Convexity techniques for BSDEs from utility indifference valuation

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Istanbul Workshop on Mathematical Finance

18–21 May 2009

Istanbul, Turkey

20.05.2009

based on joint work with

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The basic problem

Setup

- **Starting point:** Backward stochastic differential equation **(BSDE)**

$$\Gamma_s = G - \int_s^T g(Z_r)\,dr + \int_s^T Z_r\,dB_r, \quad 0 \le s \le T$$

with fully **quadratic driver**

$$g(z) = \chi + (z+\alpha)'\Lambda^{-1}(z+\alpha).$$

- **Data:** final condition G and driver $g = (\Lambda, \alpha, \chi)$ with matrix Λ, vector α and scalar χ.
- **Solution:** pair (Γ, Z) with scalar Γ and vector Z; driving Brownian motion B is also vector.
- Everything happens on $(\Omega, \mathcal{F}, I\!\!F, P)$.

Questions

- **Main question:** What happens if
  - we change the **probability measure** P **?**
  - we change the **filtration** $I\!F$ **?**
  - we change the underlying **space** Ω **?**
- Can we somehow generate and/or exploit **symmetry?**
- **Idea:**
  - Basic BSDE gives exact description of problem and solution, but often hard to solve.
  - Can we replace it by **simpler BSDE ?**
  - Can we even find another BSDE with **explicit solution** $\tilde{\Gamma}$ . . . **?**
  - . . . and then **estimate** Γ in terms of $\tilde{\Gamma}$ **?**

Precise formulation

The BSDE

- **Basic BSDE** is

$$
\begin{aligned}
d\Gamma_s &= \big(\chi_s + (Z_s + \alpha_s)' \Lambda_s^{-1} (Z_s + \alpha_s)\big)\, ds - Z_s\, dB_s, \\
\Gamma_T &= G.
\end{aligned}
$$

- **Assumptions:**
  - G, Λ, α, χ are all **uniformly bounded**.
  - Λ has eigenvalues **uniformly bounded away from 0** and ∞.
- **Solution:** pair (Γ, Z) with
  - Γ bounded semimartingale.
  - Z integrand for Brownian motion B in $\mathbb{R}^n$.
- **Explicit formula:** if $\Lambda = c\, \mathrm{I}_{n \times n}$ and $\alpha \equiv 0$, $\chi \equiv 0$, then

$$
\Gamma_s = -c \log E\big[\exp(-G/c) \,\big|\, \mathcal{F}_s\big].
$$

Basic results

- **Basic BSDE** is

$$\begin{aligned} d\Gamma_s &= \big(\chi_s + (Z_s + \alpha_s)' \Lambda_s^{-1} (Z_s + \alpha_s)\big)\, ds - Z_s\, dB_s, \\ \Gamma_T &= G. \end{aligned}$$

- Basic BSDE has **unique solution** (Γ, Z).
- For any solution (Γ, Z) with Γ bounded, Z is in $BMO(B)$.
- **Key property:** The function $(A, z) \mapsto f(A, z) = z' A^{-1} z$ in the driver is (jointly) **convex**.
- Γ from solution (Γ, Z) of basic BSDE is **jointly concave** as a function of $(G, \Lambda, \alpha, \chi)$.
- $\longrightarrow$ **Kobylanski (2000), Mania/S (2005)**

Some motivation

PDE motivation

- Start with **PDE**

$$u_s + \frac{1}{2}\Delta u - f(\Lambda, \alpha - \nabla u) - \chi = 0, \quad u(T, x) = h(x).$$

- **Link to BSDE:** solution to BSDE is $\Gamma_. = u(\cdot, B_.)$, $Z_. = -\nabla u(\cdot, B_.)$.
- **Symmetrise:** $\tilde{\Lambda} := \frac{1}{n!} \sum_{O \in \mathrm{Perm}} O' \Lambda O$, $\tilde{\alpha} := \frac{1}{n!} \sum_{O \in \mathrm{Perm}} O' \alpha$,

$$\tilde{h} := \frac{1}{n!} \sum_{O \in \mathrm{Perm}} h \circ O.$$

- Symmetrised PDE then reads

$$\tilde{u}_s + \frac{1}{2}\Delta \tilde{u} - f(\tilde{\Lambda}, \tilde{\alpha} - \nabla \tilde{u}) - \chi = 0, \quad \tilde{u}(T, x) = \tilde{h}(x).$$

- **Comparison result:** $\tilde{u}(0,0) \geq u(0,0)$.

