CONTRACTING FOR OPTIMAL INVESTMENT WITH RISK CONTROL

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Overview

- Optimal investment under constraint on law of wealth
- First best contract
- Second best contract
- Robust contract
- Law-invariant coherent risk measures.

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Solution is easy:

$$U'(w_T) = \lambda\zeta_T$$

where λ is chosen to match the budget constraint.

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Of course, generating terminal wealth X will cost more than w_0; the principal wants to hire the agent for as little as possible, subject to the participation constraint of the agent:

$$EU_A(\varphi(X)) \geq \underline{u}$$

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First best is a rather unrealistic solution concept ...

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from the second best solution we have

$$X - \varphi(X) = (u')^{-1}\left(\frac{v'(X)}{\gamma}\right).$$

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$$\begin{aligned} \rho^\mu(X) &= -\int \psi(z)\Big\{\int_{1-F_\zeta(z)}^1 a^{-1}\ \mu(da)\Big\}\ F_\zeta(dz) \\ &= -E[\psi(\zeta)g_\mu(\zeta)] \end{aligned}$$

for some non-negative increasing g_μ.

The optimization problem.

$$\max_{\psi\downarrow} EU(\psi(\zeta_T)), \qquad w_0 = E[\zeta_T \psi(\zeta_T)], \quad E[\psi(\zeta_T) g_\mu(\zeta_T)] \geq b \quad \forall \mu \in \mathcal{M}$$

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$$\begin{aligned} L(\psi, z) &\equiv E\Big[U(\psi(\zeta)) + \lambda(w_0 - \zeta\psi(\zeta)) + \sum_{i=1}^n \alpha_i\{\psi(\zeta)g_i(\zeta) - b_i - z_i\}\Big] \\ &= E\Big[U(\psi(\zeta)) - \psi(\zeta)\{\lambda\zeta - \sum_{i=1}^n \alpha_i\, g_i(\zeta)\} - \alpha\cdot(z+b)\Big] + \lambda w_0. \end{aligned}$$

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Dual-feasibility: $\alpha \geq 0$,

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$$\max_{\psi\downarrow} EU(\psi(\zeta_T)), \qquad w_0 = E[\zeta_T \psi(\zeta_T)], \quad E[\psi(\zeta_T) g_\mu(\zeta_T)] \geq b \quad \forall \mu \in \mathcal{M}$$

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Complementary slackness: $\alpha \cdot z = 0$.

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$$\begin{aligned}\Psi &= \int_0^1 \{U(\tilde\psi(x)) - \tilde\psi(x)(\tilde h(x) + \eta'(x))\}\, dx + \int_0^1 \tilde\psi(x)\eta'(x)\, dx \\ &\le \int_0^1 \tilde U(\tilde h(x) + \eta'(x))\, dx + [\tilde\psi(x)\eta(x)]_0^1 - \int_0^1 \eta(x)\, d\tilde\psi(x).\end{aligned}$$

How does it look?

Take $\sigma = 0.35$, $\mu = 0.2$, $r = 0.05$, $T = 1$, $w_0 = 1$; $\quad \underline{u} = U_A(0.05)$ and

