martingale $N$ can be written as

$$N_t = N_0 + \int_0^{\tau \wedge t} \xi_u \, dW_u + \int_{0+}^{\tau \wedge t} \zeta_u \, dM_u,$$

for predictable processes $\xi$ and $\zeta$.

- Moving now to the representation of the mortality bond, we first observe that

$$\int_0^{\tau \wedge T} G_s \, ds = H_T \int_0^{\tau} G_s \, ds + (1 - H_T) \int_0^{T} G_s \, ds.$$

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- While $(1 - H_T) \int_0^T G_s\, ds$ is again a simple claim, it remains to study the term $H_T \int_0^{\tau} G_s\, ds$, which is the principal difficulty.

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- While it would be tempting to condition on $\mathcal{G}_t$, and then to use a Feynman-Kac type argument, it seems not at all obvious how to obtain the Markov property in this approach. The main idea is to reduce the conditional $\mathcal{G}_t$-expectation via several steps to a conditional $\mathcal{F}_t$-expectation which allows us finally to work in a Brownian setting.

- In the following, we set for $t \in [0, T]$,

$$Z_t := \int_0^t G_s \, ds.$$

Assume that $E\left[\int_0^T \mu_s^2 \, ds\right] < \infty$ and set

$$\omega_t := E\left[\int_0^T Z_s \, e^{-\Gamma_s} \mu_s \, ds \middle| \mathcal{F}_t\right].$$

Then $\omega$ is a square-integrable $(\mathcal{F}_t)$-martingale, and admits the following integral representation with respect to the Brownian motion $W$ which generates $(\mathcal{F}_t)$:

$$\omega_t = \omega_0 + \int_0^t w_s \, dW_s, \quad t \in [0, T],$$

for some $(\mathcal{F}_t)$-predictable process $w$ such that $E\left[\int_0^T w_s^2 \, ds\right] < \infty$.

- With

$$N_t := e^{\Gamma_t}\, E\left[\int_t^T Z_s e^{-\Gamma_s} \mu_s\, ds \,\middle|\, \mathcal{F}_t\right],$$

we get the decomposition

$$H_T \int_0^\tau G_s\, ds = \omega_0 + \int_0^t \xi_s^{(2)}\, dW_s + \int_{0+}^t \zeta_s^{(2)}\, dM_s,$$

where $\xi_s^{(2)} := (1 - H_s)\, e^{\Gamma_s} w_s$ and $\zeta_s^{(2)} := (Z_s - N_s)$.

- With

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  where $\xi_s^{(2)} := (1 - H_s) \, e^{\Gamma_s} w_s$ and $\zeta_s^{(2)} := (Z_s - N_s)$.
- Together with known results about the decomposition of simple claims, this allows us to find the Kunita-Watanabe decomposition of the mortality bond.

- As illustration, we take the stochastic Gompertz model with a CIR-dynamics for the stochastic factor. By the Markov property, there is a function $v(t, u, x, z)$ such that, with $X_t = G_t$,

$$v(t, \mu_t, X_t, Z_t) = E\left[ \int_t^T Z_s G_s \mu_s \, ds \middle| \mathcal{F}_t \right].$$

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- Assuming that $v$ is smooth enough, a Feynman-Kac type argument yields the PDE

$$uxz + v_t + \left(e^{\gamma t}\overline{\mu} - (1-\gamma)\, u\right) v_u - uxv_x + \frac{1}{2} e^{\gamma t} u v_{uu} = 0,$$

with the terminal condition

$$v(T, u, x, z) = 0.$$

# A market model approach

- We model the price process $\widetilde{S}$ of a mortality-linked security under $P$ as

$$d\widetilde{S}_t / \widetilde{S}_{t-} = a_t \, dt + b_t \, dW_t + c_t \, dM_t$$

  where $W$ is a Brownian motion, and $M$ is the counting process martingale associated to the one-jump process $\mathbb{I}_{\tau < t}$ which we assume has deterministic intensity $\mu_t$. The functions $a$, $b$, and $c$ are as well assumed for simplicity to be deterministic but can possibly be time-inhomogenous (which we will often suppress in the notation). We moreover assume that they are bounded on $[0, T]$.
- As for the foreseeable future there will not be many actively traded derivatives written on $\widetilde{S}$, finding a pricing measure by calibration is impossible. Therefore, we will choose martingale measures according to optimality criteria.
- Our main point is that the presence of two different noise terms $dW$ and $dM$ makes the analysis much more complicated compared to a scenario with only a Brownian driver.