Argument

- Replace x by Ox and compute to get

$$f\big(\Lambda, \alpha - O\nabla u(s, Ox)\big) = f\big(O'\Lambda O, O'\alpha - \nabla u(s, Ox)\big).$$

- **Symmetrised function** $\bar{u}(s,x) := \frac{1}{n!} \sum_{O \in \mathrm{Perm}} u(s, Ox)$ solves

$$\bar{u}_s(s,x) + \frac{1}{2}\Delta\bar{u}(s,x) - \chi - \frac{1}{n!} \sum_{O \in \mathrm{Perm}} f\big(O'\Lambda O, O'\alpha - \nabla u(s, Ox)\big) = 0,$$

$$\bar{u}(T,x) = \tilde{h}(x).$$

- By joint **convexity** of f,

$$\frac{1}{n!} \sum_{O \in \mathrm{Perm}} f\big(O'\Lambda O, O'\alpha - \nabla u(s, Ox)\big) \geq f\big(\tilde{\Lambda}, \tilde{\alpha} - \nabla\bar{u}(s,x)\big).$$

- So $\tilde{u}(0,0) \geq \bar{u}(0,0) = u(0,0)$.

Changing basics of BSDEs

Changing the measure

Setup and notations

- **Key idea:** parametric family of measures via **Girsanov**.
- **Parameters** for measure change are from suitable space $\mathcal{K}$ of processes κ (integrability conditions):
  - $dP^\kappa = \mathcal{E}\left(-\int \kappa \, dB\right) \, dP$.
  - $B^\kappa = B + \int \kappa_s \, ds$ is P^κ-Brownian motion.
  - Auxiliary variables

$$G^\kappa := G - \int_0^T \left(\chi_s + \tfrac{1}{2}\kappa_s' \Lambda_s \kappa_s\right) ds - \int_0^T \left(\alpha_s + \Lambda_s \kappa_s\right) dB_s$$
$$= G - \int_0^T \chi_s \, ds - \int_0^T \alpha_s \, dB_s - \int_0^T \kappa_s \, dB_s - \frac{1}{2}\int_0^T \kappa_s' \Lambda_s \kappa_s \, ds.$$

  - **Important quantities:** $\delta^{\max} := \sup_{0 \le s \le T} \| \max \operatorname{spec}(\Lambda_s) \|_\infty$,

$$\delta^{\min} := \inf_{0 \le s \le T} \frac{1}{\| \max \operatorname{spec}(\Lambda_s^{-1}) \|_\infty}.$$

Varying the measure

- **Theorem 1:**

$$\begin{aligned}\Gamma_t &= -\operatorname*{ess\,sup}_{\kappa\in\mathcal{K}} \log E_{P^\kappa}[\exp(-G^\kappa_t/\delta^{\max}_t)\,|\,\mathcal{F}_t]^{\delta^{\max}_t} \\ &= -\operatorname*{ess\,inf}_{\kappa\in\mathcal{K}} \log E_{P^\kappa}[\exp(-G^\kappa_t/\delta^{\min}_t)\,|\,\mathcal{F}_t]^{\delta^{\min}_t},\end{aligned}$$

  and for every $\kappa \in \mathcal{K}$,

$$\Gamma_t = \log E_{P^\kappa}[\exp(-G^\kappa_t/\delta)\,|\,\mathcal{F}_t]^{\delta}\Big|_{\delta=\delta^{\kappa,G}_t}$$

  for some $\mathcal{F}_t$-measurable $\delta^{\kappa,G}_t$ with values in $[\delta^{\min}_t, \delta^{\max}_t]$.