$$U(x) = f_R(x + a) - f_R(a), \quad f_R(x) = x^{1-R}/(1 - R),$$

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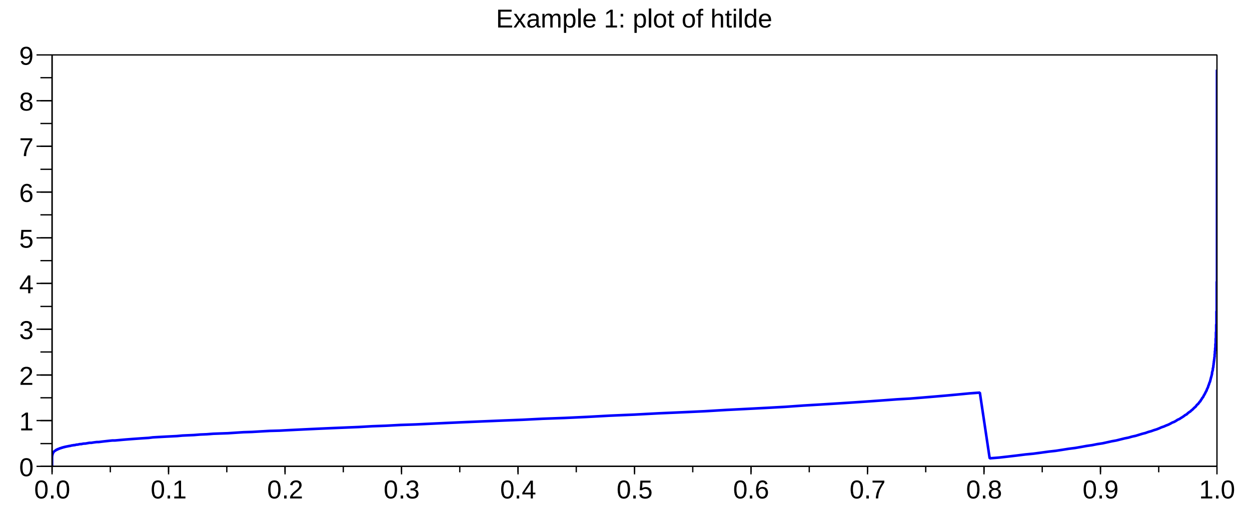
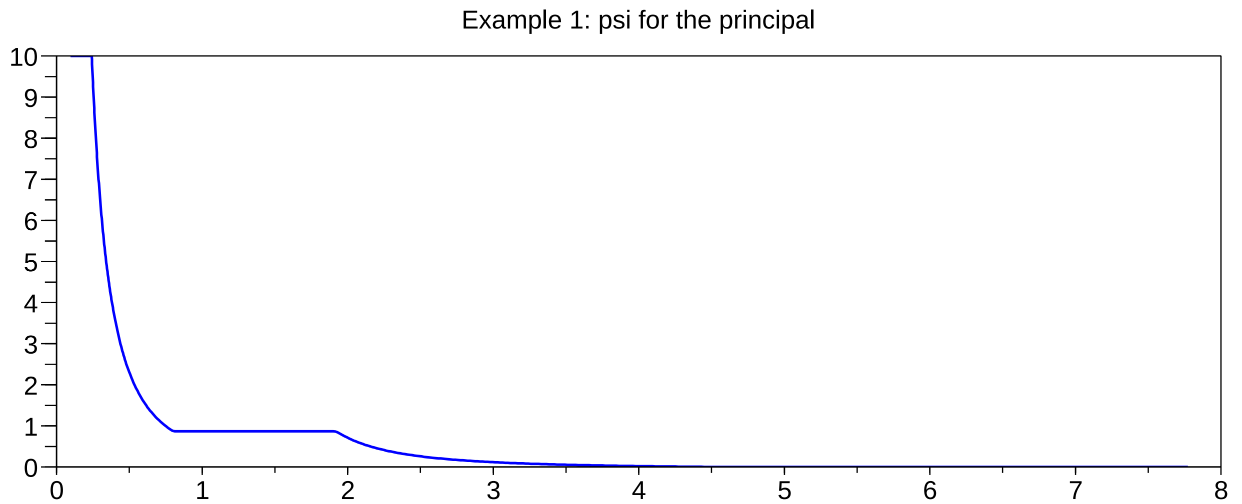