- Our goal is now to calculate several optimal martingale measures, i.e. probability measures $Q$ such that the price process $\widetilde{S}$ is a local $Q$-martingale. This is equivalent to the statement that the stochastic logarithm $S = \int d\widetilde{S}/\widetilde{S}_-$ is a local $Q$-martingale.

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- Assuming the existence of an equivalent martingale measure and a structure condition, we can write the semi-martingale decomposition of $S$ uniquely in the form

$$S = N + \int \lambda \, d\langle S \rangle$$

for a local martingale $N$ and a predictable process $\lambda$; here $\langle S \rangle$ denotes the predictable compensator of the quadratic variation process $[S]$. In our concrete market model, it is readily computed that

$$dN = b\,dW + c\,dM, \quad \lambda = \frac{a}{b^2 + \mu c^2}.$$

Moreover, we have

$$\begin{aligned} \int \lambda \, dN &= \int \frac{a}{b^2 + \mu c^2} \cdot (b\,dW + c\,dM), \\ \lambda \Delta N &= -\lambda c \Delta M. \end{aligned}$$

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- Its density is given by the Doléans-Dade stochastic exponential

$$\begin{aligned} \mathcal{E}\left(-\int \lambda \, dN\right)_T &= \exp\left(-\int_0^T \lambda_t \, b_t \, dW_t - \frac{1}{2}\int_0^T \lambda_t^2 b_t^2 \, dt\right) \\ &\quad \times \prod_{0<t\leq T} \left(1 - \lambda_t c_t \, \Delta M_t\right). \end{aligned}$$

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- As there is one single jump of $M$ with jump size one, the density of the minimal martingale measure gets negative with non-zero probability in case a jump occurs at $\tau \leq T$ and $\lambda_\tau c_\tau > 1$.
- Therefore, $\widehat{P}$ is in general a signed measure, and we conclude that the minimal martingale measure is not a good choice in our situation.

- To calculate the Esscher measure, we refer to *Kallsen & Shiryaev (2001)*, we write the price process as

$$\widetilde{S}_t = \exp\left(\int_0^t a_s ds + \int_0^t b_s dW_s + \int_0^t c_s dH_s\right) =: \exp\left(X_t\right).$$

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- For every strategy $\vartheta$ such that $\int \vartheta\, dX$ is exponentially special, we define the modified Laplace cumulant process $K^X(\vartheta)$ as exponential compensator of $\int \vartheta\, dX$. In particular,

$$Z^{\vartheta} = \exp\left(\int \vartheta\, dX - K^X(\vartheta)\right)$$

  is a local martingale with $Z_0^{\vartheta} = 1$. In case it is a UI martingale, it is the density process of a probability measure $P^{\vartheta}$.

- We have that $\exp(X)$ is a local $P^{\vartheta}$-martingale iff there is a solution $\vartheta^{\sharp}$ to
$$K^X(\vartheta+1) = K^X(\vartheta).$$
This translates into the equation
$$\begin{aligned} & a_t(\vartheta_t+1) + \frac{1}{2}b_t^2(\vartheta_t+1)^2 + \left(e^{(\vartheta_t+1)c_t}-1\right)\mu_t \\ = \ & a_t\vartheta_t + \frac{1}{2}b_t^2\vartheta_t^2 + \left(e^{\vartheta_t c_t}-1\right)\mu_t. \end{aligned}$$
By the boundedness of the co-efficients, a bounded solution always exists and gives a martingale measure by the above exponential tilting due to a verification result by *Lepingle & Mémin (1978)*.

