Dual Formulation of Second Order Target Problems

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Outline

1. Introduction
2. BSDEs under singular probability measures
3. Second Order Target Problems and Duality
   - an alternative Formulation
   - Relaxations
   - Weak version of the Second Order Target Problem

Backward SDEs

Pardoux and Peng (1990, 1992) : W BM on $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{F} = \mathbb{F}^W$

- For $\xi \in \mathbb{L}^2$, $H_t(y,z)$ Lipschitz in (y,z), $H_.(0,0) \in \mathbb{H}^2$ the BSDE

$$Y_t = -H_t(Y_t, Z_t)dt + Z_t dW_t, \qquad Y_1 = \xi$$

has a unique solution $(Y, Z) \in \mathcal{S}^2 \times \mathbb{H}^2$

- Moreover if $H_t(y,z) = h(t, X_t, y, z)$ and $\xi = g(X_1)$, where

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t$$

Then $Y_t = V(t, X_t)$ for some deterministic measurable function V

- V is a viscosity solution of the semilinear PDE

$$\partial_t V + \frac{1}{2}\sigma^2 D^2 V + bDV + h(t, x, V, \sigma DV) = 0, \quad V(1,x) = g(x).$$

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Forward-Backward SDEs

• Antonelli (1993), Ma-Protter-Yong (1994), Hu-Peng (1995), Yong (1997), Peng-Wu (1999), Pardoux-Tang (1999), Delarue (2002), Z. (2006), $\cdots$

$$\begin{aligned} X_t &= x + \int_0^t b(s, X_s, Y_s, Z_s)ds + \int_0^t \sigma(s, X_s, Y_s, Z_s)dW_s, \\ Y_t &= g(X_T) + \int_t^T h(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s \end{aligned}$$

• Together with other conditions, if the coefficients are deterministic and $\sigma = \sigma(t, x, y)$, then $Y_t = V(t, X_t)$ where V is a viscosity solution of the quasi-linear PDE

$$\begin{aligned} &\partial_t V + \frac{1}{2}\sigma^2(t, x, V)D^2 V \\ &\quad + b(t, x, V, \sigma DV)DV + h(t, x, V, \sigma DV) = 0, \end{aligned}$$

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Second Order Backward SDEs

Cheridito, Soner, Touzi and Victoir (2007) :

• 2BSDE :

$$dY_t \;=\; H_t(Y_t, Z_t, \Gamma_t)dt + Z_t \circ dW_t, \qquad Y_1 = \xi \qquad (1)$$

where

$$Z_t \circ dW_t = Z_t dW_t + \frac{1}{2} d\langle Z, W\rangle_t = Z_t dW_t + \frac{1}{2}\Gamma_t dt$$

is the Fisk-Stratonovich stochastic integration.

• If $H_t = h(t, W_t, Y_t, Z_t, \Gamma_t)$ and $\xi = g(W_1)$, then $Y_t = V(t, W_t)$, where V is associated with the fully nonlinear PDE :

$$\partial_t V + h(t, x, V, DV, D^2 V) = 0 \quad \text{and} \quad V(1, x) = g(x). \qquad (2)$$

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Motivation from Probabilistic Numerical Methods

Bally and Pagès (2003), Z. (2004), Bouchard and Touzi (2004), Gobet, Lemor and Warin (2005), $\cdots$, Fahim, Touzi and Warin (2009)

$$
\begin{aligned}
\hat{Y}^n_{t_n} &= g\left(X^n_{t_n}\right) \text{ and for } 1 \leq i \leq n: \\
\hat{Y}^n_{t_{i-1}} &= \hat{\boldsymbol{E}}^n_{i-1}\left[\hat{Y}^n_{t_i}\right] + \Delta t_i \, f\left(t_i, X^n_{t_{i-1}}, \hat{Y}^n_{t_{i-1}}, \hat{Z}^n_{t_{i-1}}, \hat{\Gamma}^n_{t_{i-1}}\right) \\
\hat{Z}^n_{t_{i-1}} &= \hat{\boldsymbol{E}}^n_{i-1}\left[\hat{Y}^n_{t_i} \frac{\Delta W_{t_i}}{\Delta t_i}\right] \\
\hat{\Gamma}^n_{t_{i-1}} &= \hat{\boldsymbol{E}}^n_{i-1}\left[\hat{Y}^n_{t_i} \frac{|\Delta W_{t_i}|^2 - \Delta t_i}{|\Delta t_i|^2}\right]
\end{aligned}
$$

