Dual Formulation of Second Order Target Problems

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Outline

1. Introduction

2. BSDEs under singular probability measures

3. Second Order Target Problems and Duality
   - an alternative Formulation
   - Relaxations
   - Weak version of the Second Order Target Problem
Backward SDEs

Pardoux and Peng (1990, 1992): W BM on \((\Omega, \mathcal{F}, \mathbb{P}), \mathbb{F} = \mathbb{F}^W\)

- For \(\xi \in \mathbb{L}^2\), \(H_t(y, z)\) Lipschitz in \((y, z)\), \(H(0, 0) \in \mathbb{H}^2\) the BSDE

\[
Y_t = -H_t(Y_t, Z_t)dt + Z_t dW_t, \quad Y_1 = \xi
\]

has a unique solution \((Y, Z) \in \mathcal{S}^2 \times \mathbb{H}^2\)

- Moreover if \(H_t(y, z) = h(t, X_t, y, z)\) and \(\xi = g(X_1)\), where

\[
dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t
\]

Then \(Y_t = V(t, X_t)\) for some deterministic measurable function \(V\)

- \(V\) is a viscosity solution of the semilinear PDE

\[
\partial_t V + \frac{1}{2} \sigma^2 D^2 V + bDV + h(t, x, V, \sigma DV) = 0, \quad V(1, x) = g(x).
\]
Introduction
BSDEs under singular probability measures
Second Order Target Problems and Duality

Backward SDEs

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\mathcal{W} \text{ BM on } (\Omega, \mathcal{F}, \mathbb{P}), \mathcal{F} = \mathcal{F}^\mathcal{W}
\)

- For \(\xi \in \mathbb{L}^2\), \(H_t(y, z)\) Lipschitz in \((y, z)\), \(H(0, 0) \in \mathbb{H}^2\) the BSDE

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- Moreover if \(H_t(y, z) = h(t, X_t, y, z)\) and \(\xi = g(X_1)\), where

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Forward-Backward SDEs


\[ X_t = x + \int_0^t b(s, X_s, Y_s, Z_s) ds + \int_0^t \sigma(s, X_s, Y_s, Z_s) dW_s, \]

\[ Y_t = g(X_T) + \int_t^T h(s, X_s, Y_s, Z_s) ds - \int_t^T Z_s dW_s \]

- Together with other conditions, if the coefficients are deterministic and \( \sigma = \sigma(t, x, y) \), then \( Y_t = V(t, X_t) \) where \( V \) is a viscosity solution of the quasi-linear PDE

\[ \partial_t V + \frac{1}{2} \sigma^2(t, x, V) D^2 V + b(t, x, V, \sigma DV)DV + h(t, x, V, \sigma DV) = 0, \]
Forward-Backward SDEs


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X_t = x + \int_0^t b(s, X_s, Y_s, Z_s)ds + \int_0^t \sigma(s, X_s, Y_s, Z_s)dW_s,
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\[
\partial_t V + \frac{1}{2} \sigma^2(t, x, V) D^2 V + b(t, x, V, \sigma D V) D V + h(t, x, V, \sigma D V) = 0,
\]
Second Order Backward SDEs

Cheridito, Soner, Touzi and Victoir (2007):

- 2BSDE:

\[ dY_t = H_t(Y_t, Z_t, \Gamma_t)dt + Z_t \circ dW_t, \quad Y_1 = \xi \]  \quad (1)

where

\[ Z_t \circ dW_t = Z_t dW_t + \frac{1}{2} d\langle Z, W \rangle_t = Z_t dW_t + \frac{1}{2} \Gamma_t dt \]

is the Fisk-Stratonovich stochastic integration.