- **Comments:**
  - Quasi-explicit **structural formula** via interpolation.
  - **Upper and lower bounds** for Γ_t via choices of κ.
  - Key quantities are **extremal eigenvalues** of Λ.

Outline of proof 1

- Explicitly, $\Gamma^{\delta,\kappa} := -\log E_{P^\kappa}[\exp(-\cdot/\delta)\,|\,\mathbb{F}]^\delta$ has dynamics

$$d\Gamma^{\delta,\kappa} = \frac{1}{2\delta}|Z^{\delta,\kappa}|^2\,dt - Z^{\delta,\kappa}\,dB^\kappa.$$

- Under P^κ, Γ has dynamics

$$\begin{aligned} d\Gamma &= \tfrac{1}{2}(Z+\alpha+\Lambda\kappa)'\Lambda^{-1}(Z+\alpha+\Lambda\kappa)\,dt \\ &\quad +(\chi-\kappa'\alpha-\tfrac{1}{2}\kappa'\Lambda\kappa)\,dt - Z\,dB^\kappa, \end{aligned}$$

and first dt-term is, by **estimating eigenvalues**,

$$\geq \frac{1}{2\delta^{\max}}|Z+\alpha+\Lambda\kappa|^2\,dt.$$

- Rewrite

$$\begin{aligned} -Z\,dB^\kappa &= -(Z+\alpha+\Lambda\kappa)\,dB^\kappa + (\alpha+\Lambda\kappa)\,dB^\kappa \\ &= -(Z+\alpha+\Lambda\kappa)\,dB^\kappa + (\alpha+\Lambda\kappa)\,dB \\ &\quad +(\alpha'\kappa+\kappa'\Lambda\kappa)\,dt. \end{aligned}$$

Outline of proof 2

- Put things together to obtain

$$d\Gamma \geq \frac{1}{2\delta^{\max}}|Z+\alpha+\Lambda\kappa|^2\,dt-(Z+\alpha+\Lambda\kappa)\,dB^\kappa + (\chi+\tfrac{1}{2}\kappa'\Lambda\kappa)\,dt+(\alpha+\Lambda\kappa)\,dB.$$

- Recall $d\Gamma^{\delta,\kappa}=\frac{1}{2\delta}|Z^{\delta,\kappa}|^2\,dt-Z^{\delta,\kappa}\,dB^\kappa$.
- Final value of original Γ is $\Gamma_T=G$. Using for BSDE inequality instead final value

$$G-\int_t^T\big((\chi_s+\tfrac{1}{2}\kappa_s'\Lambda_s\kappa_s)\,ds+(\alpha_s+\Lambda_s\kappa_s)\,dB_s\big)=G_t^\kappa$$

therefore gives $\Gamma_t\leq\Gamma_t^{\delta_t^{\max},\kappa}$ from **BSDE comparison**.
- Equality for choice of κ with $Z+\alpha+\Lambda\kappa=0$.

Changing the filtration

Setup and notations

- **Key idea:** **split space** into subspaces with $n = \overline{n} + \underline{n}$. Then **project** on upper half.
- **Brownian motion** $B = (\overline{B}, \underline{B})$ in $\mathbb{R}^n = \mathbb{R}^{\overline{n}} \times \mathbb{R}^{\underline{n}}$.
- **Filtrations** $\mathbb{F} = \mathbb{F}^B$, $\overline{\mathbb{F}} = \mathbb{F}^{\overline{B}}$, $\underline{\mathbb{F}} = \mathbb{F}^{\underline{B}}$.
- **Generators** $g, \overline{g}$ with $f(A, z) = z' A^{-1} z$ and $\overline{f}(\overline{A}, \overline{z}) = \overline{z}' \overline{A}^{-1} \overline{z}$, where $\overline{A}$ is the upper left $\overline{n} \times \overline{n}$ corner of A.
- **Final values** G and $\overline{G} = E[G \,|\, \overline{\mathcal{F}}_T]$.
- **BSDEs**