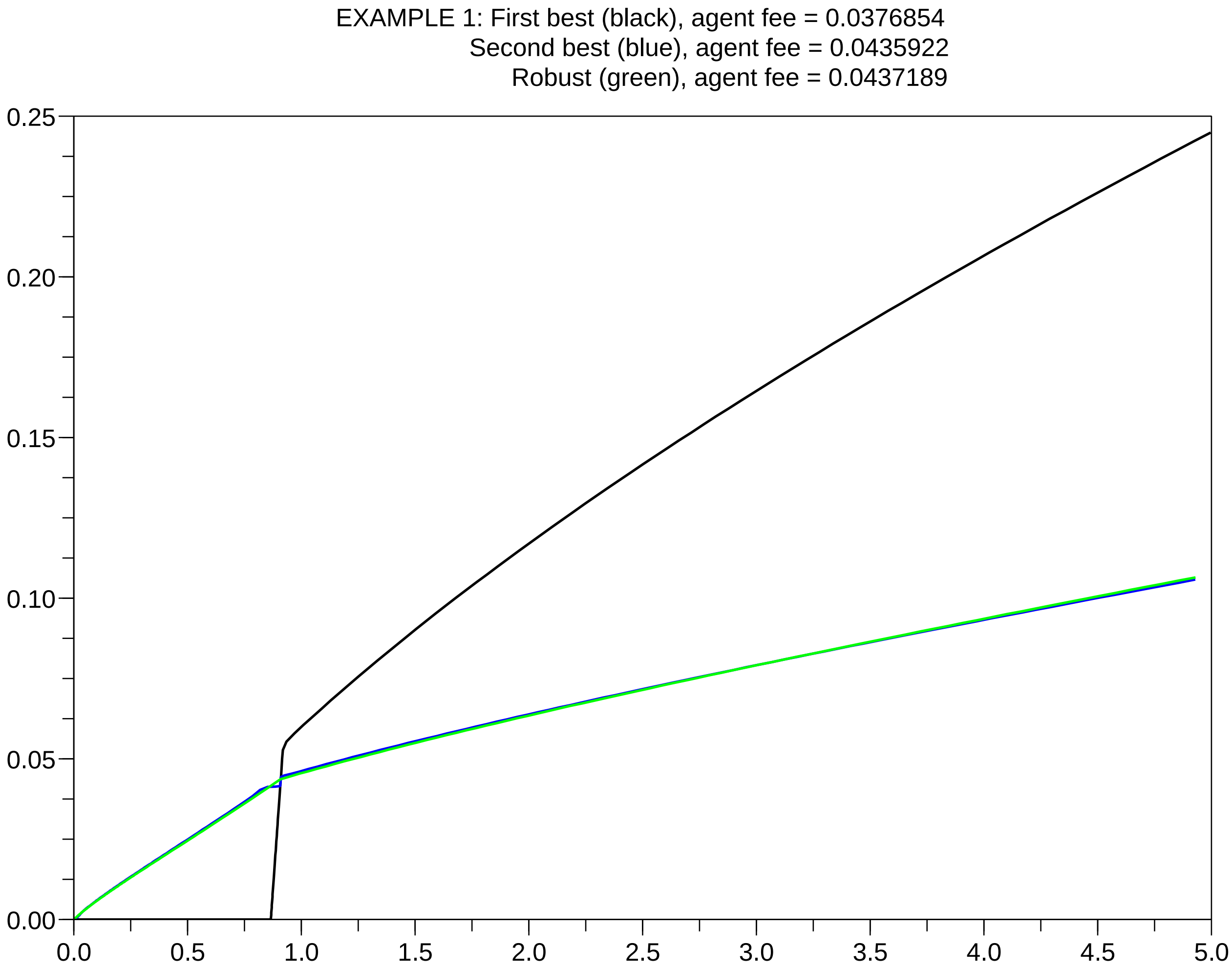
- **Example 5:** μ_1, μ_2 of the same form, $a_1 = 0.65$, $a_2 = 0.05$,and $b_1 = 1$, $b_2 = 0.3$. $\mu_3(dx) = \frac{3}{2}\sqrt{x}\,dx$, $b_3 = 0.525$.