- If \( H_t = h(t, W_t, Y_t, Z_t, \Gamma_t) \) and \( \xi = g(W_1) \), then \( Y_t = V(t, W_t) \), where \( V \) is associated with the fully nonlinear PDE:

\[ \partial_t V + h(t, x, V, DV, D^2 V) = 0 \quad \text{and} \quad V(1, x) = g(x). \]  \quad (2)
Second Order Backward SDEs

Cheridito, Soner, Touzi and Victoir (2007):

- 2BSDE:

\[ dY_t = H_t(Y_t, Z_t, \Gamma_t)dt + Z_t \circ dW_t, \quad Y_1 = \xi \quad (1) \]

where

\[ Z_t \circ dW_t = Z_t dW_t + \frac{1}{2} d\langle Z, W \rangle_t = Z_t dW_t + \frac{1}{2} \Gamma_t dt \]

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- If \( H_t = h(t, W_t, Y_t, Z_t, \Gamma_t) \) and \( \xi = g(W_1) \), then \( Y_t = V(t, W_t) \),

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\[ \partial_t V + h(t, x, V, D V, D^2 V) = 0 \quad \text{and} \quad V(1, x) = g(x). \quad (2) \]
Motivation from Probabilistic Numerical Methods


\[
\hat{Y}_{t_n} = g(X_{t_n}) \quad \text{and for } 1 \leq i \leq n:
\]
\[
\hat{Y}_{t_{i-1}} = \hat{E}_{i-1} \left[ \hat{Y}_{t_i} + \Delta t_i \ f \left( t_i, X_{t_{i-1}}, \hat{Y}_{t_{i-1}}, \hat{Z}_{t_{i-1}}, \hat{\Gamma}_{t_{i-1}} \right) \right]
\]
\[
\hat{Z}_{t_{i-1}} = \hat{E}_{i-1} \left[ \hat{Y}_{t_i} \frac{\Delta W_{t_i}}{\Delta t_i} \right]
\]
\[
\hat{\Gamma}_{t_{i-1}} = \hat{E}_{i-1} \left[ \hat{Y}_{t_i} \frac{\Delta W_{t_i}^2 - \Delta t_i}{|\Delta t_i|^2} \right]
\]
More motivations

Motivation from finance

- Hedging under Gamma constraints (Soner and Touzi 1999, Cheridito, Soner and Touzi 2005)
- Hedging under liquidity cost in the Cetin-Jarrow-Protter model (Cetin, Sonet and Touzi 2006)
- Uncertain volatility models (Denis and Martini 2006)

Peng’s $G$–expectation
Cheridito, Soner, Touzi and Victoir (2007) : In Markov case only

- **Existence** : if the PDE (2) has a smooth solution, then

  \[ Y_t = V(t, W_t), \quad Z_t = D V(t, W_t), \quad \Gamma_t = D^2 V(t, W_t). \]

- **Uniqueness** : Second Order Stochastic Target Problem

  \[ V(t, x) := \inf \left\{ y : Y_t^{y, Z} \geq g(W_1) \text{ for some } Z \in \mathcal{Z} \right\} \]

Under certain conditions, in particular if the comparison principle for viscosity solution of PDE (2) holds, then \( V \) is the viscosity solution of the PDE (2). Consequently, 2BSDE (1) has a unique solution in class \( \mathcal{Z} \).
The admissibility set \( Z \) in CSTV

**Definition** \( Z \in \mathcal{Z} \) if it is of the form

\[
Z_t = \sum_{n=0}^{N-1} z_n 1_{\{t < \tau_{n+1}\}} + \int_0^t \alpha_s ds + \int_0^t \Gamma_s dW_s
\]

- \((\tau_n)\) is a \( \nearrow \) seq. of stop. times, \( z_n \) are \( \mathcal{F}_{\tau_n} \)-measurable, \( \|N\|_{\infty} < \infty \)
- \( Z_t \) and \( \Gamma_t \) are \( L_{\infty} \)-bounded up to some polynomial of \( X_t \)
- \( \Gamma_t = \Gamma_0 + \int_0^t a_s ds + \int_0^t \xi_s dW_s, \; 0 \leq t \leq T \), and

\[
\|\alpha\|_{B,b} + \|a\|_{B,b} + \|\xi\|_{B,2} < \infty, \quad \|\phi\|_{B,b} := \sup_{0 \leq t \leq T} \frac{|\phi_r|}{1 + X_t^B} \}
\]

\( \mathbb{I} \)
"Theorem" If the following linear 2BSDE with constant coefficients has only zero solution in $\mathbb{L}^2$, then, under very mild conditions, uniqueness holds for the general non-Markovian 2BSDE (1) in essentially $\mathbb{L}^2$ space:

$$dY_t = -c \Gamma_t dt + Z_t \circ dW_t, \quad Y_1 = 0.$$ (3)
"Theorem" If the following linear 2BSDE with constant coefficients has only zero solution in $\mathbb{L}^2$, then, under very mild conditions, uniqueness holds for the general non-Markovian 2BSDE (1) in essentially $\mathbb{L}^2$ space:

$$dY_t = -c\Gamma_t dt + Z_t \circ dW_t, \quad Y_1 = 0.$$ (3)