More motivations

Motivation from finance

- Hedging under Gamma constraints (Soner and Touzi 1999, Cheridito, Soner and Touzi 2005)
- Hedging under liquidity cost in the Cetin-Jarrow-Protter model (Cetin, Sonet and Touzi 2006)
- Uncertain volatility models (Denis and Martini 2006)

Peng's $G-$expectation

Wellposedness of 2BSDEs

Cheridito, Soner, Touzi and Victoir (2007) : In Markov case only

- Existence : if the PDE (2) has a smooth solution, then

$$Y_t = V(t, W_t), \quad Z_t = DV(t, W_t), \quad \Gamma_t = D^2 V(t, W_t).$$

- Uniqueness : Second Order Stochastic Target Problem

$$V(t,x) \quad := \quad \inf \left\{ y : \; Y_1^{y,Z} \geq g(W_1) \text{ for some } Z \in \mathcal{Z} \right\}$$

Under certain conditions, in particular if the comparison principle for viscosity solution of PDE (2) holds, then V is the viscosity solution of the PDE (2). Consequently, 2BSDE (1) has a unique solution in class $\mathcal{Z}$.

The admissibility set $\mathcal{Z}$ in CSTV

Definition $Z \in \mathcal{Z}$ if it is of the form

$$Z_t = \sum_{n=0}^{N-1} z_n \mathbf{1}_{\{t<\tau_{n+1}\}} + \int_0^t \alpha_s ds + \int_0^t \Gamma_s dW_s$$

- (τ_n) is an $\nearrow$ seq. of stop. times, z_n are $\mathcal{F}_{\tau_n}-$measurable, $\|N\|_\infty < \infty$
- Z_t and Γ_t are $\mathbb{L}^\infty-$bounded up to some polynomial of X_t
- $\Gamma_t = \Gamma_0 + \int_0^t a_s ds + \int_0^t \xi_s dW_s$, $0 \le t \le T$, and

$$\|\alpha\|_{B,b} + \|a\|_{B,b} + \|\xi\|_{B,2} < \infty, \qquad \|\phi\|_{B,b} := \left\| \sup_{0\le t\le T} \frac{|\phi_r|}{1+X_t^B} \right\|_{\mathbb{L}^b}$$

Uniqueness in larger class

"Theorem" If the following linear 2BSDE with constant coefficients has only zero solution in $\mathbb{L}^2$, then, under very mild conditions, uniqueness holds for the general non-Markovian 2BSDE (1) in essentially $\mathbb{L}^2$ space :

$$dY_t = -c\Gamma_t dt + Z_t \circ dW_t, \qquad Y_1 = 0. \tag{3}$$

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• Unfortunately, unless $c = \frac{1}{2}$, the 2BSDE (3) has nonzero solutions in $\mathbb{L}^2$!

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Motivation

Consider BSDE : $Y_t = g(W_1) + \int_t^1 h(s, W_s, Y_s, Z_s)ds - \int_t^1 Z_s dW_s$
and PDE : $\partial_t V + \frac{1}{2}D^2 V + h(t, x, V, DV) = 0, \quad V(1, x) = g(x)$

- If $h(t, x, y, z) = \sup_{u \in U}[uz - f(t, x, y, u)]$, then $V = \sup_u V^u$, where

$$\partial_t V^u + \frac{1}{2}D^2 V^u + uDV^u - f(t, x, V^u, u) = 0, \quad V^u(1, x) = g(x)$$

- Consequently, $Y_0 = \sup Y_0^u$ where

$$\begin{aligned} X_t^u &= \int_0^t u_s ds + W_t; \\ Y_t^u &= g(X_1^u) - \int_t^1 f(s, X_s^u, Y_s^u, u_s)ds - \int_t^1 Z_s^u dW_s. \end{aligned}$$

- Drift control, and $P^u := P \circ (X^u)^{-1}$ is equivalent to P.