$$d\Gamma_s = \big(\chi_s + f(\Lambda_s, Z_s + \alpha_s)\big)\, ds - Z_s\, dB_s, \quad \Gamma_T = G$$

and

$$d\check{\Gamma}_s = \big(\chi_s^o + \overline{f}(\overline{\Lambda}_s^o, \check{Z}_s + \overline{\alpha}_s^o)\big)\, ds - \check{Z}_s\, d\overline{B}_s, \quad \check{\Gamma}_T = \overline{G}.$$

- Superscript ^{o} for **optional projection** under P on $\overline{\mathbb{F}}$.

Shrinking the filtration

- **Theorem 2:** For the solutions (Γ, Z) and $(\check{\Gamma}, \check{Z})$ of the BSDEs

$$d\Gamma_s = \big(\chi_s + f(\Lambda_s, Z_s + \alpha_s)\big)\, ds - Z_s\, dB_s, \quad \Gamma_T = G$$

and

$$d\check{\Gamma}_s = \big(\chi^o_s + \overline{f}(\overline{\Lambda}^o_s, \check{Z}_s + \overline{\alpha}^o_s)\big)\, ds - \check{Z}_s\, d\overline{B}_s, \quad \check{\Gamma}_T = \overline{G}.$$

we have

$$\Gamma^o \le \check{\Gamma}.$$

- **Comments:**
    - **Jensen-type inequality** for our class of BSDEs.
    - In simplest case, $\Gamma_0 = -c \log E[\exp(-G/c)]$ and $\check{\Gamma}_0 = -c \log E[\exp(-\overline{G}/c)]$, with $\overline{G} = E[G \,|\, \overline{\mathcal{F}}_T]$.

Outline of proof

- (Γ, Z) solves BSDE

$$d\Gamma_s = \big(\chi_s + f(\Lambda_s, Z_s + \alpha_s)\big)\, ds - Z_s\, dB_s, \quad \Gamma_T = G.$$

- Projecting on filtration $\bar{\mathbb{F}}$ gives

$$d\Gamma^o_s = \big(\chi^o_s + f(\Lambda_s, Z_s + \alpha_s)\big)^o\, ds - Z^o_s\, d\overline{B}_s, \quad \Gamma^o_T = \overline{G}.$$

- **Convexity of f** yields

$$\big(f(\Lambda_s, Z_s + \alpha_s)\big)^o \geq f(\Lambda^o_s, Z^o_s + \alpha^o_s),$$

and $f(A, z) \geq \overline{f}(\overline{A}, \overline{z})$.

- $(\check{\Gamma}, \check{Z})$ solves BSDE

$$d\check{\Gamma}_s = \big(\chi^o_s + \overline{f}(\overline{\Lambda}^o_s, \check{Z}_s + \overline{\alpha}^o_s)\big)\, ds - \check{Z}_s\, d\overline{B}_s, \quad \check{\Gamma}_T = \overline{G},$$

- Result follows from **BSDE comparison theorem**.

Changing the underlying space

Setup and notations

- **Key idea:** normal distribution is **rotation-invariant**; so can transform Brownian increments without doing harm.
- $\Omega = C([0, T]; \mathbb{R}^n)$ Wiener space with Wiener measure P.
- For $u \in \mathbf{O}(n)$ and fixed t, define transformation on Ω by

$$U_t(g)(s) := \begin{cases} \omega(s) & \text{for } s \leq t, \\ \omega(t) + u\big(g\omega(s) - \omega(t)\big) & \text{for } s > t. \end{cases}$$

  (**rotate increments** after time t by u).
- **Key point:** Wiener measure and transformation U_t **commute**; so
  - $B^u := U_t \circ B$ is again Brownian motion.
  - $B \circ U_t = U_t \circ B$.
  - $\int (Z \circ U_t)\, dB^u = (\int Z\, dB) \circ U_t$, i.e., stochastic integrals also "commute" with U_t.

