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Zastosowanie metody programowania liniowego do wyceny opcji amerykańskich w modelu dyfuzji ze skokami

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The Application of Linear Programming to American Option Valuation in the Jump-Diffusion Model

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Contents

Summary

In this paper, we consider the problem of pricing American vanilla options in an incomplete market in which the stock price process is driven by a diffusion with jumps of random magnitude. We use Schweizer's minimal equvalent martingale measure as the pricing measure.

We formulate the problem as a variational inequality, whose discretization leads to a linear complementarity problem (LCP). We introduce a significant modification to the discretization scheme proposed by Zhang [Zha97]. Moreover, we propose a new efficient linear programming (LP) algorithm for solving the LCP's which arise, and show that, on the whole, it performs better than the standard iterative PSOR method.

Furthermore, we analyse how the numerical solution to the American put pricing problem depends on the parameters of the model. We also investigate how the distribution of the optimal exercise time for the American put varies with the parameters.

The original ideas and results presented in this paper are the following.

1. The computation of Schweizer's minimal martingale measure for the model in question, and the application of this measure (instead of Merton's measure) to define the value of a derivative security in this model.

2. The modification of the discretization scheme for the implied variational inequality, introduced to eliminate the instability of the numerical solution and to improve its properties.

3. The construction of an efficient LP algorithm for solving the arising LCP's, its detailed analysis and comparison with the PSOR algorithm.

4. A thorough analysis of the numerical solution to the American put pricing problem (dependence on the parameters, comparison with the Black-Scholes prices, distribution of the optimal exercise times).

Chapter 1

The Valuation of American Options in the Jump-Diffusion Model

1.1 Introduction

American stock options are the most widely traded derivative securities in the world. Since the seventies, when they appeared on public exchanges, their valuation has been a crucial problem in financial management. Due to the early exercise feature of these financial products, it is impossible to value them analytically for most payoff functions, even in the Black-Scholes framework[1]. In order to price them, it is necessary to resort to numerical methods.

In the Black-Scholes case, the problem of valuing both American vanillas and American exotics numerically has been examined closely, and various methods have been proposed. Karatzas and Shreve [KS98] enumerate the following methods as the most popular ones currently in use:

- numerical solutions of partial differential equations and variational inequalities,
- binomial trees and their extensions,
- analytic approximations,

[1] The important exception are *perpetuities*, i. e. options with infinite expiry date. They can be priced analytically in both the Black-Scholes and the jump-diffusion model (see [KS98] and [Mor99] for explicit formulae in the former and the latter model, respectively).

- Monte Carlo simulation.

A survey of recent numerical techniques of pricing American options is presented in the paper by Broadie and Detemple [BD97].

Jaillet, Lamberton and Lapeyre [JLL90] were the first to introduce the *variational inequality* approach to pricing American options in the Black-Scholes model. They rely on the methodology of Bensoussan and Lions [BL78] to obtain the main results. For an overview of variational inequality methods in the context of American option pricing in the Black-Scholes model, see Wilmott, Dewynne and Howison [WDH93].

The *jump-diffusion* model of the market was first introduced by Merton [Mer76]. Merton, by assuming that the risk associated with the stock is unpriced, attempts at valuing European options in his model, despite its incompleteness. It is now well known that in incomplete markets option prices are not unique and each equivalent martingale measure defines an arbitrage-precluding price. Instead of using Merton's equivalent martingale measure (jump risk unpriced, diffusion risk priced like in the Black-Scholes model), Schweizer [Sch95] suggests using the so-called *minimal martingale measure* (MMM), which has the advantage of leaving the discounted stock price semimartingale "nearly intact".

Zhang [Zha93], [Zha97] uses the methodology of variational inequalities to price American options in Merton's model. Her approach is based on the theory of Bensoussan and Lions [BL78], [BL82], and Jaillet *et al.* [JLL90].

The discretization of the variational formulation leads to a *linear complementarity problem* (LCP), on which there is vast literature (see, for example, Cottle, Pang and Stone [CPS92] or Murty [Mur97]). In principle, there are two main approaches to solving LCP's — one iterative and the other direct. Probably the most popular iterative method is the *projected successive over-relaxation* (PSOR), considered in [WDH93] in the context of American option valuation in the Black-Scholes model. *Parametric principal pivoting* appears to be the most popular direct method. It is discussed at length in [CPS92].

The numerical treatment of the LCP's arising in Merton's jump-diffusion model is discussed in the paper by Huang and Pang [HP98]. The authors argue that both algorithms (PSOR and parametric principal pivoting) can be used to solve the LCP's in question.

Some LCP's can be equivalently formulated as *linear programming* (LP) problems. Dempster *et al.* [DHR98], [DH97], [DH99] take advantage of this equivalence by employing the LP methodology to price American options in the

Black-Scholes model.

Our aim in this paper is to value American vanilla options in the jump-diffusion model with jumps of random magnitude, using the linear complementarity approach. We use Schweizer's minimal martingale measure as the pricing measure. We modify the discretization scheme proposed by Zhang [Zha97], and show that, after the modification, the numerical solutions to the American option pricing problems behave "better" than those resulting from the unmodified scheme. The arising LCP's are solved by means of both the standard PSOR method and a new LP technique.

The paper is organized as follows. In Chapter 1, we describe the jump-diffusion model, define the option pricing problem, and then explicitly compute Schweizer's minimal martingale measure for the stock price process in question. Subsequently, we formulate the problem in the language of linear complementarity, using a different discrete approximation of the pdf of the jump relative size than that proposed by Zhang [Zha97]. We solve the LCP's which arise by means of the standard PSOR method. In Chapter 2, we propose a new linear programming method for solving the above-mentioned LCP. We show its advantages and disadvantages, and conclude that the new LP algorithm, while being comparable to PSOR in terms of accuracy, significantly outperforms PSOR in terms of speed. Subsequently, we apply the new algorithm to the valuation of the American put written on a dividend-paying stock, using Schweizer's measure as the pricing measure. In Chapter 3, we discuss the most important numerical results. Namely, we analyse how the computed price of the option depends on the parameters of the model, and we study the optimal exercise times for the option. Finally, we discuss the advantages of our approximation scheme over Zhang's scheme.

The original ideas and results presented in this paper are the following.

1. The explicit computation of Schweizer's minimal martingale measure for the model in question, and the application of this measure (instead of Merton's measure) to define the value of an American option in this model (Section 1.4).

2. The modification of the discretization scheme for the implied variational inequality, introduced to eliminate the instability of the numerical solution and to improve its properties (subsection 1.6.3, Section 3.3).

3. The construction of a new efficient LP algorithm for solving the arising LCP's, its detailed analysis and comparison with the well-established PSOR algorithm (Chapter 2).

4. A thorough analysis of the numerical solution to the American put pricing problem (dependence on the parameters, comparison with the Black-Scholes prices, distribution of the optimal exercise times: Chapter 3).

1.2 The Jump-Diffusion Model

Throughout the paper, we are concerned with a financial market on which there are two assets, $\boldsymbol{S^0}$ and $\boldsymbol{S}$, traded continuously up to time $T > 0$. The asset $\boldsymbol{S^0}$ is a bond and its price at time $t \in [0,T]$ is given by

$$S_t^0 = \exp(rt),$$

where r is the constant risk-free interest rate. The uncertainty on the market is generated by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, equipped with a filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$, such that $\mathcal{F}_T \subset \mathcal{F}$. The filtration $\{\mathcal{F}_t\}$ is assumed to satisfy the usual conditions. The asset $\boldsymbol{S}$ is a stock whose price is governed by the following stochastic differential equation:

$$\left\{ \begin{array}{rcl} S_0 & = & y, \\ \frac{dS_t}{S_{t^-}} & = & \mu dt + \sigma dW_t + d\left(\sum_{j=1}^{N_t} U_j\right), \quad t \in (0,T], \end{array} \right. \tag{1.1}$$

where y is the spot price at time 0, $\boldsymbol{W}$ is a standard one-dimensional Brownian motion, $\boldsymbol{N}$ is a Poisson process with constant intensity $\lambda \geq 0$, and $\{U_j\}_{j\geq 1}$ is a sequence of iid square integrable random variables with values in $(-1, \infty)$ (so as to keep the stock price positive). The drift coefficient μ and the volatility σ are both constant. The variables U_j represent the relative amplitudes of jumps and the parameter λ accounts for their frequency.

The processes $\{W_t\}_{t\in[0,T]}$, $\{N_t\}_{t\in[0,T]}$, and $\{U_j\}_{j\geq 1}$ (the "process" $\{U_j\}_{j\geq 1}$ being in fact a sequence of random variables) are independent. We assume that the stock $\boldsymbol{S}$ pays dividends at the (constant) rate δ, where $r > \delta \geq 0$.

Equivalently, the process $\boldsymbol{S}$ can be written in the following form:

$$\left\{ \begin{array}{rcl} S_0 & = & y, \\ \frac{dS_t}{S_{t^-}} & = & \mu^* dt + \sigma dW_t + d\left(\sum_{j=1}^{N_t} U_j - \lambda \mathbb{E}^{\mathbb{P}}(U_1)t\right), \quad t \in (0,T], \end{array} \right. \tag{1.2}$$

where $\mu^* = \mu + \lambda \mathbb{E}^{\mathbb{P}}(U_1)$.

The generalized Poisson process $\{\sum_{j=1}^{N_t} U_j\}_{t\in[0,T]}$ can be identified to a random measure $v(dt, dy)$, defined on $[0,T] \times \mathbb{R}$. As a result, the compensated process $\{\sum_{j=1}^{N_t} U_j - \lambda \mathbb{E}(U_1)t\}_{t\in[0,T]}$ is identified to the measure

$$\tilde{v}(dt, dy) = v(dt, dy) - \lambda f_{U_1}(y)dtdy,$$

where f_{U_1} is the common pdf of the random variables $U_1, U_2, \ldots$. Using this notation, we can rewrite the model (1.2) as a stochastic integral equation:

$$S_t = y + \int_0^t \mu^* S_{s^-} ds + \int_0^t \sigma S_{s^-} dW_s + \int_0^t \int_{\mathbb{R}} S_{s^-} y \tilde{v}(ds, dy), \quad t \in [0, T]. \tag{1.3}$$

The model defined by the equivalent formulae (1.1), (1.2) and (1.3), is called the *jump-diffusion* model of the market. It generalizes the standard Black-Scholes model, obtained upon setting $\lambda = 0$. Contrary to the latter, it is not complete (provided that $\lambda > 0$ and the jumps are non-zero with positive probability), which essentially means that under the absence of arbitrage oppurtunities, there are many equivalent martingale measures, i. e. probability measures equivalent to $\mathbb{P}$ under which the process $\{\exp(\delta t) S_t / S_t^0\}_{t\in[0,T]}$ is a martingale [2]. Due to its incompleteness, the model is naturally used for modelling stock prices whose jumps arise from exogeneous events (such as natural disasters or interest rate announcements), rather than those whose jumps are intrinsic to the market. For a construction of non-Poissonian complete markets with discontinuous stock prices (with jumps induced by the trading noise), see the paper by Dritschel and Protter [DP99].

1.3 Equivalent Martingale Measures and Option Pricing

We begin by recalling a few definitions.

Definition 1.3.1 *Let* $\boldsymbol{X}$ *be a semimartingale with* $X_0 = 0$. *The* quadratic variation process *of* $\boldsymbol{X}$, *denoted by* $[\boldsymbol{X}, \boldsymbol{X}] = \{[X, X]_t\}_{t\geq 0}$ *is defined by*

$$[X, X]_t = X_t^2 - 2\int_0^t X_{s^-} dX_s.$$

Definition 1.3.2 *Let* $\boldsymbol{A}$ *be a finite variation process with* $A_0 = 0$, *with locally integrable total variation* [3]. *The unique predictable finite variation process* $\tilde{\boldsymbol{A}}$ *such that* $\boldsymbol{A} - \tilde{\boldsymbol{A}}$ *is a local martingale, is called the* compensator *of* $\boldsymbol{A}$.

[2] For the terminology, see Karatzas and Shreve [KS98].
[3] For the terminology, see Protter [Pro90].

Definition 1.3.3 *Let $\boldsymbol{X}$ be a semimartingale such that its quadratic variation process $[\boldsymbol{X}, \boldsymbol{X}]$ is locally integrable. Then the* conditional quadratic variation *of $\boldsymbol{X}$, denoted by $\langle \boldsymbol{X} \rangle = \{\langle X \rangle_t\}_{t \geq 0}$, exists and it is defined to be the compensator of $[\boldsymbol{X}, \boldsymbol{X}]$.*

Next, we define the stochastic exponential (also called the Doléans-Dade exponential) of a semimartingale.

Definition 1.3.4 *The stochastic exponential (the Doléans-Dade exponential) $\boldsymbol{Y}$ of a semimartingale $\boldsymbol{X}$, $X_0 = 0$, is denoted by $\boldsymbol{Y} = \mathcal{E}(\boldsymbol{X})$, and is defined as the unique solution of the stochastic integral equation*

$$Y_t = 1 + \int_0^t Y_{s^-} dX_s.$$

Explicitly, $\boldsymbol{Y} = \mathcal{E}(\boldsymbol{X})$ is given by the following formula

$$Y_t = \exp\left(X_t - \frac{1}{2}[X, X]_t\right) \prod_{0<s\leq t} (1 + \Delta X_s) \exp\left(-\Delta X_s + \frac{1}{2}(\Delta X_s)^2\right).$$

As mentioned in the paper by Pham [Pha97], each equivalent martingale measure $\mathbb{Q}^{\boldsymbol{p}}$, which turns the process $\{\exp(\delta t) S_t / S_t^0\}_{t\in[0,T]}$ into a martingale, can be characterized (in terms of its Radon-Nikodym density with respect to the original measure $\mathbb{P}$) in the following way:

$$\frac{d\mathbb{Q}^{\boldsymbol{p}}}{d\mathbb{P}} = \mathcal{E}(\boldsymbol{D})_T \mathcal{E}(\boldsymbol{J})_T, \tag{1.4}$$

where

$$D_t = -\int_0^t \vartheta_s dW_s \tag{1.5}$$

$$J_t = \int_0^t \int_{\mathbb{R}} (p_s(y) - 1)\tilde{v}(ds, dy). \tag{1.6}$$

The predictable processes $\boldsymbol{\vartheta} = \{\vartheta_t\}_{t\in[0,T]}$ and $\boldsymbol{p} = \{p_t(y)\}_{(t,y)\in[0,T]\times\mathbb{R}}$ are linked by

$$\mu^* - r + \delta = \vartheta_t \sigma + \lambda \int_{\mathbb{R}} y(1 - p_t(y)) f_{U_1}(y) dy, \tag{1.7}$$

together with the conditions $p_t(y) > 0$ and

$$\mathbb{E}^{\mathbb{P}}\left(\frac{d\mathbb{Q}^{\boldsymbol{p}}}{d\mathbb{P}}\right) = 1.$$

The process $\boldsymbol{\vartheta}$ is called the *market price of diffusion risk* and the process $\boldsymbol{p}$ — the *market price of jump risk*. In the sequel, we limit ourselves to such equivalent measures $\mathbb{Q}^{\boldsymbol{p}}$ that the corresponding process $\boldsymbol{p}$ satisfies

$$\forall\, \omega \in \Omega \quad p_t(y) = p(y) \quad \text{and} \quad \boldsymbol{p} \in L^2(\mathbb{P}).$$

By a Girsanov-type theorem, $\tilde{v}$ continues to be a homogeneous compensated random measure under $\mathbb{Q}^{\boldsymbol{p}}$. Its characteristics become

$$\lambda^{\boldsymbol{p}} = \lambda \int_{\mathbb{R}} p(y) f_{U_1}(y)dy \tag{1.8}$$

$$f_{U_1}^{\boldsymbol{p}}(y) = \frac{p(y)}{\int_{\mathbb{R}} p(y) f_{U_1}(y)dy} f_{U_1}(y). \tag{1.9}$$

Furthermore,

$$W_t^{\boldsymbol{p}} = W_t + \int_0^t \vartheta_s ds \tag{1.10}$$

is a Brownian motion under $\mathbb{Q}^{\boldsymbol{p}}$.

We denote by $\{S_s^t(y)\}_{s\geq t}$ the càdlàg version of the flow of the SDE (1.1). Changing measures and applying the Itô formula to the equation (1.1), we have almost surely under $\mathbb{Q}^{\boldsymbol{p}}$

$$\begin{aligned} S_s^t(y) &= y \exp\left(\left(r - \delta - \frac{\sigma^2}{2} - \lambda^{\boldsymbol{p}} \mathbb{E}^{\mathbb{Q}^{\boldsymbol{p}}}(U_1)\right)(s-t) + \sigma(W_s^{\boldsymbol{p}} - W_t^{\boldsymbol{p}})\right) \\ &\quad \times \prod_{j=N_t^{\boldsymbol{p}}+1}^{N_s^{\boldsymbol{p}}} (1+U_j), \end{aligned}$$

where $\boldsymbol{N}^{\boldsymbol{p}}$ is a homogeneous Poisson process with intensity $\lambda^{\boldsymbol{p}}$. The variables $\{U_j\}_{j\geq 1}$ have the common pdf $f_{U_1}^{\boldsymbol{p}}$ under $\mathbb{Q}^{\boldsymbol{p}}$.