Principal's solution, first example

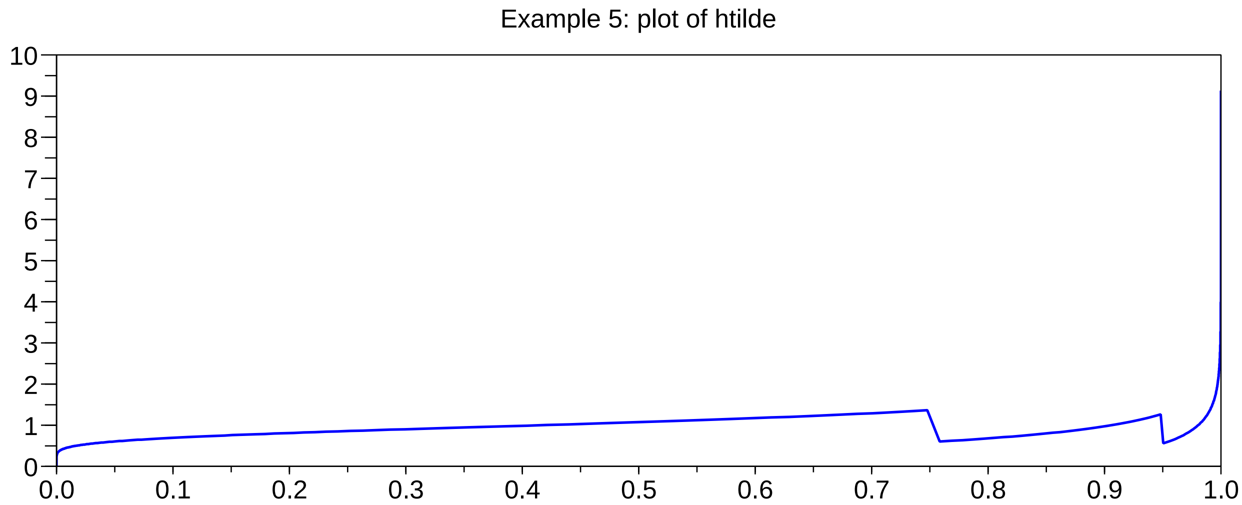


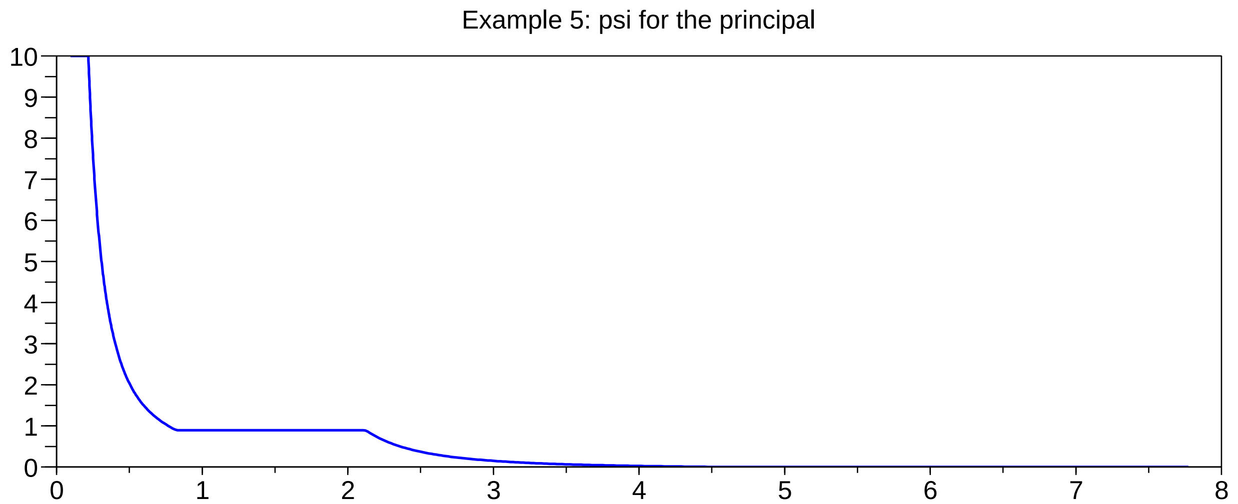


Contracts, first example



