Naturally Light Fermions from Dimensional Reduction

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We consider the 3-d Gross-Neveu model in the broken phase and construct a stable brane world by means of a domain wall and an anti-wall. Fermions of opposite chirality are localized on the walls and coupled through the 3-d bulk. At large wall separation β the 2-d correlation length diverges exponentially, hence a 2-d observer cannot distinguish this situation from a 2-d space-time. The 3-d 4-fermion coupling and β fix the effective 2-d coupling such that the asymptotic freedom of the 2-d model arises. This mechanism provides criticality without fine tuning.

1. Motivation

In Yang-Mills gauge theory on a lattice of spacing a, one naturally obtains for the correlation length $\xi_g \gg a$ based on asymptotic freedom. However, if one includes quarks, say as Wilson fermions, one would naturally arrive at $\xi_q = O(a)$, i.e. a small quark mass ξ_q^{-1} requires an unnatural process of fine tuning. In this sense, Domain Wall Fermions [1] approach the chiral limit in a "natural" way, starting from a higher dimension. However, there the "extra dimension" is just technical: it does not host gauge fields, and one separates the walls at fixed glueball mass ξ_g^{-1}.

Here we try to construct a brane world in an odd dimension, which is endowed with a mechanism for dimensional reduction to an even dimension with light fermions. This would represent an all natural mechanism for the hierarchy of scales.

In particular we are going to study the Gross-Neveu model at large N and its reduction from $d = 3$ to 2. Details will be presented in Ref. [2].

2. The 2-d Gross-Neveu model

The Euclidean action of our target theory reads

$$S[\bar{\Psi},\Psi] = \int d^2x \Big[\bar{\Psi}\gamma_\mu\partial_\mu\Psi - \frac{g}{2N}(\bar{\Psi}\Psi)^2\Big] \ , \qquad (1)$$

where we suppress the flavor index $1 \dots N$. It has a discrete chiral $Z(2)$ symmetry

$$(\bar{\Psi}_L,\Psi_L) \to \pm(\bar{\Psi}_L,\Psi_L),\ (\bar{\Psi}_R,\Psi_R) \to \mp(\bar{\Psi}_R,\Psi_R).$$

With an auxiliary scalar field $\Phi = \frac{G}{N}\bar{\Psi}\Psi$ the action (1) is equivalent to

$$S[\bar{\Psi},\Psi,\Phi] = \int d^2x \Big[\bar{\Psi}\gamma_\mu\partial_\mu\Psi - \Phi\bar{\Psi}\Psi + \frac{N}{2g}\Phi^2\Big] \ .$$

In the limit $N \to \infty$, Φ freezes to a constant, and $\bar{\Psi},\Psi$ can be integrated out. This yields an effective potential $V_{eff}(\Phi)$ with the minima $\pm\Phi_0$, which obey the gap equation

$$\frac{1}{g} = \frac{1}{\pi}\int_0^{\Lambda_2} dk\, \frac{k}{k^2+\Phi_0^2} \ . \qquad (2)$$

At weak coupling $g \ll 1$ we have $\Lambda_2 \gg \Phi_0$ and

$$m = \Phi_0 = \Lambda_2 e^{-\pi/g} \qquad (3)$$

represents the fermion mass, which is generated by the spontaneous breaking of the $Z(2)$ symmetry. The exponent expresses asymptotic freedom.

3. The 3-d Gross-Neveu model

The 3-d action

$$S[\bar{\Psi},\Psi] = \int d^3x \Big[\bar{\Psi}\gamma_\mu\partial_\mu\Psi + \bar{\Psi}\gamma_3\partial_3\Psi - \frac{G}{2N}(\bar{\Psi}\Psi)^2\Big]$$

still has a $Z(2)$ symmetry,

$$\begin{aligned}(\bar{\Psi}_L,\Psi_L)|_{(\vec{x},x_3)} &\to \pm(\bar{\Psi}_L,\Psi_L)|_{(\vec{x},-x_3)} \ ,\\ (\bar{\Psi}_R,\Psi_R)|_{(\vec{x},x_3)} &\to \mp(\bar{\Psi}_R,\Psi_R)|_{(\vec{x},-x_3)} \ ,\end{aligned} \qquad (4)$$

which turns into the discrete chiral symmetry after dimensional reduction.