Each equivalent martingale measure $\mathbb{Q}^{\boldsymbol{p}}$ defines an admissible (i. e. arbitrage-precluding) price of a given contingent claim. Given that $S_t = y$, the price at time t of a European option expiring at time T is equal to

$$V_E^{\boldsymbol{p}}(t,y) = \mathbb{E}^{\mathbb{Q}^{\boldsymbol{p}}}[e^{-r(T-t)} f(S_T^t(y))].$$

The function f defines the payoff from the option, eg in the case of the call option we have $f(x) = (x-K)^+$, and in the case of the put option — $f(x) = (K-x)^+$, where K is the preset strike price. The arbitrage-precluding price of the corresponding American option is given by

$$V_A^{\boldsymbol{p}}(t,y) = \sup_{\tau \in \mathcal{S}_{t,T}} \mathbb{E}^{\mathbb{Q}^{\boldsymbol{p}}}[e^{-r(\tau-t)} f(S_\tau^t(y))]. \tag{1.11}$$

Here, $\mathcal{S}_{t,T}$ is the set of all stopping times with values in $[t,T]$. It is known that $V_A^{\boldsymbol{p}}(t,y) \geq f(y)$, and that the optimal stopping time for the problem (1.11) is

$$\tau_{\boldsymbol{p}}^*(t,y) = \inf\{s \in [t,T] : V_A^{\boldsymbol{p}}(s, S_s^t(y)) = f(S_s^t(y))\}.$$

In other words, it is optimal to exercise the option the moment the option value falls to that of the payoff for immediate exercise. The domain $[0,T) \times \mathbb{R}_+$ is divided into the *continuation region* $\mathcal{C}^{\boldsymbol{p}}$:

$$\mathcal{C}^{\boldsymbol{p}} = \{(t,y) \in [0,T) \times \mathbb{R}_+ : V_A^{\boldsymbol{p}}(t,y) > f(y)\}$$

and the *stopping region* $\mathcal{S}^{\boldsymbol{p}}$:

$$\mathcal{S}^{\boldsymbol{p}} = [0,T) \times \mathbb{R}_+ \setminus \mathcal{C}^{\boldsymbol{p}}.$$

We denote by $V_{AC}^{\boldsymbol{p}}$ the price of the American call option:

$$V_{AC}^{\boldsymbol{p}}(t,y) = \sup_{\tau \in \mathcal{S}_{t,T}} \mathbb{E}^{\mathbb{Q}^{\boldsymbol{p}}}[e^{-r(\tau-t)}(S_\tau^t(y) - K)^+],$$

and by $V_{AP}^{\boldsymbol{p}}$ the price of the American put option:

$$V_{AP}^{\boldsymbol{p}}(t,y) = \sup_{\tau \in \mathcal{S}_{t,T}} \mathbb{E}^{\mathbb{Q}^{\boldsymbol{p}}}[e^{-r(\tau-t)}(K - S_\tau^t(y))^+].$$

The continuation region and the stopping region for the American call are denoted by $\mathcal{C}_{AC}^{\boldsymbol{p}}$ and $\mathcal{S}_{AC}^{\boldsymbol{p}}$, respectively. The respective regions for the American put are denoted by $\mathcal{C}_{AP}^{\boldsymbol{p}}$ and $\mathcal{S}_{AP}^{\boldsymbol{p}}$. We have the following proposition.

Proposition 1.3.1 *If $\delta = 0$, then the price of the American call option is equal to the price of the European call option:*

$$V_{AC}^{\boldsymbol{p}}(t,y) = \mathbb{E}^{\mathbb{Q}^{\boldsymbol{p}}}[e^{-r(T-t)}(S_T^t(y) - K)^+].$$

We then have $\tau_{\boldsymbol{p}}^*(t,y) \equiv T$ and $\mathcal{S}_{AC}^{\boldsymbol{p}} = \emptyset$.

The proof can be found in Merton [Mer73].

The following propositions can be proved by methods of Pham [Pha97].

Proposition 1.3.2 *If $\delta > 0$, then for all $t \in [0,T)$ there exists a critical stock price $b_{AC}^{\boldsymbol{p}}(t)$ above which the American call should be exercised early. We have*

$$\mathcal{C}_{AC}^{\boldsymbol{p}} = \{(t,y) \in [0,T) \times \mathbb{R}_+ : y \in (0, b_{AC}^{\boldsymbol{p}}(t))\}.$$

Proposition 1.3.3 *Similarly, for all $t \in [0,T)$, there exists a critical stock price $b^{\boldsymbol{p}}_{AP}(t)$ below which the American put should be exercised early. We have*

$$\mathcal{C}^{\boldsymbol{p}}_{AP} = \{(t,y) \in [0,T) \times \mathbb{R}_+ : y > b^{\boldsymbol{p}}_{AP}(t)\}.$$

Proposition 1.3.4 *The function $b^{\boldsymbol{p}}_{AC}(t)$ is nonincreasing, and the function $b^{\boldsymbol{p}}_{AP}(t)$ is nondecreasing on $[0,T)$.*

The functions $b^{\boldsymbol{p}}_{AC}$ and $b^{\boldsymbol{p}}_{AP}$ will be referred to as *optimal exercise boundaries* or *free boundaries.*

1.4 Schweizer's Minimal Martingale Measure

In order to price an American option in the jump-diffusion model, we have to first select an appropriate measure $\mathbb{Q}^{\boldsymbol{p}}$, which boils down to choosing the market price of jump risk $\boldsymbol{p}$. Merton [Mer76] sets $\boldsymbol{p} \equiv 1$ (the jump risk is "unpriced"). By the equation (1.7), the market price of diffusion risk $\boldsymbol{\vartheta}$ becomes

$$\boldsymbol{\vartheta} \equiv \frac{\mu^* - r + \delta}{\sigma},$$

which is identical to the market price of (diffusion) risk in the Black-Scholes model (see Karatzas and Shreve [KS98]). Note that the characteristics of the random measure $\tilde{v}$ (equations (1.8), (1.9)) do not change.

Recently it has become standard to adopt Schweizer's approach to select an "optimal" equivalent martingale measure in an incomplete market. In his paper [Sch95], Schweizer constructs the so-called *minimal martingale measure* (MMM) and suggests it as the optimal pricing measure. For a given semimartingale $\boldsymbol{Y}$ (satisfying a mild structure condition), the MMM $\hat{\mathbb{Q}}$ equivalent to the original measure $\mathbb{P}$ is characterized as the one which minimizes

$$D(\mathbb{Q},\mathbb{P}) := \left\| \frac{d\mathbb{Q}}{d\mathbb{P}} - 1 \right\|_{L^2_{\mathbb{P}}} = \sqrt{\mathrm{Var}\left(\frac{d\mathbb{P}}{d\mathbb{Q}}\right)}$$

over all signed local martingale measures $\mathbb{Q}$ for $\boldsymbol{Y}$. The MMM $\hat{\mathbb{Q}}$ is, in a sense, "as close as possible" to $\mathbb{P}$, and therefore the process $\boldsymbol{Y}$ "changes as little as possible" under $\hat{\mathbb{Q}}$. It is a strong argument in favour of the minimal martingale measure as the measure used for converting discounted stock price processes into martingales.

The minimal martingale measure for the jump-diffusion model with predictable coefficients (i. e. such that the jump component is modelled by $\rho_t dN_t$, where $\boldsymbol{\rho}$ is a predictable process and $\boldsymbol{N}$ is a Poisson process) is computed explicitly, for example, in the paper by Wiesenberg [Wie98]. In this section, we compute the minimal martingale measure for the jump-diffusion model with jumps of random magnitude (i. e. the model defined by the equivalent formulae (1.1), (1.2) and (1.3)).

Schweizer's algorithm [Sch95] for finding the minimal equivalent martingale measure for the semimartingale $\boldsymbol{Y}$ goes as follows.

1. Make sure that the semimartingale $\boldsymbol{Y}$ admits the decomposition

$$Y_t = Y_0 + M_t + \int_0^t \alpha_s d\langle M\rangle_s, \tag{1.12}$$

   where $\boldsymbol{M}$ is a $\mathbb{P}$-square-integrable local martingale with $M_0 = 0$. The process $\boldsymbol{\alpha}$ is predictable.

2. The Radon-Nikodym density of the minimal martingale measure $\hat{\mathbb{Q}}$ for $\boldsymbol{Y}$ is given by

$$\left.\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \mathcal{E}(\boldsymbol{L})_t, \tag{1.13}$$

   where

$$L_t = -\int_0^t \alpha_s dM_s. \tag{1.14}$$

3. If the process $\{\int_0^t \alpha_s^2 d\langle M\rangle_s\}_{t\in[0,T]}$ is deterministic, then the MMM $\hat{\mathbb{Q}}$ minimizes

$$D(\mathbb{Q},\mathbb{P}) = \left\|\frac{d\mathbb{Q}}{d\mathbb{P}} - 1\right\|_{L^2_{\mathbb{P}}} = \sqrt{\operatorname{Var}\left(\frac{d\mathbb{P}}{d\mathbb{Q}}\right)} \tag{1.15}$$

   over all signed local martingale measures for $\boldsymbol{Y}$.

In our case, we have $Y_t = \exp(\delta t)S_t/S_t^0$, which can be decomposed into

$$Y_t = S_0 + M_t + A_t,$$

where the martingale part M is equal to

$$M_t = \int_0^t S_{u^-}\sigma dW_u + \int_0^t S_{u^-} d\left(\sum_{j=1}^{N_u} U_j - \lambda\mathbb{E}^{\mathbb{P}}(U_1)u\right),$$

and the process A is equal to

$$A_t = \int_0^t S_{u^-}(\mu^* - r + \delta)du.$$

Denoting by $[\boldsymbol{X}, \boldsymbol{X}]^c$ the continuous part of $[\boldsymbol{X}, \boldsymbol{X}]$, we have

$$\begin{aligned} d\langle M\rangle_t &= d\widetilde{[M,M]}_t = \\ &= d\widetilde{\left([M,M]_t^c + \sum_{0<s\leq t}(\Delta M_s)^2\right)} = \\ &= S_{t^-}^2(\sigma^2 + \lambda\mathbb{E}^{\mathbb{P}}(U_1^2))dt. \end{aligned}$$

By (1.12), we must have

$$dA_t = \alpha_t d\langle M\rangle_t,$$

which yields

$$\alpha_t = \frac{1}{S_{t^-}}\frac{\mu^* - r + \delta}{\sigma^2 + \lambda\mathbb{E}^{\mathbb{P}}(U_1^2)}.$$

We denote

$$\rho = \frac{\mu^* - r + \delta}{\sigma^2 + \lambda\mathbb{E}^{\mathbb{P}}(U_1^2)}.$$

Since the process $\{\int_0^t \alpha_s^2 d\langle M\rangle_s\}_{t\in[0,T]}$ is deterministic, the MMM $\hat{\mathbb{Q}}$ indeed minimizes the "distance" defined by (1.15).

We now explicitly compute the Radon-Nikodym density of Schweizer's MMM for the process $\{\exp(\delta t)S_t/S_t^0\}_{t\in[0,T]}$. Substituting for $\boldsymbol{\alpha}$ and $\boldsymbol{M}$ in (1.14), we obtain

$$\begin{aligned} L_t &= -\int_0^t \alpha_s dM_s \\ &= -\int_0^t \rho\left(\sigma dW_s + d\left(\sum_{j=1}^{N_s} U_j - \lambda\mathbb{E}^{\mathbb{P}}(U_1)s\right)\right) \\ &= -\int_0^t \rho\sigma dW_s + \int_0^t -\rho d\left(\sum_{j=1}^{N_s} U_j - \lambda\mathbb{E}^{\mathbb{P}}(U_1)s\right) \\ &= -\int_0^t \rho\sigma dW_s + \int_0^t\int_{\mathbb{R}} -\rho y\tilde{v}(ds, dy). \end{aligned}$$

Comparing with (1.5) and (1.6), we get

$$\begin{aligned} D_t &= -\int_0^t \rho\sigma dW_s \\ J_t &= \int_0^t\int_{\mathbb{R}} -\rho y\tilde{v}(ds, dy), \end{aligned}$$

which yields

$$\begin{aligned}\vartheta_t &\equiv \rho\sigma \\ p(y) &= 1-\rho y.\end{aligned}$$

Denote by $\hat{\lambda}$ and $\hat{f}_{U_1}$ the characteristics of the random measure $\tilde{v}$ under the MMM $\hat{\mathbb{Q}}$. By (1.8) and (1.9) we have

$$\begin{aligned}\hat{\lambda} &= \lambda(1-\rho\mathbb{E}^{\mathbb{P}}(U_1)) \\ \hat{f}_{U_1}(y) &= \frac{1-\rho y}{1-\rho\mathbb{E}^{\mathbb{P}}(U_1)} f_{U_1}(y).\end{aligned}$$

Furthermore, $\hat{W}_t = W_t + \rho\sigma t$ is a Brownian motion under $\hat{\mathbb{Q}}$ (by (1.10)). Denoting

$$\hat{\mu} = r - \delta - \hat{\lambda}\mathbb{E}^{\hat{\mathbb{Q}}}(U_1),$$

the flow of the equation (1.1) becomes

$$S_s^t(y) = y\exp\left(\left(\hat{\mu} - \frac{\sigma^2}{2}\right)(s-t) + \sigma(\hat{W}_s - \hat{W}_t)\right)\prod_{j=\hat{N}_t+1}^{\hat{N}_s}(1+U_j), \quad s \geq t, \tag{1.16}$$

where $\{\hat{N}_t\}_{t\in[0,T]}$ is a Poisson process with intensity $\hat{\lambda}$. The jump relative sizes $U_1, U_2, \ldots$ have the common pdf $\hat{f}_{U_1}$ under $\hat{\mathbb{Q}}$.

Motivated by the above consideration, we introduce the following definition of the price of an American option in the jump-diffusion model defined by the eqivalent formulae (1.1), (1.2) and (1.3).

Definition 1.4.1 *The price $\hat{V}_A(t,y)$ of an American option at time $t \in [0,T]$, given that the price of the underlying instrument at time t is equal to y, is defined as follows*

$$\hat{V}_A(t,y) = \sup_{\tau\in\mathcal{S}_{t,T}} \mathbb{E}^{\hat{\mathbb{Q}}}[e^{-r(\tau-t)}f(S_\tau^t(y))],$$

where $\mathcal{S}_{t,T}$ is the set of all stopping times taking values in $[t,T]$, $\hat{\mathbb{Q}}$ is Schweizer's minimal martingale measure, r is the constant risk-free rate, f is the payoff from the option, and $\{S_s^t(y)\}_{s\geq t}$ is the process defined by the equation (1.16).

In other words, we define the price of an American option to be the arbitrage-precluding price under Schweizer's minimal martingale measure.

The prices of the American call and the American put computed under $\hat{\mathbb{Q}}$ are denoted by $\hat{V}_{AC}$ and $\hat{V}_{AP}$, respectively. The respective free boundaries defined in Propositions (1.3.2) and (1.3.3) are denoted by $\hat{b}_{AC}$ and $\hat{b}_{AP}$. The continuation region is denoted by $\hat{\mathcal{C}}$ and the stopping region — by $\hat{\mathcal{S}}$. We add the subscript AC for the American call and AP for the American put.

In the sequel, it will be our aim to compute an accurate approximation of $\hat{V}_A$, using the linear complementarity approach (see [HP98] for more details on linear complementarity in the context of option pricing).

1.5 The Variational Inequality and the Complementarity System

To introduce the variational inequality and the complementarity system appropriate for our problem, we first make the usual logarithmic change of variable. We set $X_t = \log(S_t)$ and $x = \log(y)$, and denote

$$\begin{aligned} Z_j &= \log(1+U_j) \\ \psi(x) &= f(e^x) \\ X_s^t(x) &= x + \left(\hat{\mu} - \frac{\sigma^2}{2}\right)(s-t) + \sigma(\hat{W}_s - \hat{W}_t) + \sum_{j=\hat{N}_t+1}^{\hat{N}_s} Z_j, \quad s \geq t \\ u^*(t,x) &= \sup_{\tau \in \mathcal{S}_{t,T}} \mathbb{E}^{\hat{\mathbb{Q}}}[e^{-r(\tau-t)}\psi(X_\tau^t(x))]. \end{aligned} \tag{1.17}$$

We have $\hat{V}_A(t,y) = u^*(t,\log(y))$.

We denote by $\hat{C}$ the *logarithmic continuation region*:

$$\hat{C} = \{(t,x) \in [0,T) \times \mathbb{R} : u^*(t,x) > \psi(x)\}.$$

The *logarithmic stopping region* is denoted by $\hat{S}$ and defined to be the complement of $\hat{C}$. We add the subscript AC for the American call and AP for the American put. We have

$$\begin{aligned} \hat{C}_{AC} &= \{(t,x) \in [0,T) \times \mathbb{R} : x < \log(\hat{b}_{AC}(t))\} \\ \hat{C}_{AP} &= \{(t,x) \in [0,T) \times \mathbb{R} : x > \log(\hat{b}_{AP}(t))\}. \end{aligned}$$

The functions $\log(\hat{b}_{AC})$ and $\log(\hat{b}_{AP})$ will be referred to as *logarithmic optimal exercise boundaries* or *logarithmic free boundaries.*

Assume that the random variables Z_j have the common pdf $g = g(x)$ under $\hat{\mathbb{Q}}$. Since $\hat{f}_{U_1}$ is the pdf of the variables U_j under $\hat{\mathbb{Q}}$, we have

$$g(x) = \hat{f}_{U_1}(e^x - 1)e^x.$$

The function $u^* = u^*(t,x)$ *formally* satisfies the following complementarity system

$$\left\{ \begin{array}{rcl} u(T,x) - \psi(x) & = & 0 \\ u(t,x) - \psi(x) & \geq & 0 \\ \frac{\partial u}{\partial t} + \mathcal{L}_{BS}u + Bu & \leq & 0 \\ \left(\frac{\partial u}{\partial t} + \mathcal{L}_{BS}u + Bu\right)(u - \psi) & = & 0, \end{array} \right. \tag{1.18}$$

where $\mathcal{L}_{BS}$ is the standard Black-Scholes operator

$$\mathcal{L}_{BS}u = \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} + \left(\hat{\mu} - \frac{\sigma^2}{2}\right)\frac{\partial u}{\partial x} - ru$$

and B is an integral operator resulting from jumps

$$(Bu)(t,x) = \hat{\lambda}\left(\int_{-\infty}^{\infty} u(t,x+z)g(z)dz - u(t,x)\right).$$

The proof of this fact follows by dynamic programming and can be accomplished by methods of Bensoussan and Lions [BL78], [BL82], and Jaillet *et al.* [JLL90].