• Unfortunately, unless $c = \frac{1}{2}$, the 2BSDE (3) has nonzero solutions in $\mathbb{L}^2$!
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Motivation

Consider BSDE:

\[ Y_t = g(W_1) + \int_t^1 h(s, W_s, Y_s, Z_s) ds - \int_t^1 Z_s dW_s \]

and PDE:

\[ \partial_t V + \frac{1}{2} D^2 V + h(t, x, V, D V) = 0, \quad V(1, x) = g(x) \]

- If \( h(t, x, y, z) = \sup_{u \in U} [u z - f(t, x, y, u)] \), then \( V = \sup_u V^u \), where

\[ \partial_t V^u + \frac{1}{2} D^2 V^u + u D V^u - f(t, x, V^u, u) = 0, \quad V^u(1, x) = g(x) \]

- Consequently, \( Y_0 = \sup Y_0^u \) where

\[
\begin{align*}
X_t^u &= \int_0^t u_s ds + W_t; \\
Y_t^u &= g(X_1^u) - \int_t^1 f(s, X_s^u, Y_s^u, u_s) ds - \int_t^1 Z_s^u dW_s.
\end{align*}
\]

- Drift control, and \( P^u := P \circ (X^u)^{-1} \) is equivalent to \( P \).
Consider
2BSDE : \( Y_t = g(W_1) + \int_t^1 h(s, W_s, Y_s, Z_s, \Gamma_s) ds - \int_t^1 Z_s \circ dW_s \)
and PDE : \( \partial_t V + h(t, x, V, D V, D^2 V) = 0, \quad V(1, x) = g(x) \)

• If \( h(t, x, y, z, \gamma) = \sup_{a \in A} \left[ \frac{1}{2} a \gamma - f(t, x, y, z, a) \right] \), then \( V = \sup_a V^a \),
where
\( \partial_t V^a + \frac{1}{2} a D^2 V^a - f(t, x, V^a, D V^a, a) = 0, \quad V^a(1, x) = g(x) \)

• Consequently, \( Y_0 = \sup_a Y^a_0 \), where
\( X^a_t = \int_0^t a^{1/2}_s dW_s; \)
\( Y^a_t = g(X^a_1) - \int_t^1 f(s, X^a_s, Y^a_s, Z^a_s, a_s) ds - \int_t^1 Z^a_s dX^a_s. \)

• Volatility control, and \( P^a := P \circ (X^a)^{-1} \) are mutually singular for different \( a \).
General Framework

- $\Omega := C([0,1])$, $B$ the canonical process, $\mathbb{P}_0$ the Wiener measure, $\mathcal{F} := \{\mathcal{F}_t\}_{0 \leq t \leq 1}$ the filtration generated by $B$
- $\mathcal{F}^+ := \{\mathcal{F}_t^+\}_{0 \leq t \leq 1}$
- $\overline{\mathcal{A}}$: set of all $\mathcal{F}$–adapted process $a$ satisfying $a \leq a_t(\omega) \leq \overline{a}$, $dt \times d\mathbb{P}_0$ – a.s. for some $\overline{a} \geq a > 0$
- $\mathbb{P}^a := \mathbb{P}_0 \circ (X^a)^{-1}$, measure induced by $X^a$:
  \[ X^a_t := \int_0^1 a_s^{1/2} dB_s, \ 0 \leq t \leq 1, \ \mathbb{P}_0$ – a.s. \]

$\mathbb{P}^a$ and $\mathbb{P}^{a'}$ are mutually singular for different $a$ and $a'$ in $\overline{\mathcal{A}}$.

**Definition** (Deni and Martini) We say a property holds quasi-surely, abbreviated as q.s., if it holds $\mathbb{P}^a$–a.s. for all $a \in \overline{\mathcal{A}}$. 