Motivation (cont)

Consider
2BSDE : $Y_t = g(W_1) + \int_t^1 h(s, W_s, Y_s, Z_s, \Gamma_s) ds - \int_t^1 Z_s \circ dW_s$
and PDE : $\partial_t V + h(t, x, V, DV, D^2 V) = 0, \quad V(1, x) = g(x)$

- If $h(t, x, y, z, \gamma) = \sup_{a \in A} [\frac{1}{2} a\gamma - f(t, x, y, z, a)]$, then $V = \sup_a V^a$,

where
$\partial_t V^a + \frac{1}{2} a D^2 V^a - f(t, x, V^a, DV^a, a) = 0, \quad V^a(1, x) = g(x)$

- Consequently, $Y_0 = \sup_a Y_0^a$, where
  $X_t^a = \int_0^t a_s^{1/2} dW_s$;
  $Y_t^a = g(X_1^a) - \int_t^1 f(s, X_s^a, Y_s^a, Z_s^a, a_s) ds - \int_t^1 Z_s^a dX_s^a$.

- Volatility control, and $P^a := P \circ (X^a)^{-1}$ are mutually singular for different a.

General Framework

- $\Omega := C([0,1])$, B the canonical process, $\mathbb{P}_0$ the Wiener measure, $\mathbb{F} := \{\mathcal{F}_t\}_{0\le t\le 1}$ the filtration generated by B
- $\mathbb{F}^+ := \{\mathcal{F}_{t+}\}_{0\le t\le 1}$
- $\overline{\mathcal{A}}$: set of all $\mathbb{F}-$adapted process a satisfying

$$\underline{a} \le a_t(\omega) \le \bar{a}, \quad dt \times d\mathbb{P}_0 - \text{a.s.} \quad \text{for some} \quad \bar{a} \ge \underline{a} > 0$$

- $\mathbb{P}^a := \mathbb{P}_0 \circ (X^a)^{-1}$, measure induced by X^a :

$$X_t^a := \int_0^t a_s^{1/2} dB_s, \; 0 \le t \le 1, \qquad \mathbb{P}_0 - \text{a.s.}$$

$\mathbb{P}^a$ and $\mathbb{P}^{a'}$ are mutually singular for different a and a' in $\overline{\mathcal{A}}$.

Definition (Deni and Martini) We say a property holds quasi-surely, abbreviated as q.s., if it holds $\mathbb{P}^a$-a.s. for all $a \in \overline{\mathcal{A}}$.

stochastic integration under $(\mathbb{P}^a, \mathbb{F}^+)$, $a \in \overline{\mathcal{A}}$

• Note that $\mathbb{F}^+$ is right continuous, but not complete, and thus does not satisfy the usual hypotheses.

• For any $Y \in \mathcal{H}^0(\mathbb{P}, \mathbb{F}^{\mathbb{P}})$, there exists unique $\tilde{Y} \in \mathcal{H}^0(\mathbb{P}, \mathbb{F}^+)$ such that $\tilde{Y}$ and Y are $\mathbb{P}$-modifications.

• For $Z \in \mathcal{H}^2(\mathbb{P}^a, \mathbb{F}^a)$, let $Y_t := \int_0^t Z_s dB_s$ is well defined in the standard sense

- There exists a unique $\tilde{Y} \in \mathcal{S}^2(\mathbb{P}^a, \mathbb{F}^+)$ which is $\mathbb{P}^a$-indistinguishable from Y
- For $Z \in \mathcal{H}^2(\mathbb{P}^a, \mathbb{F}^+) \subset \mathcal{H}^2(\mathbb{P}^a, \mathbb{F}^a)$, $Y_t := \int_0^t Z_s dB_s$ is well defined as a process in $\mathcal{S}^2(\mathbb{P}^a, \mathbb{F}^+)$.

Martingale representation under $\mathbb{P}^a$, $a \in \overline{\mathcal{A}}$

Lemma For any $\xi \in \mathbb{L}^2(\mathbb{P}^a, \mathcal{F}_1)$, there exists a unique process $Z^a \in \mathcal{H}^2(\mathbb{P}^a, \mathbb{F}^+)$ such that $\xi = \boldsymbol{E}^a[\xi] + \int_0^1 Z^a_t dB_t$, $\mathbb{P}^a-$a.s.