The 3-d gap equation reads

$$\frac{1}{G} = \frac{1}{(2\pi)^3}\int d^3k\, \frac{2}{k^2+\Phi_0^2} \ , \qquad (5)$$

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and for a cut-off $\Lambda_3 \gg \Phi_0$ one identifies a critical coupling $G_c = \pi^2/\Lambda_3$. At $G > G_c$ we are in the broken phase with

$$\Phi_0 = 2\pi(1/G_c - 1/G) \ , \qquad (6)$$

whereas weak coupling $G \leq G_c$ corresponds to a symmetric phase ($\Phi_0 = 0$).

4. Dimensional reduction from the symmetric phase

We compactify x_3 with periodicity β. Summing up the momenta k_3 in the gap equation (5) leads to

$$\frac{1}{G} = \frac{1}{(2\pi)^2} \int d^2k \, \frac{\coth(\beta\sqrt{\vec{k}^2 + \phi_0^2}/2)}{\sqrt{\vec{k}^2 + \phi_0^2}} \ . \qquad (7)$$

The remaining 2-d integral involves again the cut-off Λ_2. Consistency now requires the cut-off matching $\Lambda_2 = 1/G_c = \Lambda_3/\pi^2$, and $1/g = \beta(1/G - 1/G_c)$. Thus we arrive at the 2-d fermion mass

$$m = \frac{2}{\beta} e^{-\pi\beta(1/G - 1/G_c)} = \frac{2}{\beta} e^{-\pi/g} \ , \qquad (8)$$

where $2/\beta$ takes the rôle of an effective cut-off in the reduced system. The 2-d correlation length $\xi = 1/m$ grows exponentially as $\beta \to \infty$, hence we obtain dimensional reduction at *large* β.

This result seems fine, but the procedure is not satisfactory in view of our motivation: e.g. a non-perturbative treatment at finite N (on the lattice) should not start from the symmetric phase, because this just shifts the problem of fine tuning to $d = 3$. Therefore we now focus on *dimensional reduction from the broken phase.*

5. A single domain wall

Starting from the 3-d broken phase, the limit $\lim_{\beta \to 0} \beta/\xi = 2\ln(1 + \sqrt{2})$ does not provide light fermions. Hence we proceed differently and generate a light 2-d fermion as the $k_3 = 0$-mode on a domain wall. For the latter we make the ansatz $\Phi(x_3) = \Phi_0 \mathrm{tgh}(\Phi_0 x_3)$, which is inspired by Refs. [3]. We choose x_2 as the time direction, hence the Hamiltonian reads

$$H = \gamma_2[\gamma_1 \partial_1 + \gamma_3 \partial_3 - \Phi(x_3)] \ . \qquad (9)$$

The ansatz $\Psi(x_3)e^{ik_1x_1}e^{-iEt}$ (and the chiral representation for γ_i) reveals one localized eigenstate of H,

$$\Psi_0(x_3) = \sqrt{\frac{\Phi_0}{2}} \begin{pmatrix} 0 \\ \cosh^{-1}(\Phi_0 x_3) \end{pmatrix} \qquad (10)$$

with energy $E_0 = -k_1 > 0$, i.e. a left-mover. (On an anti-wall $-\Phi(x_3)$ one would obtain a right-mover with $E_0 = k_1 > 0$ and exchanged components in $\Psi_0(x_3)$).

In addition there are bulk states (not localized in x_3),

$$\Psi_{k_3}(x_3) = \frac{e^{ik_3x_3}}{\sqrt{2E(E + k_1)}} \begin{pmatrix} i(E + k_1) \\ \Phi_0 \mathrm{tgh}(\Phi_0 x_3) - ik_3 \end{pmatrix}$$

with $E = \pm\sqrt{\vec{k}^2 + \Phi_0^2}$, which form together with Ψ_0 an orthonormal basis for the 1-particle Hilbert space.

To verify the consistency of the wall profile we have to consider the chiral condensate $\bar{\Psi}\Psi$. Ψ_0 does not contribute to it, and if we sum up the bulk contributions of $E < 0$ we reproduce exactly $\Phi(x_3)$, which confirms the self-consistency of this single brane world.