Transforming Wiener space

- **Theorem 3:** If general BSDE

$$\Gamma_s = G - \int_s^T F_r(\Gamma_r, Z_r)\,dr + \int_s^T Z_r\,dB_r$$

has unique solution (Γ, Z), then **transformation** $(\Gamma \circ U_t, Z \circ U_t)$ is unique **solution of transformed BSDE**

$$\tilde{\Gamma}_s = G \circ U_t - \int_s^T (F \circ U_t)_r(\tilde{\Gamma}_r, \tilde{Z}_r)\,dr + \int_s^T \tilde{Z}_r\,d(B \circ U_t)_r.$$

In particular: Transformed solution $(\Gamma \circ U_t, Z \circ U_t)$ agrees with (Γ, Z) on $[\![0, t]\!]$.

Averaging

- Back to our **basic BSDE**

$$d\Gamma_s = \big(\chi_s + f(\Lambda_s, Z_s + \alpha_s)\big)\, ds - Z_s\, dB_s, \quad \Gamma_T = G.$$

- For finite subset $\mathcal{O}$ of $\mathbf{O}(n)$, **average parameters**
$G^{\mathcal{O}} := \frac{1}{|\mathcal{O}|} \sum_{u \in \mathcal{O}} G \circ U_t$, $\Lambda^{\mathcal{O}} := \frac{1}{|\mathcal{O}|} \sum_{u \in \mathcal{O}} u'(\Lambda \circ U_t) u$,
$\alpha^{\mathcal{O}} := \frac{1}{|\mathcal{O}|} \sum_{u \in \mathcal{O}} u'(\alpha \circ U_t)$, $\chi^{\mathcal{O}} := \frac{1}{|\mathcal{O}|} \sum_{u \in \mathcal{O}} \chi \circ U_t$.
- **Corollary:** For solution $(\Gamma^{\mathcal{O}}, Z^{\mathcal{O}})$ of BSDE

$$d\tilde{\Gamma}_s = \big(\chi^{\mathcal{O}}_s + f(\Lambda^{\mathcal{O}}_s, \tilde{Z}_s + \alpha^{\mathcal{O}}_s)\big)\, ds - \tilde{Z}_s\, dB_s, \quad \tilde{\Gamma}_T = G^{\mathcal{O}},$$

we then have

$$\Gamma_t \le \Gamma^{\mathcal{O}}_t.$$

Symmetrising 1

- **Main messages and ideas:**
  - **Averaging convex BSDEs increases solutions**, because generator decreases by convexity.
  - So averaging gives upper bounds — but how to choose good set $\mathcal{O}$ for averaging?
  - Upper bound on Γ increases with **maximal eigenvalue** of Λ.
  - So: **reduce maximal eigenvalue** by making Λ maximally **symmetric** ...
  - ... hence choose for $\mathcal{O}$ the permutation group Perm.
- Notations: $G^{\mathrm{sym}} := \frac{1}{n!} \sum_{u \in \mathrm{Perm}} G \circ U_t,$

$$\alpha^{\mathrm{sym}} := \frac{1}{n!} \sum_{u \in \mathrm{Perm}} u'(\alpha \circ U_t),\ \chi^{\mathrm{sym}} := \frac{1}{n!} \sum_{u \in \mathrm{Perm}} \chi \circ U_t.$$

Symmetrising 2

- **Corollary:** Assume Λ is diagonal and set

$$d_t := \left\| \frac{1}{n} \sum_{j=1}^{n} \max_{u \in \mathrm{Perm}} (\Lambda_s^{jj} \circ U_t) \right\|_\infty .$$

Then (**upper bound from symmetrised setting**)

$$\Gamma_t \leq -d_t \log E\left[\exp\left(-G^{\mathrm{sym}} + \int_t^T \alpha_s^{\mathrm{sym}}\, dB_s + \int_t^T \chi_s^{\mathrm{sym}}\, ds \right)^{1/d_t} \middle| \mathcal{F}_t \right].$$

- **Idea for proof:** Combine averaging result for $\mathcal{O} = \mathrm{Perm}$ with measure change result for $\kappa \equiv 0$ and use that RHS above is (by Jensen) increasing in the argument d_t.