In order to formulate the problem (1.18) as a variational inequality, we introduce the following function spaces:

$$\begin{array}{rcl} H_\alpha & = & L^2(\mathbb{R}, e^{-\alpha|x|}), \quad \alpha > 0 \\ V_\alpha & = & \{f \in H_\alpha : f' \in H_\alpha\}. \end{array} \tag{1.19}$$

Here, f' denotes the distributional derivative of f. The space V_α is a weighted Sobolev space.

Furthermore, we introduce the following spaces of functions $u = u(t,x) : [0,T] \times \mathbb{R} \mapsto \mathbb{R}$.

$$\begin{array}{rcl} L^2([0,T]; H_\alpha) & = & \left\{u : \int_0^T \|u(t,\cdot)\|^2_{H_\alpha} dt < \infty\right\} \\ L^2([0,T]; V_\alpha) & = & \left\{u : \int_0^T \|u(t,\cdot)\|^2_{V_\alpha} dt < \infty\right\}. \end{array}$$

We denote by $(\cdot,\cdot)_\alpha$ the inner product on H_α. We further define

$$\begin{aligned} a^\alpha(u,w) &= \frac{\sigma^2}{2}\int_{\mathbb{R}} \frac{\partial u}{\partial x}\frac{\partial w}{\partial x} e^{-\alpha|x|}dx + r\int_{\mathbb{R}} uwe^{-\alpha|x|}dx \\ &\quad -\int_{\mathbb{R}} \left(\frac{\sigma^2\alpha}{2}\mathrm{sign}(x) + \hat{\mu} - \frac{\sigma^2}{2}\right)\frac{\partial u}{\partial x} we^{-\alpha|x|}dx \\ b^\alpha(u,w) &= -\int_{\mathbb{R}} (Bu)we^{-\alpha|x|}dx. \end{aligned}$$

The following theorem comes from the paper by Zhang [Zha97].

Theorem 1.5.1 *If $\psi \in V_\alpha$ for some $\alpha > 0$ and if*

$$\forall\,\alpha > 0 \quad \mathbb{E}^{\hat{\mathbb{Q}}} e^{\alpha|Z_1|} < \infty, \tag{1.20}$$

then there exists a unique function $u \in L^2([0,T];V_\alpha)$ satisfying $\partial u/\partial t \in L^2([0,T];H_\alpha)$, such that

$$\left\{ \begin{array}{rcl} u(T,x) - \psi(x) &=& 0 \\ u(t,x) - \psi(x) &\geq& 0 \\ -\left(\frac{\partial u}{\partial t}, w-u\right) + a^\alpha(u,w-u) + b^\alpha(u,w-u) &\geq& 0, \quad \forall\, w \in V_\alpha, w \geq \psi. \end{array} \right. \tag{1.21}$$

Furhtermore, the unique solution of the variational inequality (1.21) is equal to the function $u^* = u^*(t,x)$ defined by (1.17).

The proof follows by methods of Bensoussan and Lions [BL78], [BL82], and Jaillet *et al.* [JLL90].

The theorem below specifies some regularity results for the function u^*.

Theorem 1.5.2 *Assume that the condition (1.20) is satisfied. If the function ψ is Lipschitz continuous, and if the function $f(x) = \psi(\log(x))$ is convex, then the unique solution $u = u^*$ of the variational inequality (1.21) admits distributional partial derivatives $\partial u/\partial t$, $\partial u/\partial x$, and $\partial^2 u/\partial x^2$, locally bounded on $[0,T)\times\mathbb{R}$. The operator Bu is also locally bounded on $[0,T)\times\mathbb{R}$.*

Furthermore, the function $\partial u/\partial x$ is continuous on $[0,T)\times\mathbb{R}$.

This theorem is stated and proved in Zhang [Zha97].

Theorem 1.5.2 provides a justification for the "strong" formulation (1.18). Indeed, once we know that the (unique) solution u of the variational inequality (1.21) satisfies the regularity conditions specified in Theorem 1.5.2, we can rewrite the variational inequality (1.21) as the complementarity system (1.18). For details, see the books by Bensoussan and Lions [BL78], [BL82].

As the problem (1.18) is easier to deal with numerically than the problem (1.21), we shall from now on concentrate on the former one.

The following theorem concerns the behaviour of u^* in the logarithmic continuation region.

Theorem 1.5.3 *In the logarithmic continuation region $\hat{C}$, the function u^* satisfies*

$$\frac{\partial u^*}{\partial t} + \mathcal{L}_{BS} u^* + B u^* = 0.$$

The proof can be found in the paper by Pham [Pha97].

We will now localize the complementarity system (1.18), and then discretize it using the finite difference method. Then we will concentrate on the resulting linear complementarity problem (LCP).

1.6 Localization and Discretization — the Linear Complementarity Problem

1.6.1 Localization

To make the system (1.18) suitable for numerical solution, we localize it by limiting it to the rectangle $[0,T] \times [X^l, X^u]$ and introducing appropriate boundary conditions. We introduce the function $\tilde{u}(\cdot,\cdot)$, defined on $[0,T]\times\mathbb{R}$, and satisfying the following localized complementarity system:

$$\left\{ \begin{array}{rcll} \tilde{u}(T,x) - \psi(x) & = & 0 & \\ \tilde{u}(t,x) - \psi(x) & \geq & 0 & \\ \frac{\partial \tilde{u}}{\partial t} + \mathcal{L}_{BS}\tilde{u} + B\tilde{u} & \leq & 0 & \\ \left(\frac{\partial \tilde{u}}{\partial t} + \mathcal{L}_{BS}\tilde{u} + B\tilde{u}\right)(\tilde{u} - \psi) & = & 0 & \\ \tilde{u}(t,x) - \psi(x) & = & 0 & \text{on} \quad [0,T] \times \left(\mathbb{R} \setminus (X^l, X^u)\right). \end{array} \right. \tag{1.22}$$

The function $\tilde{u}(\cdot,\cdot)$ is well-defined, since the system (1.22) has a unique solution, which can be shown using the methodology outlined in Section 1.5. The following theorem holds.

Theorem 1.6.1 *The function $\tilde{u} = \tilde{u}(t,x)$ converges to $u^* = u^*(t,x)$, uniformly on compact subsets of $[0,T]\times\mathbb{R}$, as $(X^l, X^u) \to (-\infty, \infty)$.*

The proof can be found in the paper by Zhang [Zha97].

1.6.2 Discretization of the Partial Differential Operator

In order to discretize the system (1.22) using the finite difference method, we divide the interval $[X^l, X^u]$ into I subintervals of length $\Delta x = (X^u - X^l)/I$, and the interval $[0, T]$ into N subintervals of length $\Delta t = T/N$. To simplify the notation, we skip the tilde in $\tilde{u}$.

We approximate the partial derivatives $\partial u/\partial t$ and $\partial^2 u/\partial x^2$ in the following way:

$$
\begin{aligned}
\left.\frac{\partial u}{\partial t}\right|_{(t,x)} &\approx \frac{u(t+\Delta t, x) - u(t,x)}{\Delta t} \\
\left.\frac{\partial^2 u}{\partial x^2}\right|_{(t,x)} &\approx \theta \frac{u(t, x+\Delta x) - 2u(t,x) + u(t, x-\Delta x)}{(\Delta x)^2} + \\
&+ (1-\theta)\frac{u(t+\Delta t, x+\Delta x) - 2u(t+\Delta t, x)}{(\Delta x)^2} + \\
&+ (1-\theta)\frac{u(t+\Delta t, x-\Delta x)}{(\Delta x)^2}.
\end{aligned}
$$

The parameter $\theta \in [0, 1]$ is usually set to 0 (which yields the so-called *explicit* method), to 1/2 (the *Crank-Nicholson* method — particularly well suited for equations), or to 1 (the *implicit* method). In the experiments, we use the implicit method due to its good convergence properties.

The partial derivative $\partial u/\partial x$ is approximated using:

1. either the *upwind* scheme:

   - if $\hat{\mu} - \frac{\sigma^2}{2} > 0$,

$$
\begin{aligned}
\left.\frac{\partial u}{\partial x}\right|_{(t,x)} &\approx \theta \frac{4u(t, x+\Delta x) - 3u(t,x) - u(t, x+2\Delta x)}{2\Delta x} + \\
&+ (1-\theta)\frac{4u(t+\Delta t, x+\Delta x) - 3u(t+\Delta t, x)}{2\Delta x} + \\
&+ (1-\theta)\frac{-u(t+\Delta t, x+2\Delta x)}{2\Delta x}
\end{aligned}
$$

- if $\hat{\mu} - \frac{\sigma^2}{2} < 0$,

$$\begin{aligned}
\left.\frac{\partial u}{\partial x}\right|_{(t,x)} &\approx -\theta \frac{4u(t, x-\Delta x) - 3u(t,x) - u(t, x-2\Delta x)}{2\Delta x} + \\
&- (1-\theta)\frac{4u(t+\Delta t, x-\Delta x) - 3u(t+\Delta t, x)}{2\Delta x} + \\
&- (1-\theta)\frac{-u(t+\Delta t, x-2\Delta x)}{2\Delta x}
\end{aligned}$$

2. or the *no-upwind* scheme (the usual central difference):

$$\begin{aligned}
\left.\frac{\partial u}{\partial x}\right|_{(t,x)} &\approx \theta \frac{u(t, x+\Delta x) - u(t, x-\Delta x)}{2\Delta x} + \\
&+ (1-\theta)\frac{u(t+\Delta t, x+\Delta x) - u(t+\Delta t, x-\Delta x)}{2\Delta x}.
\end{aligned}$$

The upwind scheme is recommended by Huang and Pang [HP98]. It speeds up the convergence of the iterative PSOR method (see Section 1.8 for the algorithm), and forces convergence where the no-upwind scheme is unstable. The linear programming method (Chapter 2) requires the use of the latter scheme.

1.6.3 Discretization of the Integral Operator — a New Scheme

To discretize the integral operator $Bu = \hat{\lambda} \int_{-\infty}^{\infty} u(t, x+z)g(z)dz - u(t,x)$, we approximate $g = g(x)$ (the pdf of the jump relative size under Schweizer's measure) by

$$g_i = \frac{g(i\Delta x)}{\sum_{\{g(i\Delta x)>0\}} g(i\Delta x)\Delta x}, \tag{1.23}$$

so that we always have

$$\sum_{\{g_i>0\}} g_i \Delta x = 1. \tag{1.24}$$

Zhang [Zha97], and Huang and Pang [HP98] after her, simply set $g_i = g(i\Delta x)$, which results in the fact that the discrete approximation $\{g_i\}_{g_i>0}$ of the pdf $g = g(x)$ does not "integrate" exactly to one ($\sum_{\{g_i>0\}} g_i\Delta x \neq 1$). This may have adverse consequences for the accuracy of the numerical solution to the discretized system (1.22). It should be borne in mind that the sequence $\{g_i\Delta x\}_{g_i>0}$ is, in a sense, a sequence of transition probabilities, and it is therefore essential that the condition (1.24) be fulfilled. Section 3.3 describes in brief what may happen if this condition is violated (which is the case when Zhang's [Zha97] approximation is used).

Bearing in mind that g_i's are defined by the equation (1.23), and denoting $\psi_i = \psi(i\Delta x)$, we discretize the operator B in the following way:

$$\begin{aligned} Bu|_{(t,x)} &\approx \hat{\lambda}\bar{\theta}\left(\sum_{j=-\infty}^{\infty} u(t, x+j\Delta x)g_j\Delta x - u(t,x)\right) + \\ &+ \hat{\lambda}(1-\bar{\theta})\left(\sum_{j=-\infty}^{\infty} u(t+\Delta t, x+j\Delta x)g_j\Delta x - u(t+\Delta t, x)\right). \end{aligned}$$

In the experiments, we set $\bar{\theta} = 1$. Zhang [Zha97] uses the less accurate substitution $\bar{\theta} = 0$, calling the resulting scheme "semi-implicit" (with $\theta = 1$).

Each infinite sum $\sum_{j=-\infty}^{\infty} u(t+k\Delta t, x+j\Delta x)g_j\Delta x$ for $k = 0, 1$, is split up into two:

$$\begin{aligned} \sum_{j=-\infty}^{\infty} u(t+k\Delta t, x+j\Delta x)g_j\Delta x &= \sum_{j\in\{1,2,\ldots,I-1\}-i} u(t+k\Delta t, x+j\Delta x)g_j\Delta x \\ &+ \sum_{j\notin\{1,2,\ldots,I-1\}-i} \psi_{i+j}g_j\Delta x. \end{aligned}$$

Obviously enough, the second term on the right-hand side is approximated by a finite sum.

1.6.4 The Linear Complementarity Problem

The above discretization of (1.22) leads to a discrete problem, whose exact solution will be denoted by $\{u_i^n\}$, $i = 0, 1, \ldots, I$, $n = 0, 1, \ldots, N$. Each element u_i^n of the discrete solution will approximate the actual solution of the problem (1.22):

$$u_i^n \approx \tilde{u}(n\Delta t, X^l + i\Delta x).$$

The discrete problem is in fact a sequence of *linear complementarity problems* (LCP's) and has the form:

$$\left\{ \begin{aligned} u^n &\geq \psi \\ Mu^n + q^{n+1} &\geq 0 \\ (u^n - \psi)^T(Mu^n + q^{n+1}) &= 0 \end{aligned} \right. \tag{1.25}$$

for each $n = N-1, N-2, \ldots, 0$, with $u^n = (u_1^n, u_2^n, \ldots, u_{I-1}^n)^T$ and $\psi = (\psi_1, \psi_2, \ldots, \psi_{I-1})^T$. The column vector q^{n+1} contains input from the previous time step and information about the boundary conditions. The square matrix M is a sum of two matrices, $\tilde{M}$ and G, where $\tilde{M}$ is the result of discretizing the partial differential operator $-\frac{\partial}{\partial t} - \mathcal{L}_{BS}$, and G is the result of discretizing the integral operator $-B$. If the upwind scheme is used, then $\tilde{M}$ is a pentadiagonal $(I-1) \times (I-1)$ matrix of the form

$$\tilde{M} = \begin{pmatrix} c & d & e & 0 & 0 & \cdots & 0 \\ b & c & d & e & 0 & \cdots & 0 \\ a & b & c & d & e & & 0 \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 & & a & b & c & d & e \\ 0 & \cdots & 0 & a & b & c & d \\ 0 & \cdots & 0 & 0 & a & b & c \end{pmatrix}, \tag{1.26}$$

where

$$\begin{aligned} a &= \frac{\theta}{2\Delta x}\left(\frac{\sigma^2}{2} - \hat{\mu}\right)^+ \\ b &= -\frac{\theta\sigma^2}{2(\Delta x)^2} - \frac{2\theta}{\Delta x}\left(\frac{\sigma^2}{2} - \hat{\mu}\right)^+ \\ c &= \frac{1}{\Delta t} + \frac{\theta\sigma^2}{(\Delta x)^2} + \frac{3\theta}{2\Delta x}\left|\hat{\mu} - \frac{\sigma^2}{2}\right| + r \\ d &= -\frac{\theta\sigma^2}{2(\Delta x)^2} - \frac{2\theta}{\Delta x}\left(\hat{\mu} - \frac{\sigma^2}{2}\right)^+ \\ e &= \frac{\theta}{2\Delta x}\left(\hat{\mu} - \frac{\sigma^2}{2}\right)^+ . \end{aligned}$$

If the no-upwind scheme is used, then $\tilde{M}$ is a tridiagonal matrix of the form

$$\tilde{M} = \begin{pmatrix} b' & c' & 0 & \cdots & 0 \\ a' & b' & c' & & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & & a' & b' & c' \\ 0 & \cdots & 0 & a' & b' \end{pmatrix}, \tag{1.27}$$

where

$$a' = -\frac{\theta\sigma^2}{2(\Delta x)^2} + \frac{\theta}{2\Delta x}\left(\hat{\mu} - \frac{\sigma^2}{2}\right)$$

$$
\begin{aligned}
b' &= \frac{1}{\Delta t} + \frac{\theta\sigma^2}{(\Delta x)^2} + r \\
c' &= -\frac{\theta\sigma^2}{2(\Delta x)^2} - \frac{\theta}{2\Delta x}\left(\hat{\mu} - \frac{\sigma^2}{2}\right).
\end{aligned}
$$

The matrix G is given by

$$
G = \bar{\theta}\hat{\lambda}I - \bar{\theta}\hat{\lambda}\Delta x \begin{pmatrix}
g_0 & g_1 & g_2 & g_3 & \cdots & g_{I-2} \\
g_{-1} & g_0 & g_1 & g_2 & \cdots & g_{I-3} \\
g_{-2} & g_{-1} & g_0 & g_1 & \cdots & g_{I-4} \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
g_{3-I} & \cdots & g_{-2} & g_{-1} & g_0 & g_1 \\
g_{2-I} & \cdots & g_{-3} & g_{-2} & g_{-1} & g_0
\end{pmatrix}. \tag{1.28}
$$

1.7 Matrix Classes and the LCP

We first recall definitions of some matrix classes.

Definition 1.7.1 *A real matrix $M = (m_{ij})_{i,j=1}^n$ is said to be* row strictly diagonally dominant *if*

$$\forall\, i \quad m_{ii} - \sum_{j \neq i} |m_{ij}| > 0.$$

Definition 1.7.2 *A real matrix $M = (m_{ij})_{i,j=1}^n$ is said to be* column strictly diagonally dominant *if*

$$\forall\, i \quad m_{ii} - \sum_{j \neq i} |m_{ji}| > 0.$$

Definition 1.7.3 *A real square matrix is said to be* strictly diagonally dominant *if it is both row strictly diagonally dominant and column strictly diagonally dominant.*

Definition 1.7.4 *A real matrix $M = (m_{ij})_{i,j=1}^n$ is said to be* coercive *if*

$$\exists\, C > 0 \quad \forall\, x \in \mathbb{R}^n \quad x^T M x \geq C x^T x.$$

Definition 1.7.5 *A real matrix $M = (m_{ij})_{i,j=1}^n$ is said to be* type Z *if*

$$\forall\, i, j, i \neq j \quad m_{ij} \leq 0.$$

Consider an LCP of the form

$$\begin{cases} v & \geq & \psi \\ Mv+q & \geq & 0 \\ (v-\psi)^T(Mv+q) & = & 0 \end{cases} \tag{1.29}$$

We have the following properties.