- Minimal entropy martingale measure $Q^E$: Its density can always be written in the form

$$\frac{dQ^E}{dP} = \exp\left(c + \int_0^T \phi_t \, dS_t\right)$$

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- In general, a martingale measure $Q$ for $S$ has a density of the form

$$\frac{dQ}{dP} = \mathcal{E}\left(-\int \lambda \, dN + L\right)_T,$$

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where $L$, $L_0 = 0$, is a local martingale strongly orthogonal to $N$.

- According to martingale representation we can write $L$ as

$$dL = b_L \, dW + c_L \, dM$$

for some predictable processes $b_L$, $c_L$. The orthogonality relation $\langle N, L \rangle = 0$ yields

$$bb_L + cc_L \mu^2 = 0.$$

- Equating the two different representations, we obtain the optimal martingale measure equation

$$
\begin{aligned}
c \quad = \quad & \int_T \left( -\phi a - \frac{1}{2}\lambda^2 b^2 + \mu\lambda c - \frac{1}{2} b_L^2 - \mu c_L - \phi\mu c \right) dt \\
& + \int_T \left( -\phi b - \lambda b + b_L \right) dW \\
& + \log\left(1 - \left(\lambda_\tau c_\tau - c_L(\tau) - \phi_\tau c_\tau\right)\right) \times \mathbb{I}_{\tau \le T} .
\end{aligned}
$$

- Assuming for now formally that there exists a smooth function $u \equiv u(t, h)$ (where $t \in [0, T]$ and $h \in \{0, 1\}$) such that

$$
\begin{aligned}
\log\left(1-\left(\lambda_\tau c_\tau - c_L\left(\tau\right) - \phi_\tau c_\tau\right)\right) &= u\left(t, H_t\right) - u\left(t, H_{t-}\right), \\
u\left(T, h\right) &= 0, \quad h \in \left\{0, 1\right\},
\end{aligned}
$$

we write

$$
\begin{aligned}
\log\left(1-\left(\lambda_\tau c_\tau - c_L\left(\tau\right) - \phi_\tau c_\tau\right)\right) &= u\left(\tau, 1\right) - u\left(\tau, 0\right) \\
&= -\left\{u(T, 1) - u\left(\tau, 1\right) + u\left(\tau, 0\right) - \right. \\
&\quad -u\left(0, 0\right) \\
&= -\int_0^T \frac{\partial}{\partial t} u\left(t, H_t\right)\, dt - u\left(0, 0\right).
\end{aligned}
$$

- We work with the hypothesis that the integrals over the $dt$-terms as well as the $dW$-terms vanish at maturity $T$. This yields the two equations

$$\phi a + \frac{1}{2}\lambda^2 b^2 - \mu\lambda c + \frac{1}{2}b_L^2 + \mu c_L + \phi\mu c + \frac{\partial u}{\partial t} = 0,$$

$$\phi b + \lambda b - b_L = 0.$$

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- Inserting the orthogonality relation, we solve for $\phi$ as

$$\phi = -\lambda + \frac{c c_L \mu^2}{b^2}.$$

- Moreover, we get, with the notation $\Delta_t u := u(t, H_t) - u(t, H_{t-})$, that

$$\begin{aligned} c_L &= \exp(\Delta u + \phi c) + \lambda c - 1 \\ &= \exp\left(\Delta u - \lambda c + \frac{c^2\mu^2}{b^2} c_L\right) + \lambda c - 1. \end{aligned}$$

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- Introducing

$$g(t, u_t) := \left(\phi a + \frac{1}{2}\lambda^2 b^2 - \mu\lambda c + \frac{1}{2} b_L^2 + \mu c_L + \phi\mu c\right)_t,$$

we derive the system of two ordinary differential equations,

$$\frac{\partial u}{\partial t} + g(t, u_t) = 0,$$
$$u(T, 0) = u(T, 1) = 0.$$