Generalisations

- Results can be **generalised** to some degree:
  - **Can** replace boundedness of G, α, χ by appropriate **exponential moment conditions**.
  - **Can** relax assumption that eigenvalues of Λ are bounded away from infinity.
  - **Cannot** relax assumption that eigenvalues of Λ are uniformly bounded away from 0.

- **Hard question:** How about other (more general) generators **?**

Towards finance applications

Exponential utility indifference valuation

- **Given:** model S for discounted asset prices, class Θ of allowed strategies, discounted payoff G.
- **Wanted:** valuation for G.
- **Incomplete market:** value depends on subjective preferences.
- Use exponential utility function $U(x) = -\exp(-\gamma x)$.
- **Utility indifference value** b_t for buying G at time t: implicitly defined by

$$\operatorname*{ess\,sup}_{\theta\in\Theta} E\left[U\left(x_t + \int_t^T \theta_u \, dS_u \right) \middle| \mathcal{F}_t \right]$$

$$= \operatorname*{ess\,sup}_{\theta\in\Theta} E\left[U\left(x_t - b_t + \int_t^T \theta_u \, dS_u + G \right) \middle| \mathcal{F}_t \right].$$

- Indifference, in terms of expected utility under optimal investment, between buying or not buying payoff.

Specific model

- Independent BMs W and $W^{\perp}$ valued in $\mathbb{R}^m$ and $\mathbb{R}^n$.
- **Correlated** Brownian motion

$$dY_s = R_s\, dW_s + \sqrt{\mathrm{I}_{n\times n} - R_s R_s'}\, dW_s^{\perp}.$$

- Eigenvalues of RR' uniformly bounded away from 1.
- "Correlation matrix" $\Lambda = \frac{1}{\gamma}(\mathrm{I}_{n\times n} - RR')^{-1}$ has $\delta_t^{\min} \geq \frac{1}{\gamma}$.
- **Model for S** ($\mathbb{R}^m$-valued) is

$$dS_s^j = S_s^j \mu_s^j\, ds + \sum_{k=1}^{m} S_s^j \sigma_s^{jk}\, dW_s^k.$$

- Filtration $(\mathbb{G})$ generated by $W, W^{\perp}$.
- Notation: Sharpe ratio is $\lambda := \sigma^{-1}\mu$.
- Everything nicely bounded ...

(Wrong) BSDE for indifference value

- Well-known result: **maximal expected utility**

$$V_t^G := \operatorname*{ess\,sup}_{\pi\in\Pi} E\Big[-\exp\Big(-\gamma \int_t^T \pi_u' \sigma_u \, dW_u - \gamma G \Big) \Big| \mathcal{G}_t \Big]$$

satisfies $V_t^G = -\exp(-\gamma \check{\Gamma}_t)$ with **quadratic BSDE**

$$\check{\Gamma}_s = G - \int_s^T \Big(\tfrac{\gamma}{2} |\check{Z}_r|^2 - \hat{Z}_r' \lambda_r - \tfrac{1}{2\gamma} |\lambda_r|^2 \Big) \, dr + \int_s^T \hat{Z}_r \, dW_r + \int_s^T \check{Z}_r \, dW_r^{\perp}.$$

- $\longrightarrow$ **Hu/Imkeller/Müller (2005)**
- Unique solution, nice properties, . . . —
- — **but** BSDE does not have general form we need **!**
- **Problem:** Only second part $\check{Z}$ of full vector $Z = (\hat{Z}, \check{Z})$ appears in fully quadratic form.