1. If a real square matrix is strictly diagonally dominant, then it is coercive.

2. If the matrix M is coercive, then the LCP (1.29) has a unique solution (see Zhang [Zha97]).

3. If M is a strictly diagonally dominant matrix, then the iterative PSOR method for solving the LCP (1.29) (the method is described in detail in Section 1.8) converges for all values of $\omega \in (0, \bar{\omega})$, where $\bar{\omega} = \bar{\omega}(M)$ and $1 < \bar{\omega} \leq 2$ (see Huang and Pang [HP98]).

4. Under mild conditions on the discretization steps Δx and Δt, the matrix $M = \tilde{M} + G$ of the LCP (1.25) is strictly diagonally dominant for both the upwind and the no-upwind discretization scheme. Consequently, the LCP (1.25) has a unique solution for each $n = N-1, N-2, \ldots, 0$. This solution can be found by the PSOR method.

The key property concerning type Z matrices (Definition 1.7.5) will be stated in Chapter 2.

1.8 The PSOR Method

As mentioned in Section 1.1, there are two main approaches to solving LCP's of the form (1.29). The parametric principal pivoting (PPP) is probably the most popular direct method (see Cottle, Pang and Stone [CPS92] and Huang and Pang [HP98]). The projected successive over-relaxation (PSOR) is by far the most popular iterative method. It is discussed, for example, in Murty [Mur97] and, in the context of LCP's arising in option pricing in the Black-Scholes model, in Wilmott, Dewynne and Howison [WDH93].

Consider again the LCP (1.29) and assume that the $J \times J$ matrix M is strictly diagonally dominant, so that the PSOR method is bound to converge. Denote by $v^{(n)} = (v_1^{(n)}, v_2^{(n)}, \ldots, v_J^{(n)})$ the nth iterate (which approximates the original solution to (1.29)). The PSOR algorithm goes as follows.

1. Compute the (conservative) upper limit $\bar{\omega}$ on the relaxation parameter ω using the following formula:

$$\bar{\omega} = 2 \min_i \frac{m_{ii}}{\sum_j |m_{ij}|}. \tag{1.30}$$

   If M is symmetric, set $\bar{\omega} = 2$.

2. Pick any $\omega \in (0, \bar{\omega})$ and an arbitrary initial vector $v^{(0)} \geq \psi$. Choose a suitable accuracy parameter $\varepsilon > 0$.

3. For all $j = 1, 2, \ldots, J$ set

$$v_j^{(n)} = \max\left(v_j^{(n-1)} - \frac{\omega}{m_{jj}} \left(\sum_{k=1}^{j-1} m_{jk} v_k^{(n)} + \sum_{k=j}^{n} m_{jk} v_k^{(n-1)} + q_j \right), \psi_j \right)$$

   to obtain the successive iterates $v^{(1)}, v^{(2)}, \ldots$.

4. Terminate when $\operatorname{dist}(v^{(n)}, v^{(n-1)}) < \varepsilon$, where $\operatorname{dist}(\cdot, \cdot)$ is an appropriately defined distance function, and take $v^{(n)}$ to be the (approximate) solution of the LCP (1.29).

Dewynne in his "Option pricing demonstration code"[4] uses the PSOR algorithm with

$$\operatorname{dist}(v^{(n)}, v^{(n-1)}) = \frac{\sum_{j=1}^{J} j |v_j^{(n)} - v_j^{(n-1)}|}{\max\left(1, \sum_{j=1}^{J} |v_j^{(n)}|\right)}. \tag{1.31}$$

Following Dempster *et al.* [DHR98], [DH97], [DH99] who test their linear programming method against PSOR in the Black-Scholes framework, we compare these two methods in the jump-diffusion setting. In Chapter 2, we first propose a new LP algorithm appropriate for the jump-diffusion model, and then examine its efficiency by comparing it with PSOR.

1.9 The Convergence Theorem

The aim of this section is to state the theorem which specifies the conditions under which the solution of the LCP (1.25) converges to the solution of the localized problem (1.22) (which in turn converges to the solution of the

[4] "Option pricing demonstration code", ©Oxford Financial Software, 1996, is a freeware for pricing American and European options in the Black-Scholes model. It is available at `http://www.maths.soton.ac.uk/staff/Dewynne/ofs-demo1.html`.

original problem (1.18), see Theorem 1.6.1). Throughout this section, we assume that the no-upwind scheme is used to discretize the problem (1.22), and that Zhang's [Zha97] method is used to approximate the pdf of the jump relative size $g = g(x)$ (see subsection 1.6.3 for details). Similar results can probably be obtained in the case of the upwind scheme and the new approximation method for g.

After Zhang [Zha97], we introduce the following stability conditions: there exist two constants $\beta_1 > 0$, $\beta_2 > 0$ such that

$$1 - (1-\theta)\frac{8C_1^2C_2^2}{\sigma^2}\frac{\Delta t}{(\Delta x)^2} \geq \beta_1 \tag{1.32}$$

$$1 - \bar{\theta}C_3\Delta t - \theta C_2\frac{\Delta t}{(\Delta x)^2} \geq \beta_2 \tag{1.33}$$

$$1 - \left(\hat{\lambda}(1-\bar{\theta}) + r(1-\theta)\right)\Delta t - (1-\theta)\sigma^2\frac{\Delta t}{(\Delta x)^2} > 0, \tag{1.34}$$

where

$$\begin{aligned} C_1 &= \sqrt{5} \quad \text{when} \quad \Delta x < 1 \\ C_2 &= 3(\sigma^2 + |\hat{\mu}| + r) \\ C_3 &= \hat{\lambda}\sqrt{10}. \end{aligned}$$

For a given solution $(u_i^n), i = 1, 2, \ldots I-1, n = 0, 1, \ldots, N-1$ of the LCP (1.25) we define a piecewise constant function $u_I^N(t,x)$ by

$$u_I^N(t,x) = \sum_{n=1}^{N}\sum_{i=1}^{I-1} u_i^n \mathbf{1}_{(X^l+\Delta x(i-1/2), X^l+\Delta x(i+1/2)]}(x)\mathbf{1}_{((n-1)\Delta t, n\Delta t]}(t).$$

We further define the difference operator D:

$$Df(t,x) = \frac{1}{\Delta x}\left[f\left(t, x+\frac{\Delta x}{2}\right) - f\left(t, x-\frac{\Delta x}{2}\right)\right].$$

The following convergence theorem holds.

Theorem 1.9.1 *Suppose that the piecewise constant function $u_I^N(t,x)$ was created from the solution (u_i^n) of the LCP (1.25), and the function $\tilde{u}$ is the solution of the localized problem (1.22). Let $\theta, \bar{\theta} \in [0,1]$. Assume that $\Delta t/(\Delta x)^2 < \beta$, where β is a sufficiently small constant, so that the stability conditions (1.32), (1.33) and (1.34) are satisfied. Suppose further that*

$\psi \in V_\alpha$, where V_α is the weighted Sobolev space defined by (1.19). We have, as $(\Delta t, \Delta x) \to (0,0)$,

$$\begin{aligned} u_I^N(\cdot,\cdot) &\to \tilde{u}(\cdot,\cdot) \quad \textit{strongly in} \quad L^2([0,T]\times(X^l,X^u)) \\ Du_I^N(\cdot,\cdot) &\to \frac{\partial \tilde{u}}{\partial x}(\cdot,\cdot) \quad \textit{weakly in} \quad L^2([0,T]\times(X^l,X^u)). \end{aligned}$$

The proof can be found in Zhang [Zha97].

The above theorem shows that in order to value an American option in the jump-diffusion model, it suffices to solve an LCP of the form (1.25). In this chapter, we have shown that this can be done using the PSOR method. In Chapter 2, we propose a new method of solving the LCP (1.25), based on a linear programming formulation of (1.25).

Chapter 2

A New Linear Programming Method

In this chapter, we first reformulate the discrete problem (1.25) as a linear programming problem, and then propose an algorithm for solving the resulting linear program. Subsequently, we compare our algorithm with the PSOR algorithm of Section 1.8. Throughout the chapter, we use the new approximation scheme for the pdf of the relative jump size $g = g(x)$ (see subsection 1.6.3).

2.1 A Linear Programming Formulation of the LCP

The following theorem comes from Dempster and Hutton [DH99].

Theorem 2.1.1 *If the matrix M is type Z, then the linear complementarity problem (1.29) is equivalent to the following linear programming problem: for a fixed arbitrary column vector $c > 0$,*

$$\begin{cases} \textit{minimize} & c^T v \\ \textit{subject to} & \begin{cases} v & \geq & \psi \\ Mv + q & \geq & 0. \end{cases} \end{cases} \tag{2.1}$$

Under mild conditions on the discretization steps Δx and Δt, the matrix $M = \tilde{M} + G$ of the LCP (1.25) is type Z for the no-upwind discretization scheme. Therefore, the LCP (1.25) can be equivalently formulated as the sequence of linear programs

$$\left\{ \begin{array}{ll} \text{minimize} & c^T u^n \\ \text{subject to} & \left\{ \begin{array}{rcl} u^n & \geq & \psi \\ Mu^n + q^{n+1} & \geq & 0, \end{array} \right. \end{array} \right. \tag{2.2}$$

where c is a fixed arbitrary column vector such that $c > 0$.

2.2 Motivation for the LP Algorithm

We first motivate the linear programming algorithm by making the following observations.

1. The assertions of Propositions 1.3.2, 1.3.3, and 1.3.4 carry over to the discrete case.

   (a) Suppose that $\{u_i^n\}$ is the solution of the discretized American put problem of the form (1.25). For each time index $n = N-1, N-2, \ldots, 0$ there exists a space index k_n such that $u_i^n = \psi_i$ for all $i \in \{1, 2, \ldots, k_n\}$, and $u_i^n > \psi_i$ for all $i \in \{k_n + 1, k_n + 2, \ldots, I-1\}$. Moreover, the "discrete" logarithmic free boundary k_n is nondecreasing, i. e. $k_n \geq k_{n-1}$ for $n = 1, 2, \ldots, N-1$.

   (b) Similarly, if $\{u_i^n\}$ is the solution of the discretized American call problem with $\delta > 0$, then for each $n = N-1, N-2, \ldots, 0$ there exists l_n such that $u_i^n > \psi_i$ for all $i \in \{1, \ldots, l_n - 1\}$, and $u_i^n = \psi_i$ for all $i \in \{l_n, l_n + 1, \ldots, I-1\}$. The discrete logarithmic free boundary l_n is nonincreasing, i. e. $l_n \leq l_{n-1}$ for $n = 1, 2, \ldots, N-1$.

   Figure 2.1 illustrates the behaviour of the free boundary for the American put. The continuation region lies above the boundary, and the stopping region — below it.

2. Theorem 1.5.3 also has its discrete counterpart. To fix ideas, let us concentrate on the American put. Suppose that the operator $-\frac{\partial}{\partial t} - \mathcal{L}_{BS} - B$ is discretized only in the logarithmic continuation region, and let M^{k_n} denote the matrix resulting from this discretization. The matrix M^{k_n} will operate on the vector $u^{n,k_n} = (u^n_{k_n+1}, u^n_{k_n+2}, \ldots, u^n_{I-1})^T$, which is the part of the vector u^n lying in the logarithmic continuation region. Denote by q^{n+1,k_n} the vector carrying input from the previous

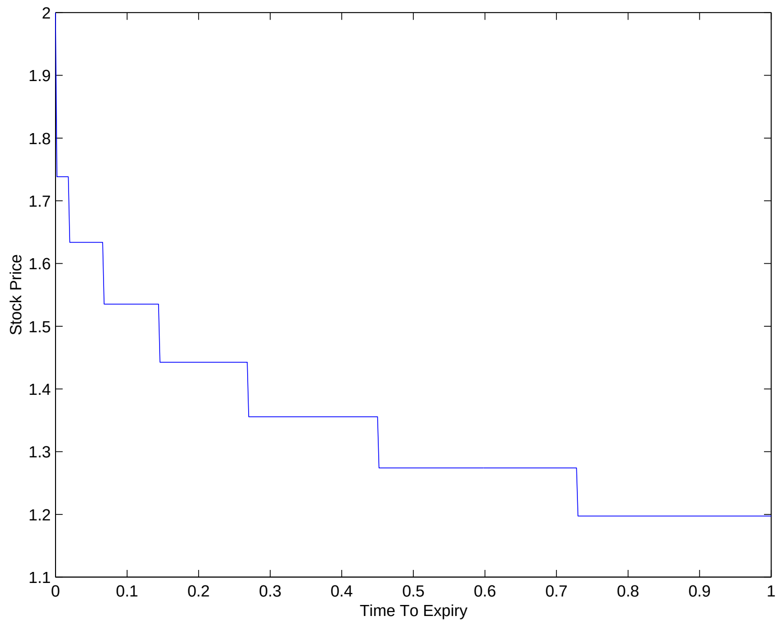

Figure 2.1: The calculated optimal exercise boundary for the American put, with $\mu = 0.15$, $r = 0.1$, $\delta = 0.02$, $\sigma = 0.3$, $\lambda = 5$, $K = 2$ and $U_1 \sim \text{Unif}[-0.2, 0.2]$. The discretization is $I = 100$, $N = 500$, and the stock price range is $(0.01, 5)$.

time step and information about the boundary conditions, resulting from the above discretization. We have

$$M^{k_n} u^{n,k_n} + q^{n+1,k_n} = 0,$$

for $n = N-1, N-2, \ldots, 0$ (a discrete counterpart of the equality in Theorem 1.5.3). Analogous equalities hold in the case of the American call.

2.3 The LP Algorithm

Motivated by the remarks of Section 2.2, we propose a new algorithm for solving the sequence of linear programs (2.2). The algorithm which we give is suitable for the American put. An analogous one can be constructed for the American call.

1. Set $c^T = (1, 1, \ldots, 1) \in \mathbb{R}^{I+1}$, where c is the constant positive vector in the problem (2.2).

2. Assume that the previous time step solution $u^{n+1} \in \mathbb{R}^{I+1}$ is known. Determine the free boundary k_{n+1}. Set $j := k_{n+1}$.

3. Temporarily set $u^n = (K, K, \ldots, K)^T \in \mathbb{R}^{I+1}$. In fact, the price of the American put option is never greater than K (the strike price).

4. Introduce a temporary vector $v = (v_0, v_1, \ldots, v_I)^T$ and set it to zero.

5. For $i = 0, 1, \ldots, j$, set $v_i = \psi_i$. It is possible that $k_n = j$. Discretize the operator $-\frac{\partial}{\partial t} - \mathcal{L}_{BS} - B$ in the hypothetical logarithmic continuation region so that it is represented by an $(I-j-1) \times (I-j-1)$ matrix M^j. Cumulate in the vector $q^{n+1,j}$ the information about the previous time step and the boundary conditions. Denote $v^j = (v_{j+1}, v_{j+2}, \ldots, v_{I-1})^T$. Solve
$$M^j v^j + q^{n+1,j} = 0.$$

6. Set
$$v = \min(K - \exp(X^l), \max(v, u^{n+1})).$$
It is safe to do so, since the actual solution u^n at time step n will have to satisfy these constraints anyway.

7. Check whether v is feasible, i. e. if
$$Mv^0 + q^{n+1} \geq 0.$$
If it is, then examine whether
$$c^T v < c^T u^n.$$
If so, then set $u^n = v$.

8. Unless $j = 0$, set $j := j - 1$ and jump back to point 5.

2.4 Computational Details

To adequately explain the functioning of the algorithm, the following remarks should be made.

(A) We have $u^N = \psi$ (the payoff from the option). In the case of the American put, we set $k_N = \max\{i : \psi_i > 0\}$. In the case of the American call, we set $k_N = \min\{i : \psi_i > 0\}$.

(B) The matrices M^j defined in point 5. of the algorithm, are in fact submatrices of the main matrix M. They are formed by removing the last $j-1$ columns and $j-1$ rows from M. Therefore, the band structure of M (see subsection 1.6.4) carries over to the matrices M^j. Denote by L the lower triangular, and by U — the upper triangular matrix, resulting from the LU factorization of M. Similarly, denote by L^j and U^j the respective matrices resulting from the LU factorization of M^j. It can be proved that L^j are formed by removing the last $j-1$ columns and rows from L, and U^j are formed by removing the last $j-1$ columns and rows from U.

The systems of linear equations

$$M^j v^j + q^{n+1,j} = 0 \tag{2.3}$$

in point 5., are solved in the following way: first (at the beginning of the algorithm), the matrix M is decomposed into the matrices L and U, and then the matrices L^j and U^j are (rapidly) formed from L and U. Equation (2.3) becomes

$$L^j U^j v^j + q^{n+1,j} = 0,$$

and this is solved by forward and backward substitution:

$$\begin{aligned} x &:= U^j v^j \\ x &= -(L^j)^{-1} q^{n+1,j} \\ v^j &= (U^j)^{-1} x. \end{aligned}$$

(C) The feasibility condition of point 7. is in fact hardly ever satisfied. In practice, we set $w = Mv^0 + q^{n+1}$ and check if

$$\min_i (w_i) \geq -\varepsilon_1^n$$

or if

$$\sum_{w_i<0} w_i \geq -\varepsilon_2^n,$$

where $\varepsilon_1^n, \varepsilon_2^n > 0$. If either of these conditions is satisfied, we say that v is feasible and proceed with the algorithm. It seems reasonable to choose, for example

$$\begin{aligned} \varepsilon_1^n &= -\min_i (z_i) \\ \varepsilon_2^n &= -\sum_{z_i<0} z_i, \end{aligned}$$

where $z_i = Mu^{n+1,0} + q^{n+1}$.