Jianfeng ZHANG  |  Dual Formulation of Second Order Target Problems
stochastic integration under \((\mathbb{P}^a, \mathbb{F}^+), a \in \bar{A}\)

• Note that \(\mathbb{F}^+\) is right continuous, but not complete, and thus does not satisfy the usual hypotheses.

• For any \(Y \in \mathcal{H}^0(\mathbb{P}, \mathbb{F}^\mathbb{P})\), there exists unique \(\tilde{Y} \in \mathcal{H}^0(\mathbb{P}, \mathbb{F}^+)\) such that \(\tilde{Y}\) and \(Y\) are \(\mathbb{P}\)-modifications.

• For \(Z \in \mathcal{H}^2(\mathbb{P}^a, \mathbb{F}^a)\), let \(Y_t := \int_0^t Z_s dB_s\) is well defined in the standard sense
  
  • There exists a unique \(\tilde{Y} \in \mathcal{S}^2(\mathbb{P}^a, \mathbb{F}^+)\) which is \(\mathbb{P}^a\)-indistinguishable from \(Y\)
  
  • For \(Z \in \mathcal{H}^2(\mathbb{P}^a, \mathbb{F}^+) \subset \mathcal{H}^2(\mathbb{P}^a, \mathbb{F}^a)\), \(Y_t := \int_0^t Z_s dB_s\) is well defined as a process in \(\mathcal{S}^2(\mathbb{P}^a, \mathbb{F}^+)\).
Lemma For any $\xi \in L^2(\mathbb{P}^a, \mathcal{F}_1)$, there exists a unique process $Z^a \in H^2(\mathbb{P}^a, \mathbb{F}^+)$ such that $\xi = \mathbb{E}^a[\xi] + \int_0^1 Z^a_t dB_t$, $\mathbb{P}^a$-a.s.

- Since $a$ is invertible, $(\mathbb{P}^X^a)\mathbb{P}_0 = \mathbb{F}\mathbb{P}_0$. Then $a_t = a_t(B.), B_t = \beta_t(X^a), dt \times d\mathbb{P}_0$-a.s. for some measurable $a, \beta$

- Denote $W_t^a := \beta_t(B.), \tilde{a}_t := a_t(W^a)$. Then

$$(\mathbb{P}_0, B, X^a, a) = (\mathbb{P}^a, W^a, B, \tilde{a}) \text{ in distribution.}$$

- Since $d\langle X^a \rangle_t = a_t dt, \mathbb{P}_0$-a.s., we have $d\langle B \rangle_t = \tilde{a}_t dt, \mathbb{P}^a$-a.s.
Under standard assumptions:

- there is a unique solution $X \in \mathcal{S}^2(\mathbb{P}^a, \mathbb{F}^+)$ to the SDE

$$X^a_t = x + \int_0^t b_s(X^a_s)ds + \int_0^t \sigma_s(X^a_s)dB_s, \quad \mathbb{P}^a \text{ a.s.}$$

- there is a unique solution $(Y^a, Z^a) \in \mathcal{S}^2(\mathbb{P}^a, \mathbb{F}^+) \times \mathcal{H}^2(\mathbb{P}^a, \mathbb{F}^+)$ to the Backward SDE

$$Y^a_t = \xi + \int_t^T f_s(Y^a_s, Z^a_s)ds - \int_t^T Z^a_s dB_s$$

Moreover, usual comparison and stability statement also hold true.
Lemma (Karandikar) Let $X, M$ be two $\mathcal{F}^+-$adapted càd-làg processes q.s. with $M$ a $\mathbb{P}^a$—semimartingale for every $a \in \overline{A}$. Then there exists a càd-làg process $N$ such that $N_t = \int_0^t X_s - dM_s$, $\mathbb{P}^a$—a.s. for every $a \in \overline{A}$.