• Since a is invertible, $(\mathbb{F}^{X^a})^{\mathbb{P}_0} = \mathbb{F}^{\mathbb{P}_0}$. Then $a_t = \mathbf{a}_t(B_\cdot)$, $B_t = \beta_t(X^a_\cdot), dt \times d\mathbb{P}_0-$a.s. for some measurable $\mathbf{a}, \beta$

• Denote $W^a_t := \beta_t(B_\cdot)$, $\tilde{a}_t := \mathbf{a}_t(W^a_\cdot)$. Then

$$(\mathbb{P}_0, B, X^a, a) = (\mathbb{P}^a, W^a, B, \tilde{a}) \text{ in distribution.}$$

• Since $d\langle X^a\rangle_t = a_t dt, \mathbb{P}_0-$a.s., we have $d\langle B\rangle_t = \tilde{a}_t dt, \mathbb{P}^a-$a.s.

Forward and Backward SDEs

Under standard assumptions :

• there is a unique solution $X \in \mathcal{S}^2(\mathbb{P}^a, \mathbb{F}^+)$ to the SDE

$$X_t^a = x + \int_0^t b_s(X_s^a)ds + \int_0^t \sigma_s(X_s^a)dB_s, \quad \mathbb{P}^a - \text{a.s.}$$

• there is a unique solution $(Y^a, Z^a) \in \mathcal{S}^2(\mathbb{P}^a, \mathbb{F}^+) \times \mathcal{H}^2(\mathbb{P}^a, \mathbb{F}^+)$ to the Backward SDE

$$Y_t^a = \xi + \int_t^T f_s(Y_s^a, Z_s^a)ds - \int_t^T Z_s^a dB_s$$

Moreover, usual comparison and stability statement also hold true

Patching processes quasi-surely

Lemma (Karandikar) Let X, M be two $\mathbb{F}^+$-adapted càd-làg processes q.s. with M a $\mathbb{P}^a$−semimartingale for every $a \in \overline{\mathcal{A}}$. Then there exists a càd-làg process N such that $N_t = \int_0^t X_{s-} dM_s$, $\mathbb{P}^a$−a.s. for every $a \in \overline{\mathcal{A}}$.

Corollary Assume M is $\mathbb{F}^+$-adapted and càd-làg q.s. and is $\mathbb{P}^a$−semimartingale for every $a \in \overline{\mathcal{A}}$. Then there exists a càd-làg process X such that $X_t = \langle M, B \rangle_t$, $\mathbb{P}^a$−a.s. for every $a \in \overline{\mathcal{A}}$. In particular, $\langle B \rangle$ can be defined q.s. and there exists a process $\hat{a}$ such that

$$d\langle B \rangle_t = \hat{a}_t dt = \tilde{a}_t dt, \quad \mathbb{P}^a - \text{a.s. for every} \quad a \in \overline{\mathcal{A}}$$

(Define $X_t := M_t B_t - \int_0^t M_s dB_s - \int_0^t B_s dM_s$)

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Nonlinearity and spaces

- $H_t(\omega, y, z, \gamma) : \Omega \times [0,1] \times \mathbb{R}^3 \longrightarrow \mathbb{R} \cup \{\infty\}$ is a given $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^3)-$measurable map, continuous with respect to the $\gamma-$component. Define

$$F_t(\omega, y, z, a) := \sup_{\gamma \in \mathbb{R}} \left\{ \frac{1}{2} a\gamma - H_t(\omega, y, z, \gamma) \right\}, \quad a \in \mathbb{R}^+ \qquad (4)$$

- Assume H and F are uniformly Lipschitz in (y, z)
- For simplicity, we assume $\text{Dom}(H_t) = \mathbb{R}$ as a function of γ, and $\text{Dom}(F_t) = \mathbb{R}^+$ as a function of a
- Define the spaces :

$$\hat{\mathcal{H}}^2 := \bigcap_{a \in \overline{\mathcal{A}}} \mathcal{H}^2(\mathbb{P}^a, \mathbb{F}^+)$$

and the corresponding subsets of càd-làg, continuous, and semimartingales, ...