(D) It has been observed that the algorithm produces the same result if it is stopped once $c^T v$ exceeds $c^T u^n$ for the first time (see point 7. of the algorithm).

2.5 The Model and an Example

In the numerical analysis which follows, we consider models in which the jump relative sizes U_j are uniformly distributed under the original measure $\mathbb{P}$:

$$U_j \sim \mathrm{Unif}[-a, a], \quad \text{where} \quad a \in [0, 1).$$

The LP algorithm of Sections 2.3, 2.4, and the PSOR algorithm of Section 1.8 were implemented in MATLAB 5.2 on a PC system with a 300 MHz processor and 64 MB RAM, running under Windows 98. The distance function in PSOR was taken to be that defined by (1.31). The parameter ω was set to 1 in order to ensure convergence irrespective of the properties of the matrix M, given that it is row strictly diagonally dominant (see Formula (1.30)). The paramter ε of point 4. of the PSOR algorithm of Section 1.8 was set to 10^{-8}.

To give an example, we apply the LP algorithm to compute the price of the American put in the model with the following set of parameters:

$$\left.\begin{array}{rcl} \mu & = & 0.12 \\ r & = & 0.1 \\ \delta & = & 0.01 \\ \sigma & = & 0.4 \\ \lambda & = & 50 \\ a & = & 0.1. \end{array}\right\} \tag{2.4}$$

The strike price $K = 2$ and the time horizon $T = 1/4$. The discretization parameters are: the stock price range $= [0.01, 10]$, $I = 150$, $N = 500$.

The jumps have intensity $\lambda = 50$ (on average 50 jumps per year), and their relative magnitudes are uniformly distributed on the interval $[-0.1, 0.1]$. The price surface of the American put option in this model is plotted in Figure 2.2. The price of the option at $1\frac{1}{2}$ and 3 months before the expiry date is plotted in Figure 2.3. The free boundary (with the continuation region above it, and the stopping region below it) is plotted in Figure 2.4.

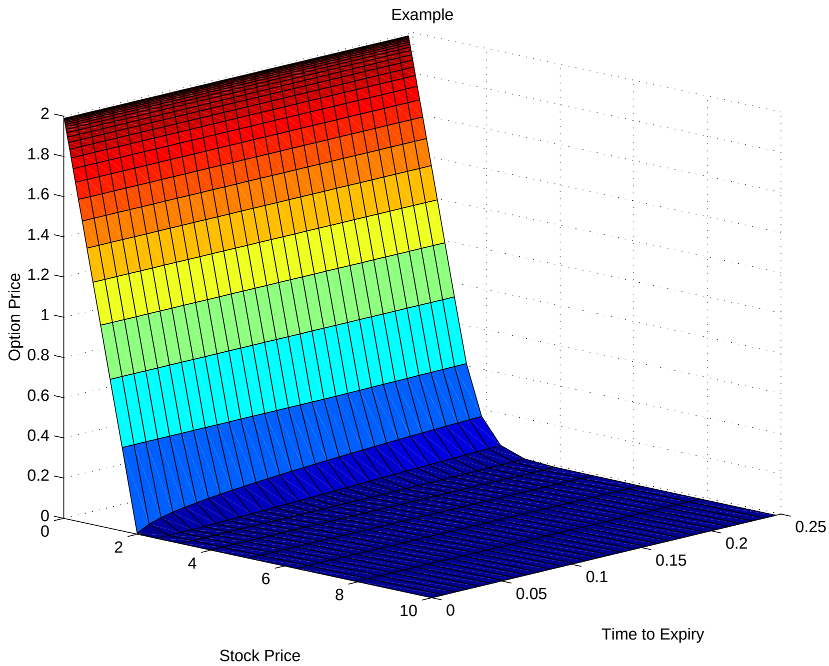

Figure 2.2: The price surface of the American put; the parameters as in (2.4).

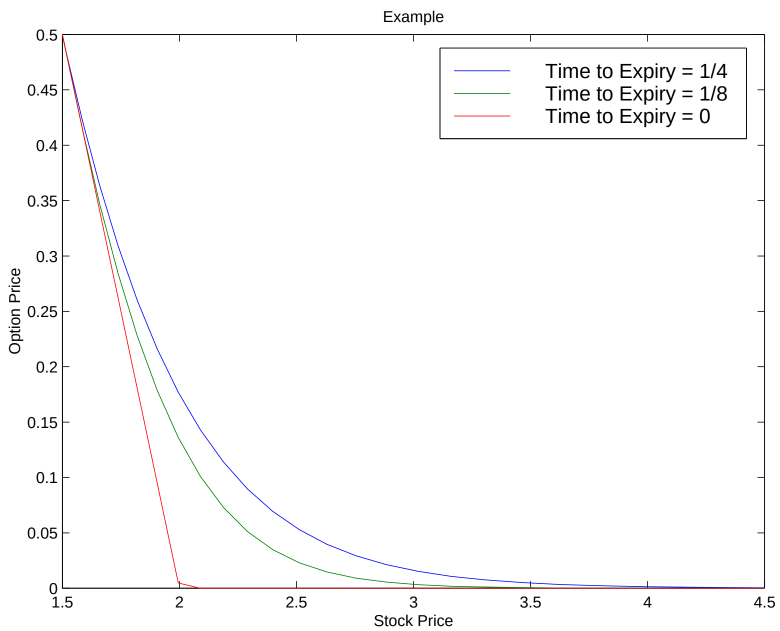

Figure 2.3: The payoff function and the price of the American put at $1\frac{1}{2}$ and 3 months before expiry; the parameters as in (2.4).

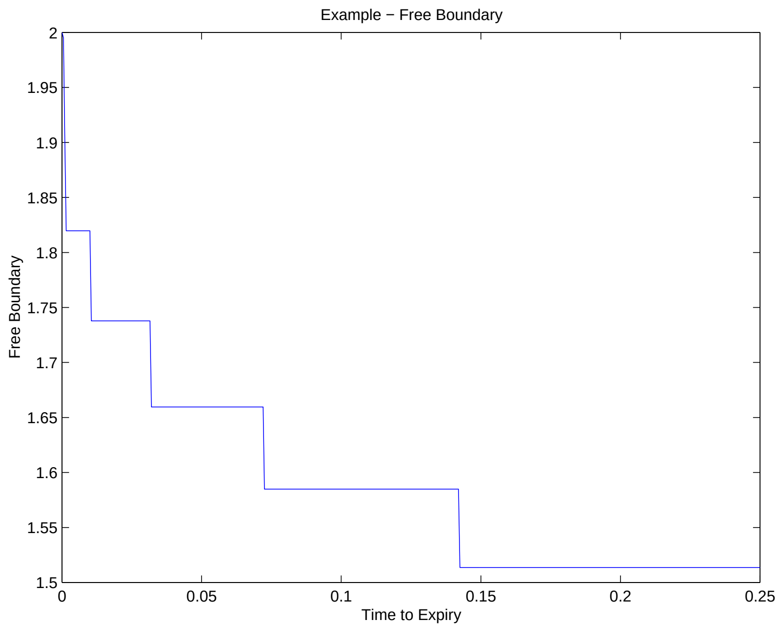

Figure 2.4: The free boundary for the American put; the parameters as in (2.4).

2.6 Accuracy of the Algorithm

Throughout the section, we denote by $\{u_i^n\}$ the solution obtained by means of the LP algorithm, and by $\{v_i^n\}$ — by means of the PSOR algorithm. To illustrate the accuracy of the LP algorithm, we consider the system with the following set of parameters:

$$\left.\begin{array}{rcl} \mu & = & 0.11 \\ r & = & 0.1 \\ \delta & = & 0.01 \\ \sigma & = & 0.3 \\ \lambda & = & 10 \\ a & = & 0.1. \end{array}\right\} \tag{2.5}$$

The strike price K is equal to 1, and the time horizon T is 1/2. We take the stock price range to be $[0.1, 10]$. We set $N = 500$ and consider various choices of I.

1. $I = 160$
   The stability condition (1.33) is not satisfied (as the actual stability

condition for the new pdf approximation scheme is unknown, the inequality (1.33) serves as a “rough approximation” of the true stability condition).

2. $I = 150$
   The stability condition (1.33) is satisfied. We have $\max|u_i^n - v_i^n| = 0.0009$. As can be seen in Figure 2.5, the difference $u_i^n - v_i^n$ is positive when the stock price is small (perhaps starting from around 1/2), which may mean that the LP solution $\{u_i^n\}$ lifts off the payoff surface “sooner” than the PSOR solution $\{v_i^n\}$ (eg the critical price implied by the LP solution may be lower than the critical price implied by the PSOR solution). Indeed, the free boundary derived from $\{u_i^n\}$ is situated below the free boundary derived from $\{v_i^n\}$ (illustration in Figure 2.6). It is clear that the free boundary obtained from $\{u_i^n\}$ is incorrect.

3. $I = 110$
   The stability condition (1.33) is satisfied. We have $\max|u_i^n - v_i^n| = 0.0004$. The difference $u_i^n - v_i^n$ is plotted in Figure 2.7. The difference is nonpositive close to the expiry date, which corresponds to the correctness of the LP free boundary close to the expiry date (see Figure 2.8). Similarly, the positivity (for small values of the stock price) of the difference for time to expiry greater than 0.17 corresponds to the incorrectness of the LP free boundary in that interval.

4. $I = 100$
   The stability condition (1.33) is satisfied. We have $\max|u_i^n - v_i^n| = 0.00014$. The difference $u_i^n - v_i^n$ is nonpositive (Figure 2.9). We may therefore expect the free boundary derived from the LP solution $\{u_i^n\}$ to be correct. Indeed, the two free boundaries nearly coincide (Figure 2.10).

The pattern described above arises in all problems which have been considered. For small values of I, the difference between the LP solution and the PSOR solution is nonpositive and both the implied free boundaries are correct (as in point 4.). For larger values of I, the difference is nonpositive close to the expiry date, and nonnegative in the remaining part of the time interval (for small values of the stock price), which corresponds to the incorrectness of the LP free boundary in this subinterval (as in point 3.). The larger I gets, the bigger part of the LP free boundary becomes “pushed down”. Eventually, the difference becomes nonnegative for small stock prices in the entire time interval. The LP free boundary is then incorrect for all $t \in [0, T]$ (as in

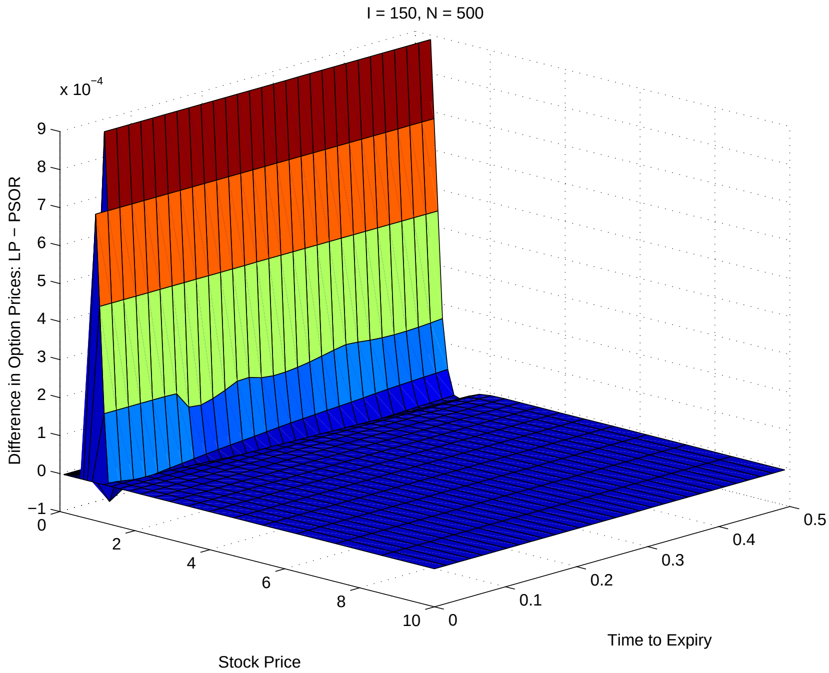

Figure 2.5: The difference in option prices obtained by LP and PSOR; the parameters as in (2.5), $I = 150$, $N = 500$.

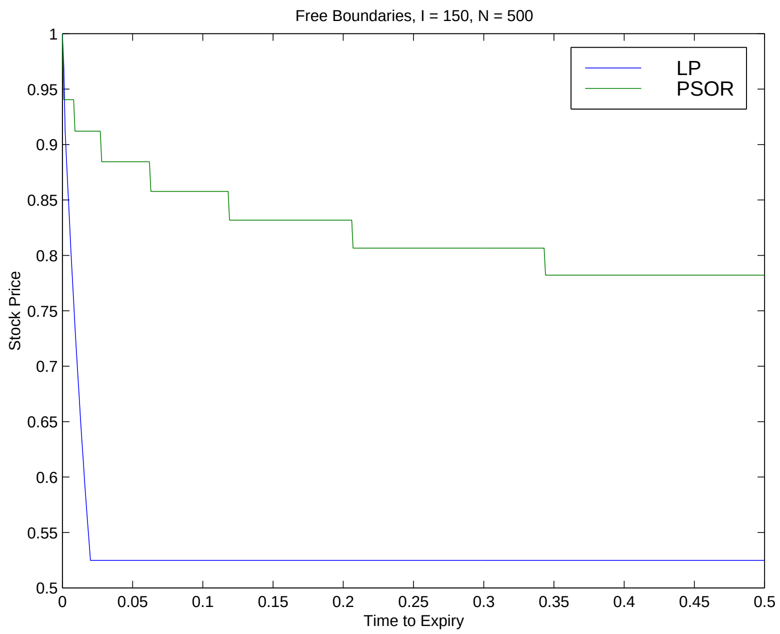

Figure 2.6: The correct PSOR free boundary and the incorrect LP free boundary; the parameters as in (2.5), $I = 150$, $N = 500$.

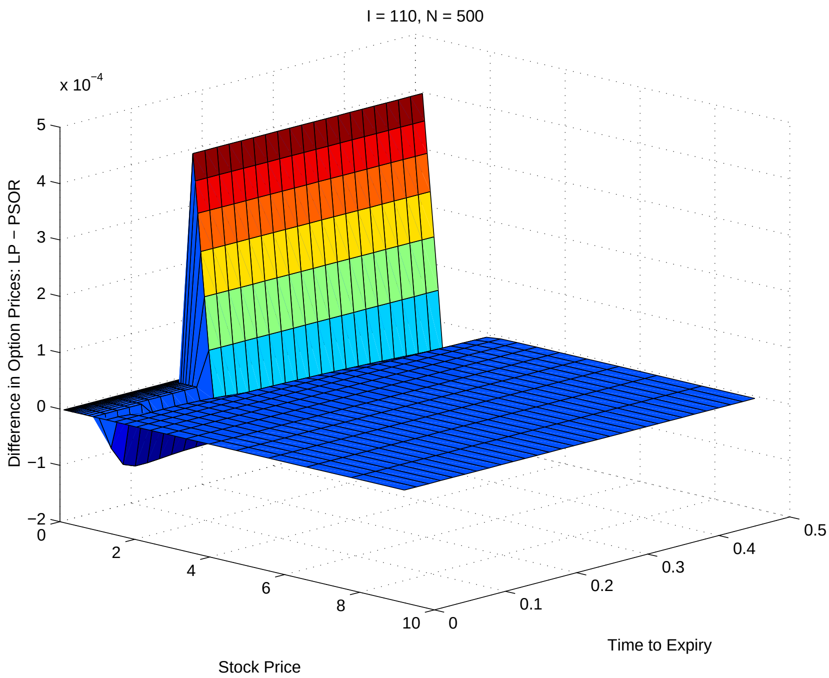

Figure 2.7: The difference in option prices obtained by LP and PSOR; the parameters as in (2.5), $I = 110$, $N = 500$.

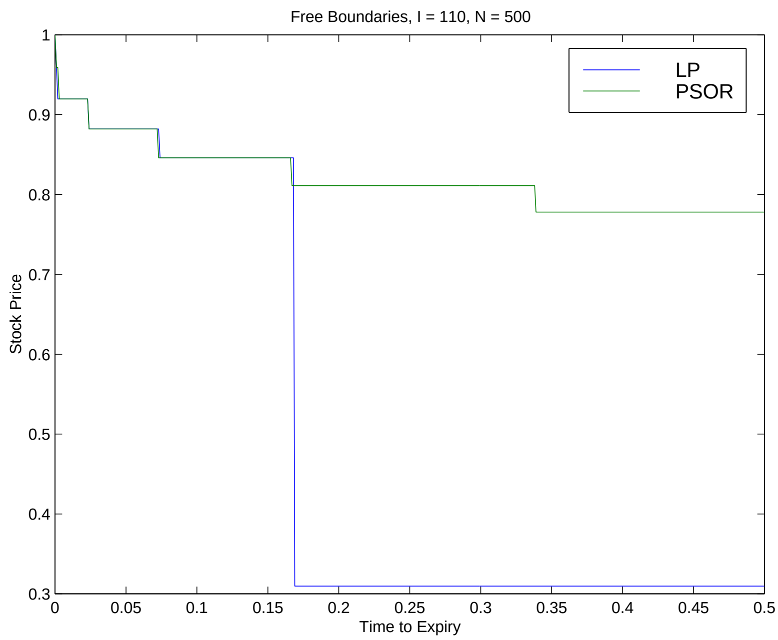

Figure 2.8: The correct PSOR free boundary and the partially correct LP free boundary; the parameters as in (2.5), $I = 110$, $N = 500$.