Corollary Assume $M$ is $\mathcal{F}^+-$adapted and càd-làg q.s. and is $\mathbb{P}^a$—semimartingale for every $a \in \overline{A}$. Then there exists a càd-làg process $X$ such that $X_t = \langle M, B \rangle_t$, $\mathbb{P}^a$—a.s. for every $a \in \overline{A}$. In particular, $\langle B \rangle$ can be defined q.s. and there exists a process $\hat{a}$ such that

$$d\langle B \rangle_t = \hat{a}_t dt = \tilde{a}_t dt,$$  $\mathbb{P}^a$ — a.s. for every $a \in \overline{A}$

(Define $X_t := M_t B_t - \int_0^t M_s dB_s - \int_0^t B_s dM_s$)
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Nonlinearity and spaces

• $H_t(\omega, y, z, \gamma) : \Omega \times [0, 1] \times \mathbb{R}^3 \to \mathbb{R} \cup \{\infty\}$ is a given $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^3)$–measurable map, continuous with respect to the $\gamma$–component. Define

\[ F_t(\omega, y, z, a) := \sup_{\gamma \in \mathbb{R}} \left\{ \frac{1}{2} a \gamma - H_t(\omega, y, z, \gamma) \right\}, \quad a \in \mathbb{R}^+ \quad (4) \]

• Assume $H$ and $F$ are uniformly Lipschitz in $(y, z)$
• For simplicity, we assume $\text{Dom}(H_t) = \mathbb{R}$ as a function of $\gamma$, and $\text{Dom}(F_t) = \mathbb{R}^+$ as a function of $a$
• Define the spaces:

\[ \hat{\mathcal{H}}^2 := \bigcap_{a \in \overline{A}} \mathcal{H}^2(\mathbb{P}^a, \mathbb{F}^+) \]

and the corresponding subsets of càd-làg, continuous, and semimartingales, ...
Definition

- For $Z \in \widehat{SM}^2$, denote by $\Gamma$ the density of the quadratic covariation between $Z$ and $B$:

$$d\langle Z, B \rangle_t = \Gamma_t d\langle B \rangle_t = \Gamma_t \hat{a}_t\,dt, \text{ q.s.}$$

- For $y \in \mathbb{R}$ and $Z \in \widehat{SM}^2$, let $Y_{y,Z} \in S^2$ be:

$$Y_t = y - \int_0^t H_s(Y_s, Z_s, \Gamma_s)\,ds + \int_0^t Z_s \circ dB_s, \quad t \leq 1, \text{ q.s.}$$

where $Z_s \circ dB_s = Z_s dB_s + \frac{1}{2} \Gamma_s \hat{a}_s\,ds, \text{ q.s.}$

- For an $\mathcal{F}_1$-measurable r.v. $\xi$, let

$$\mathcal{V}(\xi) := \inf \left\{ y : \exists Z \in \widehat{SM}^2 \text{ such that } Y_1^{y,Z} \geq \xi, \text{ q.s.} \right\}.$$
First Relaxation of Second Order Target Problems

Relax the connection between $Z$ and $\Gamma$

- Note: $dY_t^{y,Z} = \frac{1}{2}[\Gamma_t \hat{a}_t - H_t(Y_t^{y,Z}, Z_t, \Gamma_t)]dt + Z_t dB_t$, q.s.

- For $Z, G \in \hat{H}^2$, define the controlled state $\bar{Y}^a := \bar{Y}^{a,y,Z,G}$ :

$$d\bar{Y}_t^a = \left[\frac{1}{2} G_t \hat{a}_t - H_t(\bar{Y}_t^a, Z_t, G_t)\right] dt + Z_t dB_t, \quad \mathbb{P}^a - a.s.$$

for every $a \in \overline{A}$

- The relaxed problem is:

$$\bar{V}(\xi) := \inf \left\{ y : \exists Z, G \in \hat{H}^2, \bar{Y}_t^a \geq \xi \mathbb{P}^a - a.s. \text{ for every } a \in \overline{A} \right\}$$

- Peng's $G$-BSDE: for some constants $c_1 > c_0 > 0$,

$$H_t(y, z, \gamma) = \frac{1}{2}[c_1 \gamma^+ - c_0 \gamma^-] + h_t(y, z) = \frac{1}{2} \sup_{c_0 \leq a \leq c_1} a \gamma + h_t(y, z).$$
Further Relaxation of Second Order Target Problems

Second relaxation : forget $\Gamma$!