Making our results applicable 1

- **First approach:** replace BSDE by

$$\check{\Gamma}^{\epsilon}_s = G - \int_s^T \left(\tfrac{\gamma}{2}|\check{Z}^{\epsilon}_r|^2 + \epsilon|\hat{Z}^{\epsilon}_r| - (\hat{Z}^{\epsilon})'_r\lambda_r - \tfrac{1}{2\gamma}|\lambda_r|^2\right) dr + \int_s^T \hat{Z}^{\epsilon}_r \, dW_r + \int_s^T \check{Z}^{\epsilon}_r \, dW^{\perp}_r$$

(note that missing fully quadratic term $\epsilon|\hat{Z}|^2$ has been added).

- **Theorem 4:** Our desired solution $\check{\Gamma}$ is given by

$$\check{\Gamma}_t = -\operatorname*{ess\,inf}_{\epsilon\in(0,1]} \operatorname*{ess\,inf}_{\kappa\in\mathcal{K}^{(m)}} \log E_{Q^{\kappa}}[\exp(-\gamma G^{\kappa,\epsilon}_t) \,|\, \mathcal{G}_t]^{1/\gamma}.$$

- **Usefulness:** Gives **lower bounds** for $\check{\Gamma}$ and hence for V^G by choosing suitable ϵ and κ.

Making our results applicable 2

- **Special case:** $\kappa = \lambda$ gives $Q^\kappa = \hat{P}$ (the **minimal martingale measure**) and

$$\check{\Gamma}_t \geq -\log E_{\hat{P}}\left[\exp\left(-\gamma G - \frac{1}{2}\int_t^T |\lambda_r|^2\,dr\right)\Bigg|\,\mathcal{G}_t\right]$$

- $\longrightarrow$ **Zariphopoulou (2001), ...**
- Remark: Can use projection results to get **upper bounds** for V^G.

Nontradable payoffs and symmetrisation 1

- **Assume** that
  - G is $\mathcal{F}_T^Y$-measurable
  - $\lambda = \sigma^{-1}\mu$ is $\mathbb{F}^Y$-predictable
  - R is $\mathbb{F}^Y$-predictable
- **Interpretation:**
  - Think of Y as **nontradable factor** (process).
  - Payoff G only depends on factor (e.g. variance swap).
  - Sharpe ratio only depends on factor (e.g. stochastic volatility model).
  - Note that S and Y can have **stochastic correlation** via R.
  - Correlations only depend on factor.
- **Comment:** Looks restrictive — but most literature has all parameters nonrandom and constant in time!

Nontradable payoffs and symmetrisation 2

- **Key point** of assumptions: obtain $V^G = -\exp(-\gamma\Gamma)$ with

$$\Gamma_s = G - \int_s^T \left(\tfrac{1}{2} Z_r' \Lambda_r^{-1} Z_r - Z_r' R_r \lambda_r - \tfrac{1}{2\gamma}|\lambda_r|^2\right) dr + \int_s^T Z_r \, dY_r.$$

- **Consequences:**
  - **Technically:** Above BSDE falls now into our general framework, so can use all results and techniques.
  - **Economically:** Everything can be expressed in factor filtration.
- In particular: can use symmetrisation estimates.
- **Limitations:** needs technical conditions; competing impacts.

Some references

Some references

- Barrieu/El Karoui (2009)
- Briand/Hu (2008)
- Frei/S (2008a,b)
- Hu/Imkeller/Müller (2005)
- Kobylanski (2000)
- Leung/Sircar (2008)
- Mania/S (2005)
- Morlais (2007)
- Musiela/Zariphopoulou (2004)
- Rouge/El Karoui (2000)
- Zariphopoulou (2001)
- . . . (other missing links)

The end (for now . . .)

Thank you for your attention !

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