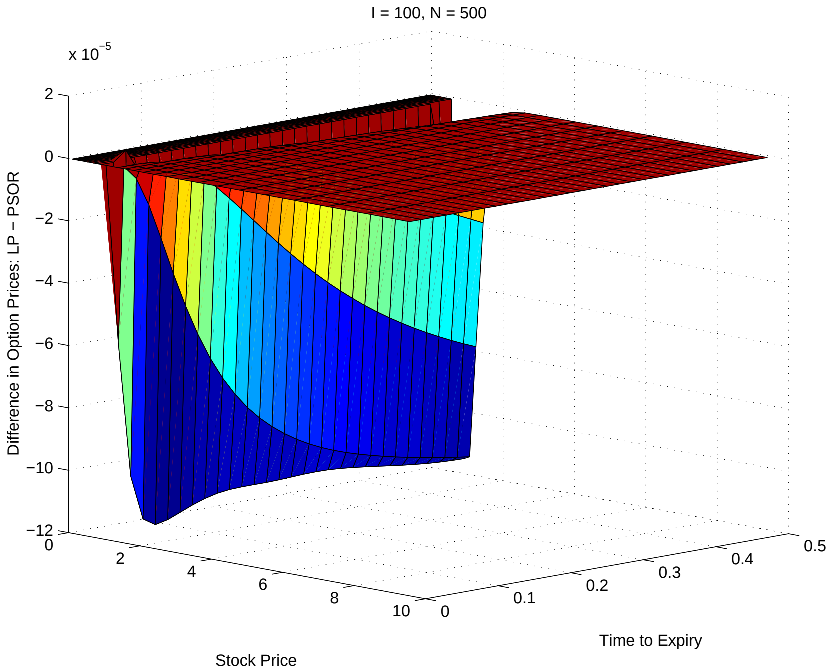

Figure 2.9: The difference in option prices obtained by LP and PSOR; the parameters as in (2.5), $I = 100$, $N = 500$.

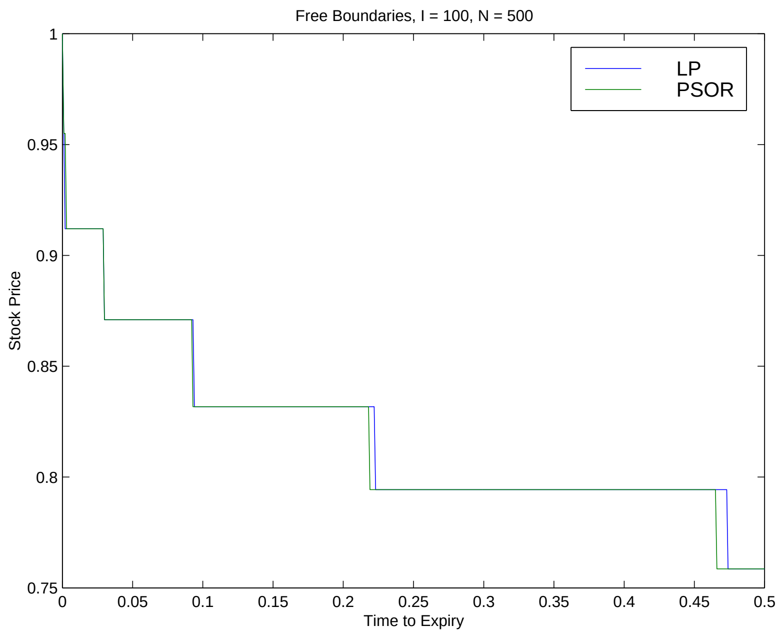

Figure 2.10: The correct LP and PSOR free boundaries; the parameters as in (2.5), $I = 100$, $N = 500$.

point 2.). When I exceeds a certain number, the stability condition (1.33) ceases to be satisfied (as in point 1.).

To summarize, the LP algorithm is comparable, but marginally inferior to PSOR in terms of accuracy. In its present version, it yields incorrect free boundaries for larger values of I, but this imperfection can probably be patched up without much trouble.

2.7 Solution Times

The tables in this section contain MATLAB solution times of selected problems. To begin with, we illustrate the dependence of the solution times on I. We set

$$\left.\begin{array}{lcl} \mu &=& 0.12 \\ r &=& 0.1 \\ \delta &=& 0.01 \\ \sigma &=& 0.3 \\ \lambda &=& 26 \\ a &=& 0.1. \end{array}\right\} \tag{2.6}$$

The strike price is $K = 1$. The stock price range is $[0.1, 20]$, the time horizon is $T = 1/4$, and the spatial discretization is $N = 1000$. Table 2.1 shows the

I	LP solution time	PSOR solution time
50	4.34	25.32
100	14.94	74.47
150	33.17	128.3
200	56.08	219.05
250	92.66	333.01
300	122.04	446.11
350	181.26	644.77

Table 2.1: Problem (2.6), solution times in seconds.

LP and PSOR solution times in seconds. The times are plotted in Figure 2.11. For both the LP and the PSOR method, we observe a nonlinearity of the solution time as a function of I. However, as we have observed in this and many other cases, the LP solution times are *near-linear* in the spatial discretization, which cannot be said of the PSOR solution times.

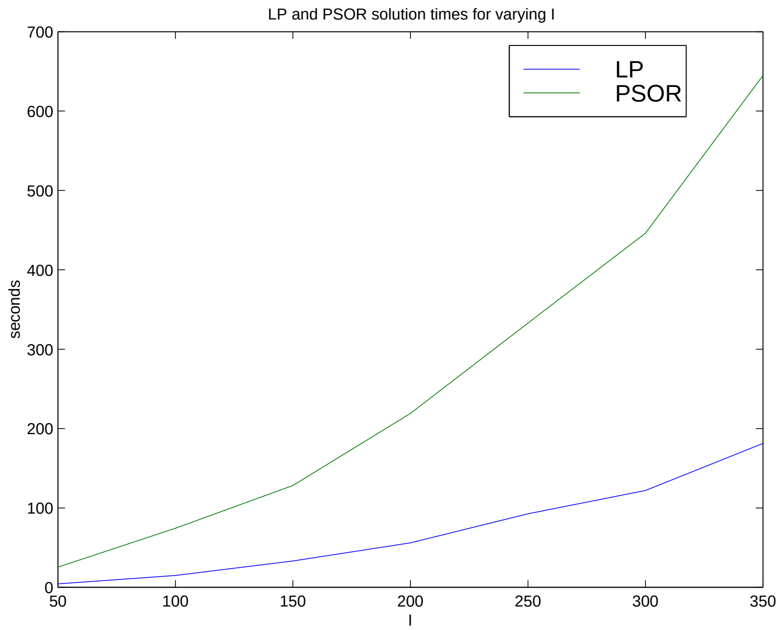

Figure 2.11: LP and PSOR solution times for problem (2.6), with varying I.

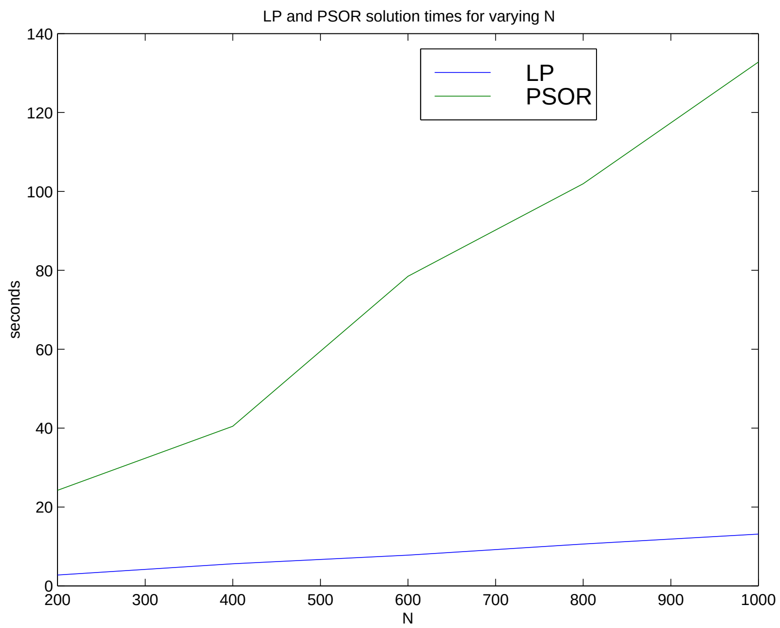

Figure 2.12: LP and PSOR solution times for problem (2.7), with varying N.

Subsequently, we illustrate the dependence of the solution times on N.
We set

$$\left.\begin{array}{rcl}\mu & = & 0.15 \\ r & = & 0.1 \\ \delta & = & 0.02 \\ \sigma & = & 0.9 \\ \lambda & = & 50 \\ a & = & 0.2.\end{array}\right\} \qquad (2.7)$$

The strike price is $K = 1$. The stock price range is $[0.01, 5]$, the time horizon is $T = 1/12$, and the time discretization is $I = 100$. Table 2.2 shows the

N	LP solution time	PSOR solution time
200	2.74	24.22
400	5.6	40.48
600	7.8	78.48
800	10.6	101.95
1000	13.13	132.81

Table 2.2: Problem (2.7), solution times in seconds.

LP and PSOR solution times in seconds. The times are plotted in Figure 2.12. The observed linearity of the LP solution time as a function of N is intuitively justifiable. It has appeared it all of the problems tested.

What is important is that LP solution times are robust to changes in those parameters which account for the volatility of the system: σ, λ and a. To illustrate this, we consider the following problem

$$\left.\begin{array}{rcl}\mu & = & 0.12 \\ r & = & 0.1 \\ \delta & = & 0.01.\end{array}\right\} \qquad (2.8)$$

We set the strike price to $K = 1$. The stock price range is $[0.1, 20]$, the time horizon $T = 1/4$. The discretization is $I = 200$, and $N = 1000$. Table 2.3 gives the solution times for problem (2.8), for varying parameters σ, λ and a. While the LP solution time is a constant function of these parameters, the PSOR solution time is an increasing function of each of them.

To summarize, the LP method significantly outperforms PSOR in terms of speed, even though an important part of the LP algorithm is the LU decomposition of a substantially large matrix. The computational complexity of the LU decomposition is $O(I^3/3)$, where $I \times I$ is the size of the matrix.

σ	λ	a	LP solution time	PSOR solution time
0.1	0		51.63	139.84
0.5	0		51.63	258.7
0.5	10	0.2	52.02	283.14
0.5	50	0.2	51.74	297.26

Table 2.3: Problem (2.8), solution times in seconds.

However, due to the stability condition (1.33) (a similar stability condition probably arises in the case of the new approximation scheme, see subsection 1.6.3), I is usually limited to a few hundred, unless the partition of the time interval is very fine. Therefore, the size of the matrix is limited. In practice, the LU decomposition of a 500×500 matrix is performed in about 3.5 seconds by means of the MATLAB routine `lu`.

Chapter 3

Numerical Results

In this chapter, we analyse the dependence of the price of the American put on a variety of parameters. We also examine the densities of the optimal stopping times for the American put problem.

Throughout the chapter, we use the new approximation scheme for the pdf of the relative jump size $g = g(x)$ (see subsection 1.6.3). The only exception is Section 3.3, where we give examples of what may happen if we apply the usual scheme (i. e. the one proposed by Zhang [Zha97]).

3.1 The Impact of the Parameters on the Option Price

It has to be emphasized that the examples quoted in this section illustrate a general pattern which has appeared in all of the problems considered, not only in the ones referred to below.

3.1.1 The Volatility Parameters

By "volatility parameters" we mean the three parameters σ, λ and a. However, the term "volatility parameter" is reserved for σ.

The results obtained from the conducted experiments suggest that *the price of the American put is an increasing function of each of the volatility parameters σ, λ and a.*

To illustrate the above statement, we consider the system with the following parameters:

$$\left.\begin{array}{rcl} \mu & = & 0.11 \\ r & = & 0.1 \\ \delta & = & 0.01. \end{array}\right\} \tag{3.1}$$

The strike price is $K = 1$. We set the stock price range to $[0.1, 20]$ and the time horizon to $T = 1/4$. The discretization is $I = 500$, $N = 2000$. We denote by $u_i^n(\sigma, \lambda, a)$ the computed price of the American put as a function of the volatility parameters.

Figures 3.1 and 3.2 illustrate the fact that $u_i^n(\sigma, \lambda, a)$ is an increasing function of σ. Similarly, Figures 3.3 and 3.4 suggest that the computed option price increases with λ, and Figures 3.5 and 3.6 — that it increases with a.

Moreover, it has been observed that the option price is a continuous function of all the volatility parameters.

3.1.2 The Drift

It is well-known that in the Black-Scholes case, the drift parameter μ does not influence the price of the derivative instrument. As should be expected, this is not the case in the model with jumps. It has been observed that the computed option price, as a function of the drift, increases with μ if the jumps in the model are big (= a is large). Conversely, it decreases with μ if the jumps are small. However, the differences between the option prices computed for different drift parameters μ (the other parameters fixed) are small. Two illustrative examples are shown in Figures 3.7 and 3.8. We set

$$\left.\begin{array}{rcl} r & = & 0.1 \\ \delta & = & 0.01 \\ \sigma & = & 0.3 \\ \lambda & = & 10 \end{array}\right\} \tag{3.2}$$

and compute the price of the option with $K = 1$ at 12 months before expiry (the stock price range = $[0.1, 20]$, $I = 500$, $N = 2000$). Figure 3.7 shows the difference between the option prices computed for $\mu = 0.5$ and $\mu = 0$, with $a = 0.1$. Figure 3.8 shows the corresponding difference computed for $a = 0.2$.

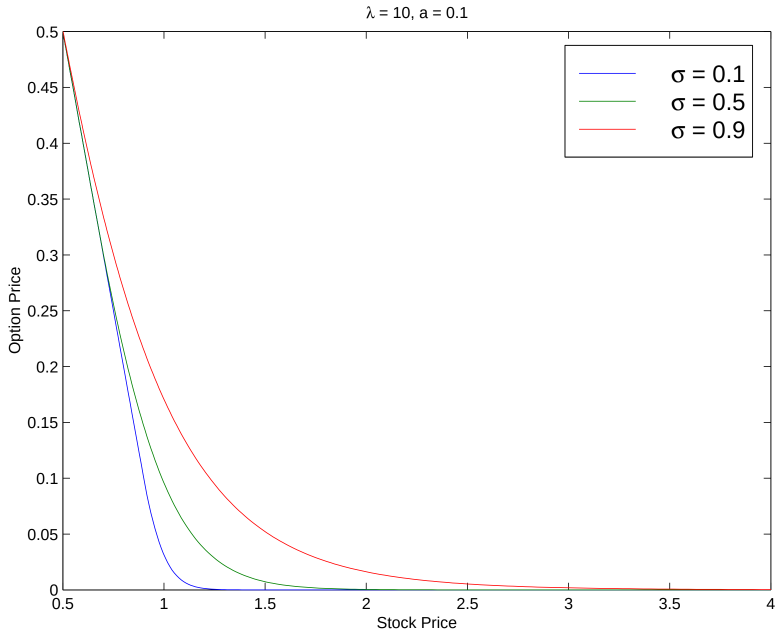

Figure 3.1: American put prices for varying σ at 3 months before expiry; the parameters as in (3.1), $\lambda = 10$, $a = 0.1$. The option price increases with σ.

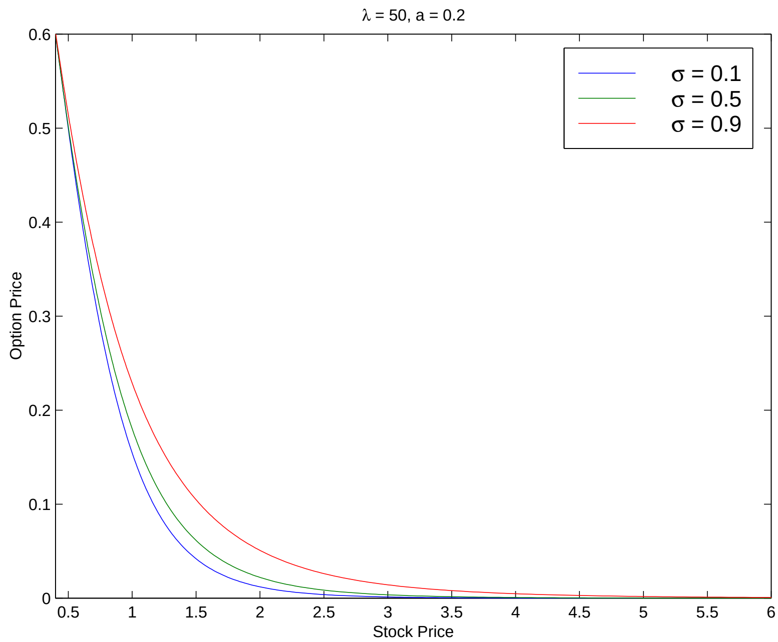

Figure 3.2: American put prices for varying σ at 3 months before expiry; the parameters as in (3.1), $\lambda = 50$, $a = 0.2$. The option price increases with σ.

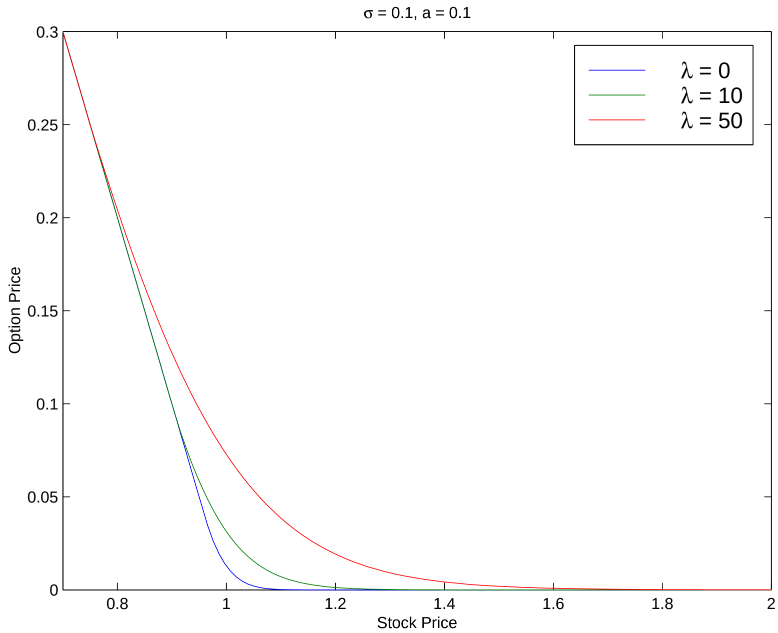

Figure 3.3: American put prices for varying λ at 3 months before expiry; the parameters as in (3.1), $\sigma = 0.1$, $a = 0.1$. The option price increases with λ.