- Recall the (partial) convex conjugate of $H$:

$$F_t(y, z, a) := \sup_{\gamma \in \mathbb{R}} \left\{ \frac{1}{2} a \gamma - H_t(y, z, \gamma) \right\}, \quad a \in \mathbb{R}^+$$

- For $Z \in \mathcal{H}^2$, define the controlled state $\hat{Y}^a_t := \hat{Y}^a_{t,y,Z}$:

$$d \hat{Y}^a_t = F_t(\hat{Y}^a_t, Z_t, \hat{a}_t)dt + Z_t dB_t, \quad \mathbb{P}^a - a.s. \text{ for every } a \in \overline{A}$$

- The further relaxed problem is:

$$\hat{V}(\xi) := \inf \left\{ y : \exists Z \in \mathcal{H}^2, \hat{Y}^a_1 \geq \xi \quad \mathbb{P}^a - a.s. \text{ for every } a \in \overline{A} \right\}$$
A Natural dual Formulation

For $\xi \in \hat{L}^2$, $a \in \overline{A}$, denote $(Y^a, Z^a) \in S^2(\mathbb{P}^a, \mathbb{F}^+) \times \mathcal{H}^2(\mathbb{P}^a, \mathbb{F}^+)$ the solution of the BSDE under $\mathbb{P}^a$:

$$Y^a_t = \xi - \int_t^1 F_s(Y^a_s, Z^a_s, \hat{a}_s)ds - \int_t^1 Z^a_s dB_t, \quad \mathbb{P}^a - \text{a.s.}$$

and define the natural dual problem:

$$v(\xi) := \sup_{a \in \overline{A}} Y^a_0$$

- For all these problem, we have the obvious relation:

$$V(\xi) \geq \bar{V}(\xi) \geq \hat{V}(\xi) \geq v(\xi)$$

- In the Markov case, if the corresponding PDE has a sufficiently smooth solution, we easily prove that $V(\xi) = \bar{V}(\xi) = \hat{V}(\xi) = v(\xi)$
The duality result

**Assumption** \( \xi = g(B_.) \) and \( F_t(y, z, a) = \phi(t, B_., y, z, a) \) for some deterministic functions \( g \) and \( \phi \) uniformly continuous w.r.t \( \omega \) + some growth conditions.

**Theorem** For any \( \xi \in \hat{L}^2 \), we have \( \hat{V}(\xi) = v(\xi) \), and the \( \hat{V}(\xi) \) problem has the optimal \( Z \).

Important tool: Peng’s nonlinear Doob-Meyer decomposition
Reference Probability Measures

- $T_0$ be a dense subset of $[0, 1]$ containing $\{0, 1\}$
- $\mathcal{A}_0 = (a^i)_{i \geq 1}$ a sequence in $\bar{A}$ satisfying the concatenation property:

  $$a^i 1_{[0,t_0)} + a^j 1_{[t_0,1]} \in \mathcal{A}_0 \quad \text{for every} \quad i, j \geq 1 \text{ and } t_0 \in T_0 \quad (7)$$

Then, we may define $\nu_i$, $i \geq 1$, such that

$$\sum_{i=1}^{\infty} \nu_i = 1 \quad \text{and} \quad \sum_{i=1}^{\infty} \nu_i \mathbb{E}^{\hat{P}_0} \int_0^1 a^i_t dt < \infty \quad (8)$$

For every such choice of $T_0$, $\mathcal{A}_0$, define the reference probability measure:

$$\hat{P} := \hat{P}^{\mathcal{A}_0, T_0} := \sum_{i=1}^{\infty} \nu_i \hat{P}^i.$$
• $\tilde{A}_0$ : set of all processes $a \in \tilde{A}$ such that, for some non-decreasing sequence $(\tau_n)_{n \geq 1} \subset T_1$ with values in $T_0$, such that $\inf\{n : \tau_n = 1\} < \infty$, $\mathbb{P}_0$–a.s.

$$a = a^i \text{ on } [\tau_n, \tau_{n+1}] \text{ for some } i \geq 1, \ dt \times \mathbb{P}_0 - \text{a.s.}$$

**Proposition** For any $a \in \tilde{A}_0$, $\mathbb{P}^a$ is absolutely continuous with respect to $\hat{\mathbb{P}}$.

• Note : $\hat{\mathbb{P}}$–a.s. iff $\mathbb{P}^a$–a.s. for all $a \in \tilde{A}_0$. 
Patching processes under $\hat{\mathbb{P}}$

- Denote

$$\overline{A}_0(i, \tau) := \left\{ a \in \overline{A}_0 : a = a^i \text{ on } [0, \tilde{\tau}] \right\}$$

for some $\mathcal{T}_1 \ni \tilde{\tau} > \tau \ dt \times d\hat{\mathbb{P}}_0 - \text{a.s.}$