Definition

- For $Z \in \widehat{\mathcal{SM}}^2$, denote by Γ the density of the quadratic covariation between Z and B :

$$d\langle Z, B\rangle_t \;=\; \Gamma_t d\langle B\rangle_t \;=\; \Gamma_t \hat{a}_t\, dt, \quad q.s., \tag{5}$$

- For $y \in \mathbb{R}$ and $Z \in \widehat{\mathcal{SM}}^2$, let $Y^{y,Z} \in \mathcal{S}^2$ be :

$$Y_t \;=\; y - \int_0^t H_s(Y_s, Z_s, \Gamma_s)ds + \int_0^t Z_s \circ dB_s,\;\; t \le 1, \quad q.s.$$

where $Z_s \circ dB_s = Z_s dB_s + \frac{1}{2}\Gamma_s \hat{a}_s ds$, q.s.

- For an $\mathcal{F}_1-$measurable r.v. ξ, let

$$\mathcal{V}(\xi) := \inf\left\{ y : \exists\, Z \in \widehat{\mathcal{SM}}^2 \text{ such that } Y_1^{y,Z} \ge \xi, \text{ q.s.}\right\}. \tag{6}$$

First Relaxation of Second Order Target Problems

Relax the connection between Z and Γ

- Note : $dY_t^{y,Z} = \frac{1}{2}[\Gamma_t \hat{a}_t - H_t(Y_t^{y,Z}, Z_t, \Gamma_t)dt + Z_t dB_t$, q.s.
- For $Z, G \in \hat{\mathcal{H}}^2$, define the controlled state $\bar{Y}^a := \bar{Y}^{a,y,Z,G}$:

$$d\bar{Y}_t^a \quad = \quad \left[\frac{1}{2} G_t \hat{a}_t - H_t(\bar{Y}_t^a, Z_t, G_t)\right] dt + Z_t dB_t, \quad \mathbb{P}^a - \text{a.s.}$$

for every $a \in \overline{\mathcal{A}}$

- The relaxed problem is :

$$\bar{\mathcal{V}}(\xi) \quad := \quad \inf \left\{ y : \exists\, Z, G \in \hat{\mathcal{H}}^2, \ \bar{Y}_1^a \geq \xi \ \mathbb{P}^a - \text{a.s. for every } a \in \overline{\mathcal{A}} \right\}$$

- Peng's G-BSDE : for some constants $c_1 > c_0 > 0$,

$$H_t(y,z,\gamma) = \frac{1}{2}[c_1\gamma^+ - c_0\gamma^-] + h_t(y,z) = \frac{1}{2} \sup_{c_0 \leq a \leq c_1} a\gamma + h_t(y,z).$$

Further Relaxation of Second Order Target Problems

Second relaxation : forget Γ !

- Recall the (partial) convex conjugate of H :

$$F_t(y,z,a) \quad := \quad \sup_{\gamma\in\mathbb{R}} \left\{ \frac{1}{2} a\gamma - H_t(y,z,\gamma) \right\}, \quad a \in \mathbb{R}^+$$

- For $Z \in \hat{\mathcal{H}}^2$, define the controlled state $\hat{Y}^a_t := \hat{Y}^{a,y,Z}_t$:

$$d\hat{Y}^a_t \quad = \quad F_t(\hat{Y}^a_t, Z_t, \hat{a}_t)dt + Z_t dB_t, \quad \mathbb{P}^a - \text{a.s. for every } a \in \overline{\mathcal{A}}$$

- The further relaxed problem is :

$$\hat{\mathcal{V}}(\xi) \quad := \quad \inf \left\{ y : \; \exists\, Z \in \hat{\mathcal{H}}^2, \; \hat{Y}^a_1 \geq \xi \;\; \mathbb{P}^a - \text{a.s. for every } a \in \overline{\mathcal{A}} \right\}$$