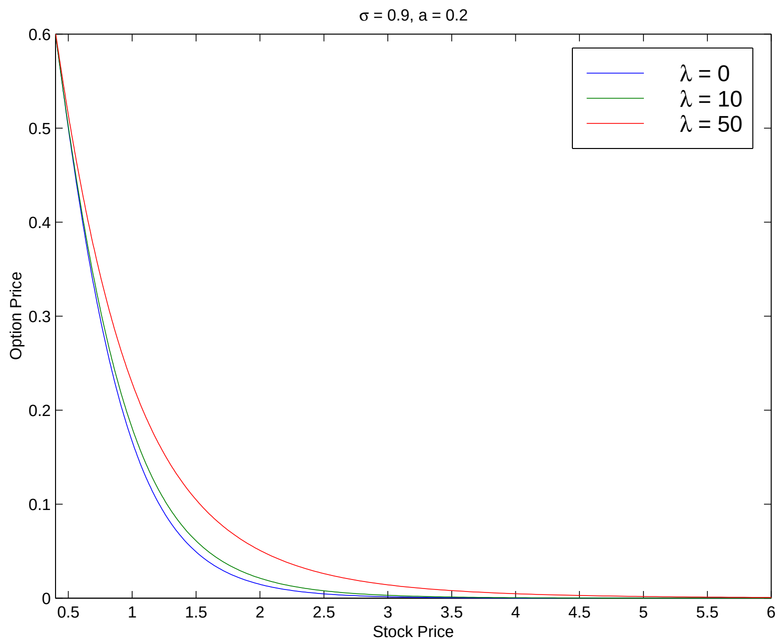

Figure 3.4: American put prices for varying λ at 3 months before expiry; the parameters as in (3.1), $\sigma = 0.9$, $a = 0.2$. The option price increases with λ.

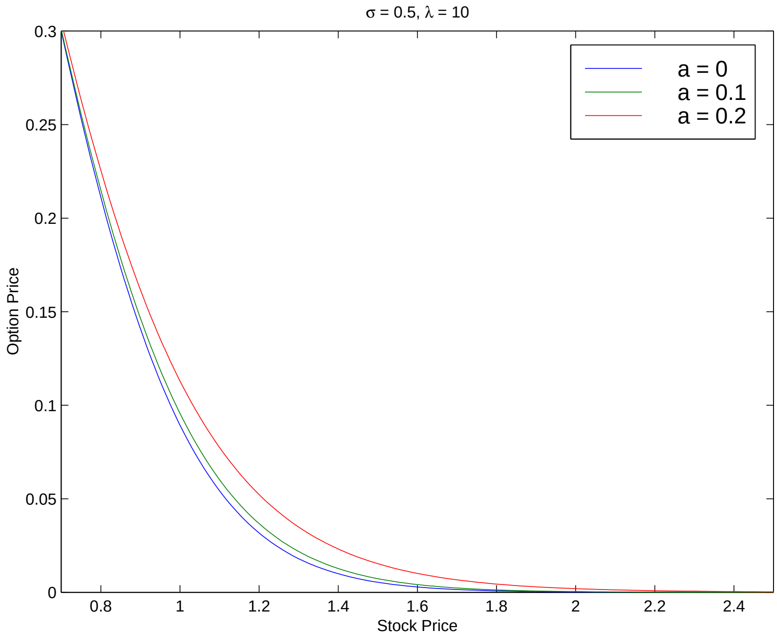

Figure 3.5: American put prices for varying a at 3 months before expiry; the parameters as in (3.1), $\sigma = 0.5$, $\lambda = 10$. The option price increases with a.

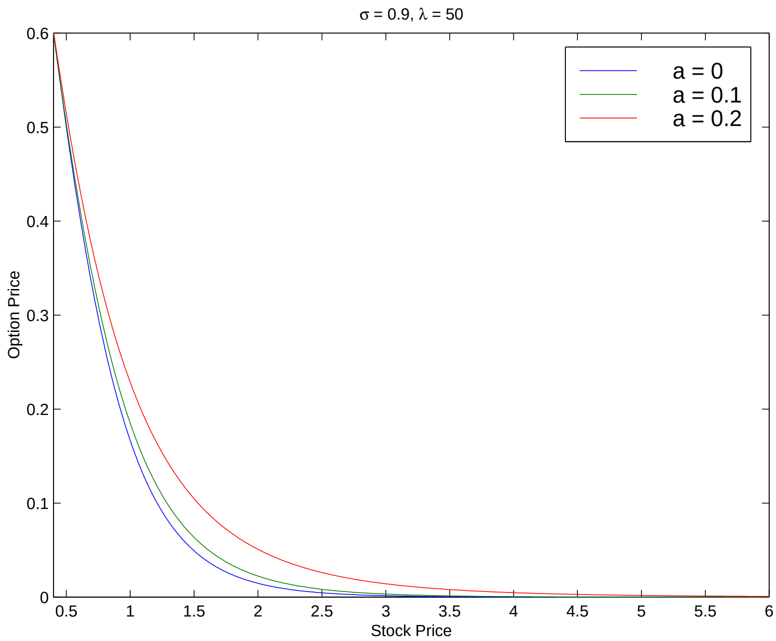

Figure 3.6: American put prices for varying a at 3 months before expiry; the parameters as in (3.1), $\sigma = 0.9$, $\lambda = 50$. The option price increases with a.

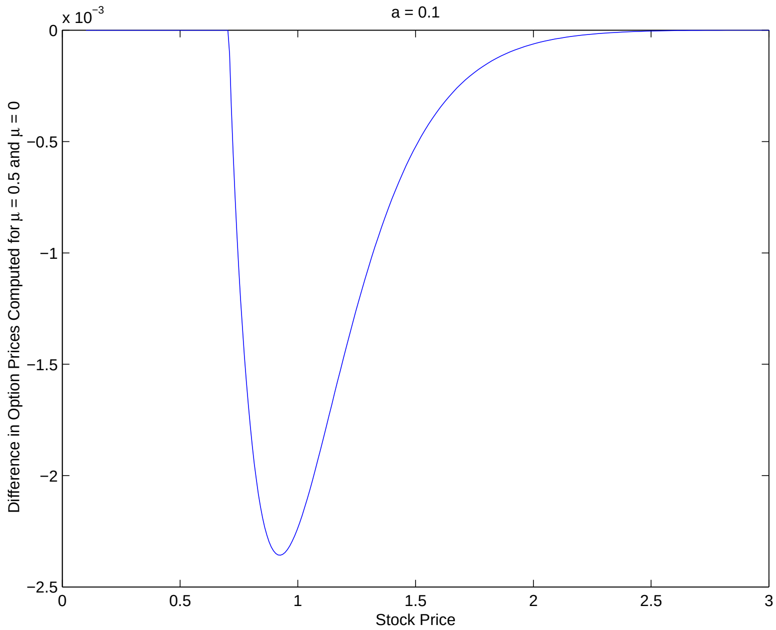

Figure 3.7: Difference between the option prices computed for $\mu = 0.5$ and $\mu = 0$; "small" jumps: $a = 0.1$, the other parameters as in (3.2).

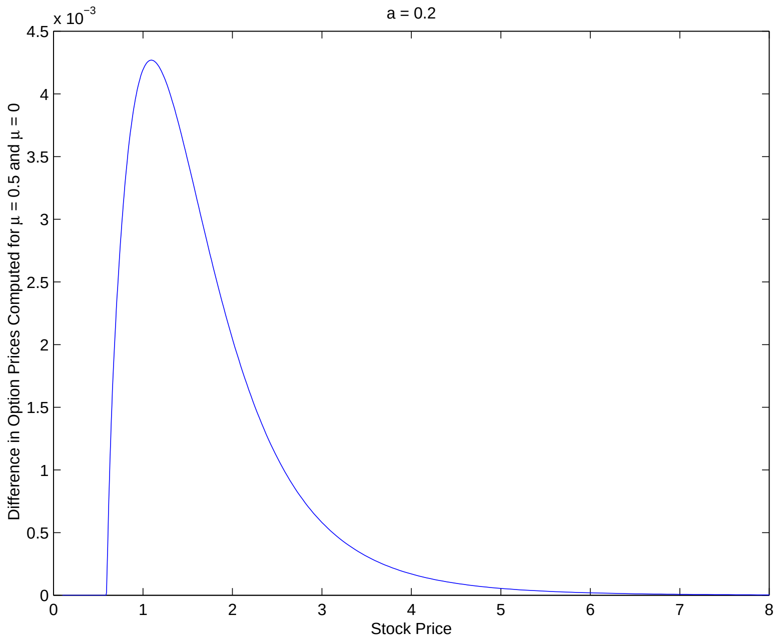

Figure 3.8: Difference between the option prices computed for $\mu = 0.5$ and $\mu = 0$; "big" jumps: $a = 0.2$, the other parameters as in (3.2).

3.2 Optimal Exercise Times

In this section, we analyse the distribution of the optimal exercise times for the American put. As was mentioned before, the optimal exercise time is the first moment the stock price enters the exercise region (located below the free boundary in the case of the American put).

We assume that the spot price of the stock at time 0 is equal to $S_0 = 1$. We further assume that the risk-free rate r is equal to 0.1, and the dividend rate δ is equal to 0.01. We consider the American put with $T = 1$, whose strike price is equal to S_0, i. e. $K = 1$. For various choices of μ, σ, λ and a, we first compute the free boundaries (the stock price range = $[0.1, 20]$, $I = 500$, $N = 2000$), and then run the stock price processes 10000 times to obtain the approximate distributions of the optimal exercise times for the options.

We denote by $\tau_{10000}(\sigma, \lambda, a, \mu)$ the samples of computed optimal exercise times obtained in this way.

The impact of the volatility parameters on the distribution of the optimal exercise time can be easily deduced from the data obtained. The following tendencies are apparent.

1. As any of the volatility parameters σ, λ or a increases, more and more options are exercised before the expiry date. An option still unexercised at time $T = 1$ is useless (and worthless). The only options which are "worth having" are those which are exercised during their lifetime. The conclusion is that it is more likely that the option will be "useful" if the market is volatile, i. e. if σ, λ or a are large.

2. As any of the volatility parameters increases, more and more options are exercised later in the year.

However, if the diffusion volatility parameter σ "dominates" over the jump volatility parameters λ and a, the influence of λ and a on the distribution of the optimal exercise time is less apparent. Conversely, if the jump part dominates, the impact of σ is less clearly visible.

Representative histograms for varying σ, the other parameters fixed, are plotted in Figure 3.9. Similarly, histograms for varying λ are shown in Figure 3.10, and histograms for varying a — in Figure 3.11.

The mean optimal exercise time $\bar{\tau}_{10000}(\sigma, \lambda, a, \mu)$ appears to increase with σ, λ and a. However, in some cases the increasing trend is not clearly visible. Tables 3.1, 3.2 and 3.3 show the evolution of the mean optimal exercise times which correspond to the respective histograms.

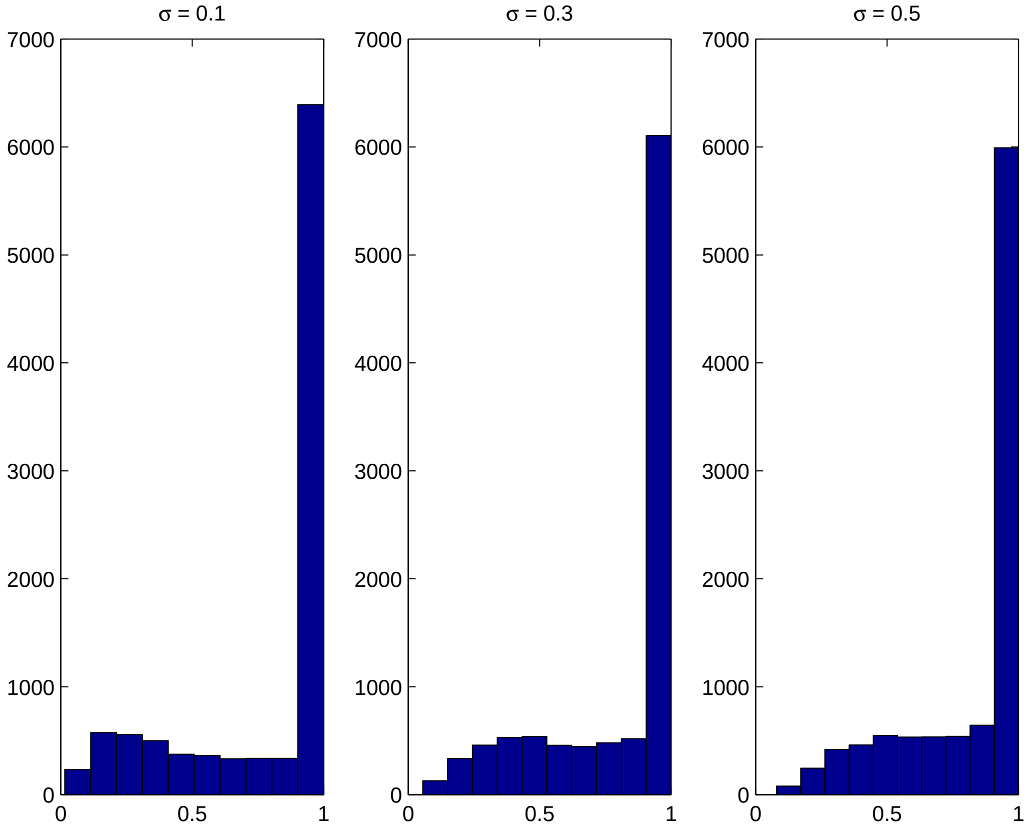

Figure 3.9: Optimal exercise time histograms for varying σ; the other parameters: $\lambda = 10$, $a = 0.1$, $\mu = 0.11$.

σ	$\bar{\tau}_{10000}(\sigma, 10, 0.1, 0.11)$
0.1	0.7944
0.3	0.8127
0.5	0.8245

Table 3.1: The mean optimal exercise times for varying σ; the other parameters: $\lambda = 10$, $a = 0.1$, $\mu = 0.11$. The inreasing tendency is apparent.

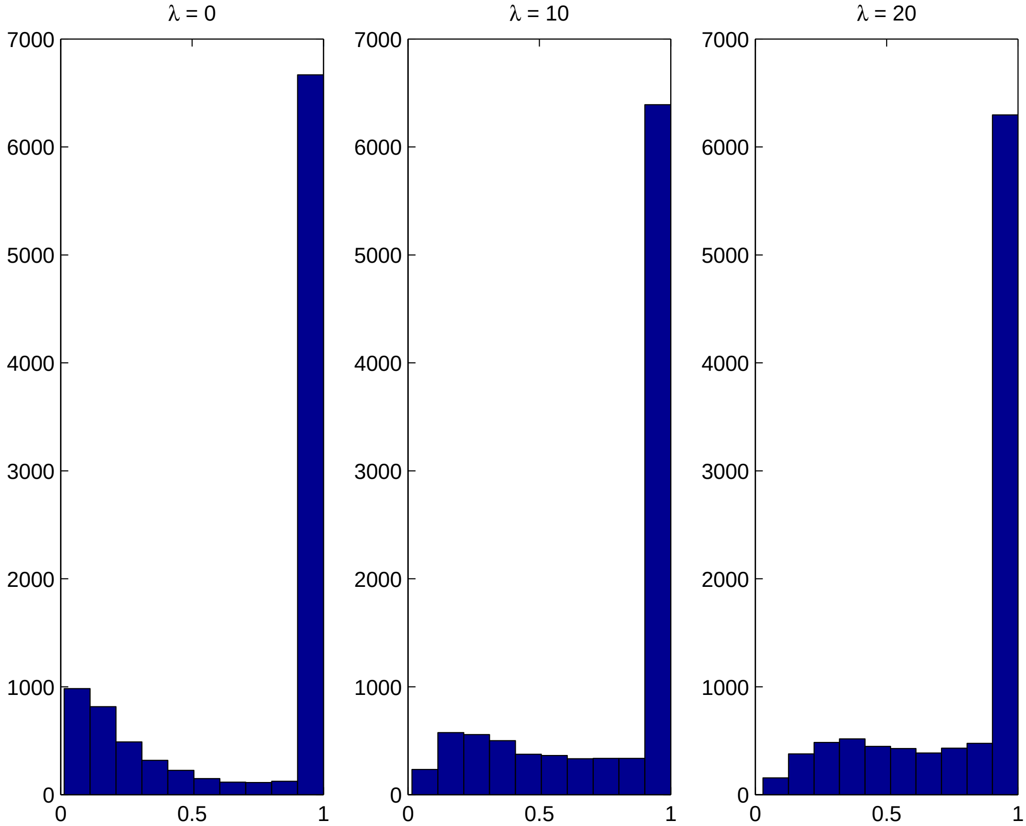

Figure 3.10: Optimal exercise time histograms for varying λ; the other parameters: $\sigma = 0.1$, $a = 0.1$, $\mu = 0.11$.

λ	$\bar{\tau}_{10000}(0.1, \lambda, 0.1, 0.11)$
0	0.7546
10	0.7944
20	0.8111

Table 3.2: The mean optimal exercise times for varying λ; the other parameters: $\sigma = 0.1$, $a = 0.1$, $\mu = 0.11$. The increasing tendency is apparent.