**Aggregation**

Let $X^i \in \mathcal{H}^0(\mathbb{P}^a_i)$ be a family of processes such that

$$X^i = X^j, \text{ on } [0, \tau] \ dt \times d\mathbb{P}^a_i - \text{a.s.} \ \text{whenever} \ a^j \in \overline{A}_0(i, \tau)$$

Then there is a unique process $X \in \mathcal{H}^0(\hat{\mathbb{P}})$ such that

$$X = X^i \ dt \times d\mathbb{P}^a_i - \text{a.s.}$$
Second Order Target Problem under $\hat{\mathbb{P}}$

- Define $\hat{\mathcal{L}}^2_0 := \bigcap_{i \geq 1} \mathcal{L}^2(\mathbb{P}^i, \mathcal{F}_1)$, $\hat{\mathcal{H}}^2_0 := \bigcap_{i \geq 1} \mathcal{H}^2(\mathbb{P}^i, \mathcal{F}^+)$, and $\hat{\mathcal{S}}\mathcal{M}^2_0 := \bigcap_{i \geq 1} \mathcal{S}\mathcal{M}^2(\mathbb{P}^i, \mathcal{F}^+)$

- Define the target problem and its relaxations:

$V_0(\xi) := \inf \left\{ y : Y^{Y^0, Z}_1 \geq \xi, \quad \hat{\mathbb{P}} \text{- a.s. for some } Z \in \hat{\mathcal{S}}\mathcal{M}^2_0 \right\}$

$\bar{V}_0(\xi) := \inf \left\{ y : \bar{Y}^{Y, Z, G}_1 \geq \xi, \quad \hat{\mathbb{P}} \text{- a.s. for some } Z, G \in \hat{\mathcal{H}}^2_0 \right\}$

$\hat{V}_0(\xi) := \inf \left\{ y : \hat{Y}^{Y^0, Z}_1 \geq \xi, \quad \hat{\mathbb{P}} \text{- a.s. for some } Z \in \hat{\mathcal{H}}^2_0 \right\}$

We also define the corresponding dual problem

$v_0(\xi) := \sup_{a \in \overline{\mathcal{A}}_0} Y^a_0$
The target problem and its relaxations

**Theorem**  Under technical conditions,

\[ \nu_0(\xi) = \overline{\nu}_0(\xi) = \hat{\nu}_0(\xi) = \nu_0(\xi) \]

Moreover, existence holds for the relaxed problems \( \overline{\nu}_0(\xi) \) and \( \hat{\nu}_0(\xi) \). To be specific, there exist process \( \overline{Y}, \overline{Z}, \overline{G} \) and an increasing càd-làg process \( \overline{K} \) with \( \overline{K}_0 = 0 \) such that

\[
d\overline{Y}_t = \left[ \frac{1}{2} \hat{a}_t \overline{G}_t - H_t(\overline{Y}_t, \overline{Z}_t, \overline{G}_t) \right] dt + \overline{Z}_t dB_t - d\overline{K}_t, \quad \mathbb{P} - \text{a.s.}
\]

\[
\overline{Y}_0 = \overline{\nu}_0(\xi), \quad \overline{Y}_1 = \xi,
\]

- Important tool: nonlinear version of Bank-Baum result
The q.s. problem and the $\hat{P}$-a.s. problem

**Theorem**  Under the continuity conditions on $\xi$ and $F$, we have $\nu_0(\xi) = \nu(\xi)$. In particular, $\nu_0(\xi)$, $\hat{\nu}_0(\xi)$ and $\hat{V}_0(\xi)$ are independent from the choice of the sets $A_0$ and $T_0$. 
Conclusion

- Second order stochastic target problems have a suitable formulation by allowing for model uncertainty
- From the dual formulation, we have obtained existence for the second relaxation of the target problem in the quasi-surely sense
- For the weak formulation under \( \hat{P} \), we have obtained existence for both the first and the second relaxation of the target problem
- Future work:
  (i) existence for the first relaxation of the target problem in the quasi-surely sense
  (ii) existence result for second order BSDEs q.s. and/or \( \hat{P} \)-a.s.