A Natural dual Formulation

For $\xi \in \hat{\mathbb{L}}^2$, $a \in \overline{\mathcal{A}}$, denote $(Y^a, Z^a) \in \mathcal{S}^2(\mathbb{P}^a, \mathbb{F}^+) \times \mathcal{H}^2(\mathbb{P}^a, \mathbb{F}^+)$ the solution of the BSDE under $\mathbb{P}^a$:

$$Y_t^a = \xi - \int_t^1 F_s(Y_s^a, Z_s^a, \hat{a}_s)ds - \int_t^1 Z_s^a dB_t, \quad \mathbb{P}^a - \text{a.s.}$$

and define the natural dual problem :

$$v(\xi) := \sup_{a \in \overline{\mathcal{A}}} Y_0^a$$

• For all these problem, we have the obvious relation :

$$\mathcal{V}(\xi) \geq \bar{\mathcal{V}}(\xi) \geq \hat{\mathcal{V}}(\xi) \geq v(\xi)$$

• In the Markov case, if the corresponding PDE has a sufficiently smooth solution, we easily prove that $\mathcal{V}(\xi) = \bar{\mathcal{V}}(\xi) = \hat{\mathcal{V}}(\xi) = v(\xi)$.

The duality result

Assumption $\xi = g(B_.)$ and $F_t(y, z, a) = \phi(t, B_., y, z, a)$ for some deterministic functions g and ϕ uniformly continuous w.r.t ω + some growth conditions

Theorem For any $\xi \in \hat{\mathbb{L}}^2$, we have $\hat{\mathcal{V}}(\xi) = v(\xi)$, and the $\hat{\mathcal{V}}(\xi)$ problem has the optimal Z.

Important tool : Peng's nonlinear Doob-Meyer decomposition

Reference Probability Measures

- T_0 be a dense subset of $[0,1]$ containing $\{0,1\}$
- $\mathcal{A}_0 = (a^i)_{i\geq 1}$ a sequence in $\bar{\mathcal{A}}$ satisfying the concatenation property :

$$a^i \mathbf{1}_{[0,t_0)} + a^j \mathbf{1}_{[t_0,1]} \in \mathcal{A}_0 \quad \text{for every} \quad i,j \geq 1 \text{ and } t_0 \in T_0 \quad (7)$$

Then, we may define ν_i, $i \geq 1$, such that

$$\sum_{i=1}^{\infty} \nu_i = 1 \quad \text{and} \quad \sum_{i=1}^{\infty} \nu_i \, \boldsymbol{E}^{\mathbb{P}_0} \int_0^1 a_t^i dt < \infty \qquad (8)$$

For every such choice of T_0, $\mathcal{A}_0$, define the reference probability measure :

$$\hat{\mathbb{P}} \; := \; \hat{\mathbb{P}}^{\mathcal{A}_0, T_0} \; := \sum_{i=1}^{\infty} \nu_i \mathbb{P}^i.$$

More singular probability measures dominated by $\hat{\mathbb{P}}$

• $\bar{\mathcal{A}}_0$: set of all processes $a \in \bar{\mathcal{A}}$ such that, for some non-decreasing sequence $(\tau_n)_{n\geq 1} \subset \mathcal{T}_1$ with values in T_0, such that $\inf\{n : \; \tau_n = 1\} < \infty$, $\mathbb{P}_0-$a.s.

$$a = a^i \quad \text{on} \quad [\tau_n, \tau_{n+1}] \text{ for some } i \geq 1, \; dt \times \mathbb{P}_0 - \text{a.s.}$$

Proposition For any $a \in \overline{\mathcal{A}}_0$, $\mathbb{P}^a$ is absolutely continuous with respect to $\hat{\mathbb{P}}$.

• Note : $\hat{\mathbb{P}}-$a.s. iff $\mathbb{P}^a-$a.s. for all $a \in \bar{\mathcal{A}}_0$.