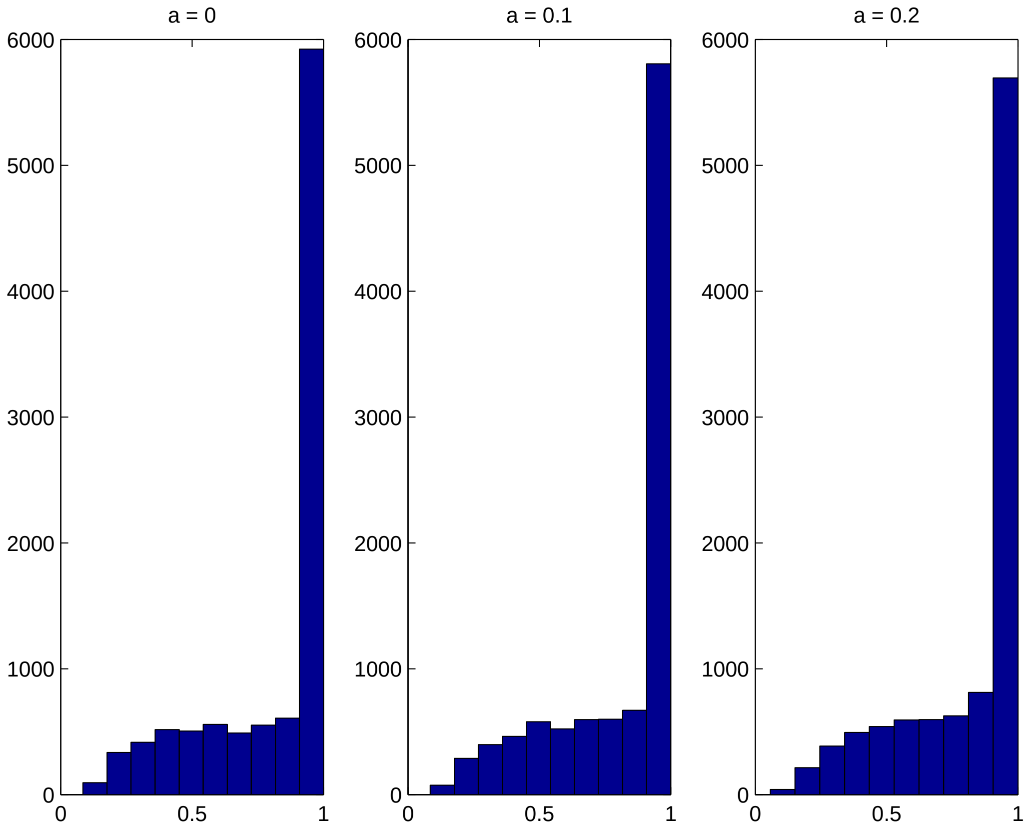

Figure 3.11: Optimal exercise time histograms for varying a; the other parameters: $\sigma = 0.5$, $\lambda = 20$, $\mu = 0.11$.

a	$\bar{\tau}_{10000}(0.5, 20, a, 0.11)$
0	0.8180
0.1	0.8197
0.2	0.8185

Table 3.3: The mean optimal exercise times for varying a; the other parameters: $\sigma = 0.5$, $\lambda = 20$, $\mu = 0.11$. We do not observe an increasing tendency; this is so because σ dominates over λ and a.

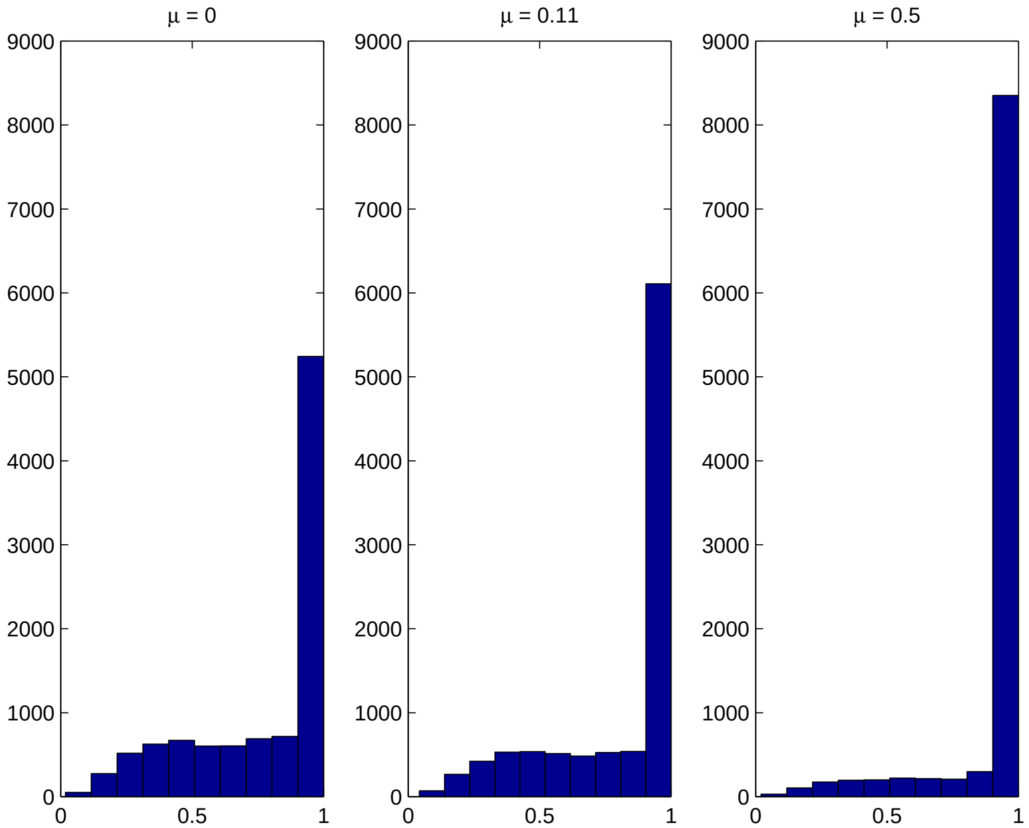

Figure 3.12: Optimal exercise time histograms for varying μ; the other parameters: $\sigma = 0.3$, $\lambda = 10$, $a = 0.2$.

μ	$\bar{\tau}_{10000}(0.3, 10, 0.2, \mu)$
0	0.7793
0.11	0.8184
0.5	0.9243

Table 3.4: The mean optimal exercise times for varying μ; the other parameters: $\sigma = 0.3$, $\lambda = 10$, $a = 0.2$. A very strong increasing tendency.

As should be expected, the infuence of the drift parameter μ on the distribution of the optimal exercise time is substantial. Obviously enough, the larger μ, the more options "survive" unexercised until the expiry date. Representative histograms for varying μ are shown in Figure 3.12. The mean optimal exercise time increases with μ and the increasing trend is very conspicuous, see Table 3.4.

3.3 Inaccuracy of Zhang's Discretization Scheme

As was mentioned in subsection 1.6.3, the approximation scheme for the pdf of the jump size $g = g(x)$ which was proposed by Zhang [Zha97], yields serious numerical errors.

Firstly, for some choices of the parameters of the model, the option price computed using Zhang's scheme

(a) exceeds the strike price K,

(b) is a non-convex function of the stock price[1],

even though the stability conditions (1.32), (1.33) and (1.34) are satisfied, and the matrix of the LCP in question is strictly diagonally dominant and type Z. A representative example is shown in Figure 3.13. The parameters are

$$\left.\begin{array}{rcl} \mu &=& 0.11 \\ r &=& 0.1 \\ \delta &=& 0.01 \\ \sigma &=& 0.3 \\ \lambda &=& 10 \\ a &=& 0.2. \end{array}\right\} \tag{3.3}$$

The strike price K is equal to 2. The stock price range is $[0.1, 10]$, the time horizon is $T = 1/2$, and the discretization is $I \times N = 130 \times 500$. In this case, Zhang's approximation scheme produces the characteristic "hump" for I's around 130. The behaviour illustrated in Figure 3.13 is typical for systems with large volatility parameters. Even if Zhang's approximation scheme is

[1] The convexity property of the American put price is well-known, see for example Pham [Pha97].

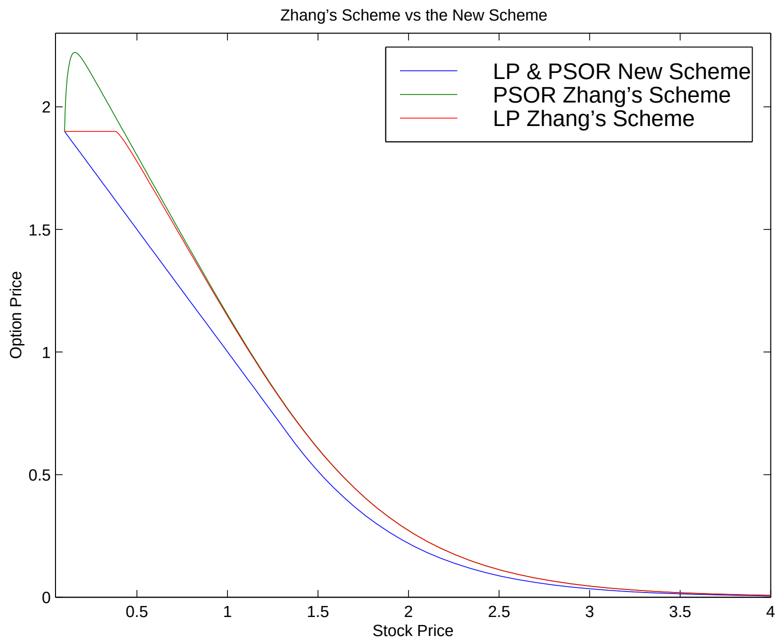

Figure 3.13: The option price computed under the "new" approximation scheme, and the (apparently incorrect) price computed using Zhang's scheme. The parameters as in (3.3). The "correct" LP and PSOR prices coincide.

used, there is no hump in the PSOR option price when the system is little volatile[2].

By contrast, it has been observed that the option price computed using our approximation scheme is always lower than K, and is a convex function of the stock price, regardless of the parameters of the model and regardless of the discretization parameters.

Moreover, the option price computed under Zhang's scheme is not always an increasing function of the jump intensity λ. To illustrate the above statement, we consider the following system:

$$\left.\begin{array}{rcl} \mu & = & 0.11 \\ r & = & 0.1 \\ \delta & = & 0.01 \\ \sigma & = & 0.9 \\ a & = & 0.1. \end{array}\right\} \tag{3.4}$$

[2]The fact that the LP option price computed under Zhang's scheme is constant for small stock prices (see Figure 3.13) is due to the cutoff performed in point 6 of the LP algorithm (Section 2.3).

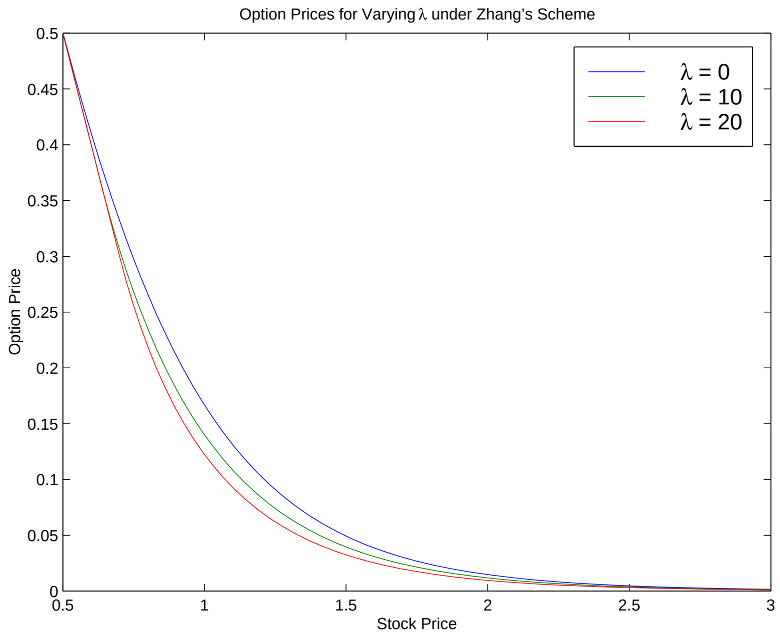

Figure 3.14: Under Zhang's approximation scheme, option prices can decrease with λ when σ is large. The parameters as in (3.4).

The strike price K is equal to 1. The stock price range is $[0.1, 20]$, the time horizon is $T = 1/4$, and the discretization is $I \times N = 150 \times 500$. The option prices for varying λ, computed under Zhang's scheme, are plotted in Figure 3.14. As λ increases, the corresponding option price decreases.

When σ is smaller, it often happens that the option price computed under Zhang's scheme intersects the Black-Scholes price (see Figures 2 and 3 in [Zha97]). Finally, for extremely small σ's, Zhang's price increases with λ.

Lastly, when Zhang's approximation scheme is used, the computed option price is not a continuous function of a at $a = 0$. Namely, as $a \downarrow 0$, the resulting option price converges to a limit which is lower than the Black-Scholes price.

To demonstrate once more the advantage of the new approximation scheme over Zhang's scheme, we discuss Proposition 5.1 from Pham [Pha97], who argues that for a fixed market price of jump risk $\boldsymbol{p}$, the price of the American put in the jump-diffusion model is a nondecreasing function of λ. We recall that in our case we have

$$p(y) \quad = \quad 1 - \rho y =$$

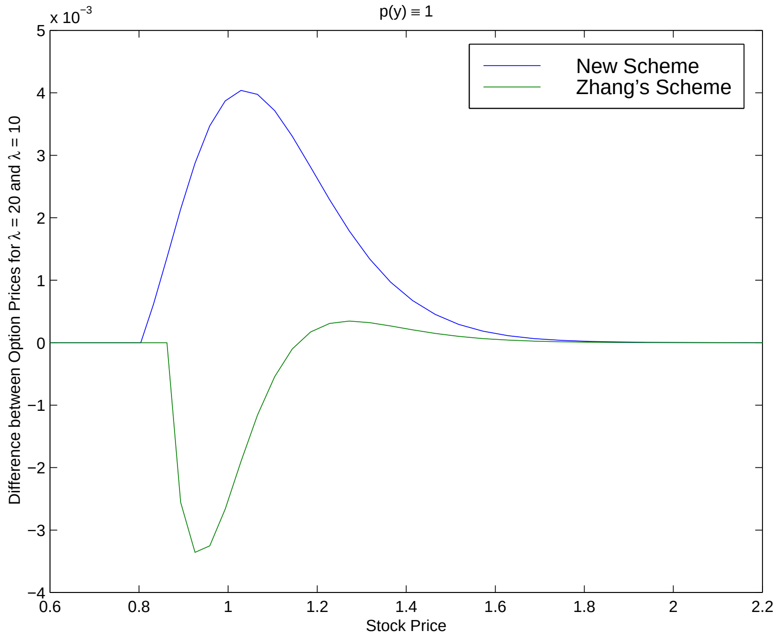

Figure 3.15: When the new scheme is applied, the option price increases with λ for $\boldsymbol{p}$ fixed, as it should according to Pham's proposition.

$$
\begin{aligned}
&= 1 - \frac{\mu^* - r + \delta}{\sigma^2 + \lambda \mathbb{E}^{\mathbb{P}}(U_1^2)} y = \\
&= 1 - \frac{\mu - r + \delta}{\sigma^2 + \lambda a^2/3} y.
\end{aligned}
$$

To keep $\boldsymbol{p}$ fixed, we set

$$
\left.\begin{aligned}
\mu &= 0.09 \\
r &= 0.1 \\
\delta &= 0.01.
\end{aligned}\right\} \tag{3.5}
$$

We further set

$$
\left.\begin{aligned}
\sigma &= 0.3 \\
a &= 0.1.
\end{aligned}\right\} \tag{3.6}
$$

The strike price is $K = 1$. The time horizon is $T = 1/4$, the stock price range is $[0.1, 20]$, and the discretization is $I \times N = 150 \times 500$. Figure 3.15 shows two differences between the option prices for $\lambda = 20$ and $\lambda = 10$, computed using Zhang's approximation scheme and our scheme. The difference under Zhang's scheme is not at all nonnegative, as it should be according to Pham's proposition. By contrast, the difference under the new scheme is nonnegative for all stock prices.

It has to be mentioned that the approximation scheme proposed in this paper is still far from ideal, since the corresponding Schweizer's measure is computed for continuously distributed jumps, and not for their discretized counterparts. The scheme would probably be even more accurate if the minimal martigale measure was computed directly for the discrete problem.

Another important remark is that for extremely fine discretizations, the difference between the both schemes ceases to be significant, since the discretized pdf $g = g(x)$ integrates *nearly* to one in Zhang's scheme (while it always integrates exactly to one in our scheme, see subsection 1.6.3 for details).

The application of Zhang's scheme also distorts the free boundary for fine discretizations when the LP algorithm is used (see Section 2.6).

Conclusions

In the paper, we have proposed a new linear programming (LP) algorithm for solving linear complementarity problems arising from the variational formulation of American option pricing problems in the jump-diffusion model. We have shown that the new method is much faster than the standard PSOR algorithm, and, more importantly, it is robust to parameter changes and near-linear in the spatial discretization. However, for certain choices of parameters, it distorts the free boundary, and therefore there is still scope for the improvement of its accuracy.

Moreover, we have explicitly computed Schweizer's minimal martingale measure for the price process in question, and we have used it used throughout the paper as the pricing measure.

We have introduced a modification to the discretization scheme proposed by Zhang [Zha97], thanks to which we have eliminated the serious numerical inaccuracy of Zhang's scheme. Namely, we have erradicated an instability which was arising for certain discretization parameters in highly volatile systems, and we have restored the correct dependence of the computed option price upon the jump intensity λ. Moreover, we have obtained continuity of the option price as a function of the maximum relative jump size a.

We have applied both algorithms (PSOR and LP) to the valuation of the American put. We have shown that the numerical solution to the put pricing problem, under the corrected approximation scheme, is an increasing function of the volatility parameters σ, λ and a. These results, while contradicting those obtained by Zhang, are consistent with the theoretical results of Pham [Pha97].

We have also analysed the distribution of the optimal exercise times for the American put and found that the higher the volatility parameters, the more options are exercised before the expiry date. An increase in any of the volatility parameters also increases the mean optimal exercise time.

We are convinced that the thought-provoking results reported here will provide much stimulation for researchers in the area.

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