Principal´s solution, second example





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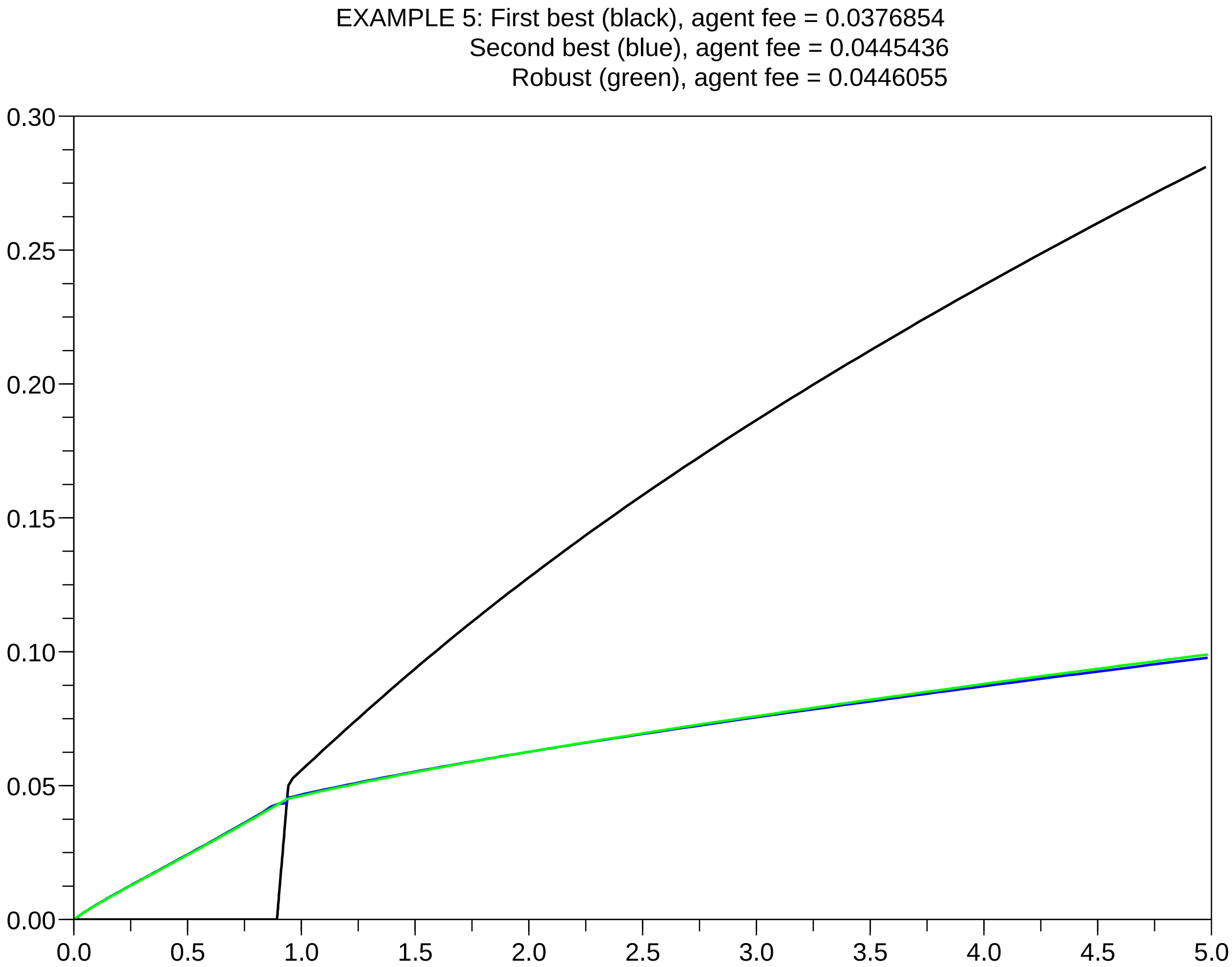

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