Patching processes under $\hat{\mathbb{P}}$

- Denote

$$\overline{\mathcal{A}}_0(i,\tau) := \Big\{ a \in \overline{\mathcal{A}}_0 : a = a^i \text{ on } [0,\tilde{\tau}] \\ \text{for some } \mathcal{T}_1 \ni \tilde{\tau} > \tau \; dt \times d\mathbb{P}_0 - \text{a.s.} \Big\}.$$

Aggregation Let $X^i \in \mathcal{H}^0(\mathbb{P}^{a_i})$ be a family of processes such that

$$X^i = X^j, \text{ on } [0,\tau] \;\; dt \times d\mathbb{P}^{a_i} - \text{a.s.} \quad \text{whenever} \quad a^j \in \overline{\mathcal{A}}_0(i,\tau)$$

Then there is a unique process $X \in \mathcal{H}^0(\hat{\mathbb{P}})$ such that

$$X = X^i \qquad dt \times d\mathbb{P}^{a_i} - \text{a.s.}$$

Second Order Target Problem under $\hat{\mathbb{P}}$

• Define $\hat{\mathbb{L}}_0^2 := \bigcap_{i\geq 1} \mathbb{L}^2(\mathbb{P}^i, \mathcal{F}_1)$, $\hat{\mathcal{H}}_0^2 := \bigcap_{i\geq 1} \mathcal{H}^2(\mathbb{P}^i, \mathbb{F}^+)$, and $\widehat{\mathcal{SM}}_0^2 := \bigcap_{i\geq 1} \mathcal{SM}^2(\mathbb{P}^i, \mathbb{F}^+)$

• Define the target problem and its relaxations :

$$\mathcal{V}_0(\xi) := \inf\left\{y : Y_1^{y,Z} \geq \xi,\ \hat{\mathbb{P}} - \text{a.s. for for some } Z \in \widehat{\mathcal{SM}}_0^2\right\}.$$

$$\bar{\mathcal{V}}_0(\xi) := \inf\left\{y : \bar{Y}_1^{y,Z,G} \geq \xi,\ \hat{\mathbb{P}} - \text{a.s. for for some } Z, G \in \hat{\mathcal{H}}_0^2\right\}$$

$$\hat{\mathcal{V}}_0(\xi) := \inf\left\{y : \hat{Y}_1^{y,Z} \geq \xi,\ \hat{\mathbb{P}} - \text{a.s. for for some } Z \in \hat{\mathcal{H}}_0^2\right\}$$

We also define the corresponding dual problem

$$v_0(\xi) := \sup_{a\in\overline{\mathcal{A}}_0} Y_0^a$$

The target problem and its relaxations

Theorem Under technical conditions,

$$\mathcal{V}_0(\xi) = \bar{\mathcal{V}}_0(\xi) = \hat{\mathcal{V}}_0(\xi) = v_0(\xi)$$

Moreover, existence holds for the relaxed problems $\bar{\mathcal{V}}_0(\xi)$ and $\hat{\mathcal{V}}_0(\xi)$. To be specific, there exist process $\bar{Y}, \bar{Z}, \bar{G}$ and an increasing càd-làg process $\bar{K}$ with $\bar{K}_0 = 0$ such that

$$\begin{aligned} d\bar{Y}_t &= [\frac{1}{2}\hat{a}_t \bar{G}_t - H_t(\bar{Y}_t, \bar{Z}_t, \bar{G}_t)]dt + \bar{Z}_t dB_t - d\bar{K}_t, \\ \bar{Y}_0 &= \bar{\mathcal{V}}_0(\xi), \quad \bar{Y}_1 = \xi, \end{aligned} \qquad \hat{\mathbb{P}} - \text{a.s.}$$

- Important tool : nonlinear version of Bank-Baum result

The q.s. problem and the $\hat{\mathbb{P}}$-a.s. problem

Theorem Under the continuity conditions on ξ and F, we have $v_0(\xi) = v(\xi)$. In particular, $\mathcal{V}_0(\xi)$, $\bar{\mathcal{V}}_0(\xi)$ and $\hat{\mathcal{V}}_0(\xi)$ are independent from the choice of the sets $\mathcal{A}_0$ and $\mathcal{T}_0$.

Conclusion

• Second order stochastic target problems have a suitable formulation by allowing for model uncertainty

• From the dual formulation, we have obtained existence for the second relaxation of the target problem in the quasi-surely sense

• For the weak formulation under $\hat{\mathbb{P}}$, we have obtained existence for both the first and the second relaxation of the target problem

• Future work :

(i) existence for the first relaxation of the target problem in the quasi-surely sense

(ii) existence result for second order BSDEs q.s. and/or $\hat{\mathbb{P}}$-a.s.