In addition we are free to fill up some of the Ψ_0 states. Those with $E_0 < \Phi_0$ are confined to the $(1 + 1)$-d world, whereas states with $E_0 \geq \Phi_0$ can escape in the 3-direction. For the low energy observer on the brane this event appears as a *fermion number violation.*

6. A brane world with wall and anti-wall

We now want to include both, Ψ_L and Ψ_R, to be localized on a wall and an anti-wall separated by β. For the profile we make the ansatz

$$\begin{aligned} \Phi(x_3) &= \Phi_0\{a[\mathrm{tgh}_+ - \mathrm{tgh}_-] - 1\} \\ \mathrm{tgh}_\pm &:= \mathrm{tgh}(a\Phi_0[x_3 \pm \beta/2]) \ , \ a \in [0, 1]. \end{aligned} \qquad (11)$$

The ansatz for a bound state with the same form as for single walls,

$$\Psi_0(x_3) = c \begin{pmatrix} \alpha_1 \cosh^{-1}(a\Phi_0[x_3 - \beta/2]) \\ \alpha_2 \cosh^{-1}(a\Phi_0[x_3 + \beta/2]) \end{pmatrix} , \qquad (12)$$

implies the condition $\mathrm{tgh}(a\Phi_0\beta) = a$. Hence the parameter a controls the brane separation, such that $a \to 0$ and $a \to 1$ correspond to $\beta \to 0$ resp. $\beta \to \infty$.

Remarkably, the Dirac equation in this background still has an analytic solution, which is given by the ansatz (12) with

$$c = \frac{1}{2}\sqrt{\frac{a\Phi_0}{E_0(E_0+k_1)}} \ , \quad \alpha_1 = -i(E_0+k_1) \ ,$$
$$\alpha_2 = m = \sqrt{1-a^2}\Phi_0 \ , \quad E_0 = \pm\sqrt{k_1^2+m^2} \ .$$

This $\Psi_0(x_3)$ represents a Dirac fermion with components Ψ_L, Ψ_R localized on the wall resp. the anti-wall. For a fast motion to the left (right) we have $0 < E_0 \simeq -k_1$ $(+k_1)$, so that the lower (upper) component dominates. The mass m measures the extent of the L, R mixing. The limit $a \to 0$ does not provide a light fermion, $m = \Phi_0$. However, the opposite limit $a \to 1$ achieves this, since L, R mixing is suppressed,

$$m \simeq 2\,\Phi_0\, e^{-\beta\Phi_0} \ . \qquad (13)$$

As in the case of the symmetric phase, *large* β implies $\xi \gg \beta$ and therefore dimensional reduction. A low energy observer in $d = 1+1$ now perceives a point-like Dirac fermion composed of $L-$ and $R-$modes. On the other hand, a high energy observer in $d = 2+1$ refers to the scale Φ_0 (the 3-d fermion mass) and observes a Dirac fermion with strongly separated $L-$ and $R-$constituents.

Also the bulk states can be determined analytically [2]. Summing up again their $E < 0$ contributions to $\bar\Psi\Psi$ leads to

$$\frac{G}{N}\int dk_3 \bar\Psi_{k_3}\Psi_{k_3}|_{E<0} = \Phi(x_3) + \mathcal{A} \ , \qquad (14)$$

i.e. the desired result up to a term $\mathcal{A}$ [2], which has to be canceled by Ψ_0. This requires all the bound states with energies $E_0 \leq \Phi_0$ to be filled, i.e. exactly up to the threshold energy for the escape into the third dimension. Hence this wall anti-wall brane world does contain naturally light fermions, but it is completely packed with them, so that its physics is blocked by Pauli's principle.

We also checked if the wall and anti-wall repel or attract each other, which could lead to a disastrous end of this brane world. However, it turns out that the brane tension energy per fermion always amounts to Φ_0, hence it does not depend on the brane separation, so this toy world is stable.

Finally we studied the possibility of adding a fermion mass term $M\bar\Psi\Psi$ to the Lagrangian, so that the $Z(2)$ symmetry is also *explicitly* broken in $d = 3$ (which is actually realistic for a lattice formulation at finite N). This lifts the degeneracy of the minima of $V_{eff}(\Phi)$. If we still insert the profile (11), the condition for $\bar\Psi\Psi$ requires the bound fermion states to be filled even beyond Φ_0, hence in this case there is no stable configuration.

7. Summary

We studied the reduction $d = 3 \to 2$ in the Gross-Neveu model at large N. In $d = 3$ it splits into a symmetric phase with massless fermions and a broken phase, while the 2-d model is always broken and asymptotically free. Our question was if a *natural* setting in $d = 3$ can reduce to $d = 2$ with light fermions ($m \ll \Lambda_2$), and thus solve the hierarchy puzzle all by itself ?

Such a reduction from the symmetric phase does occur at *large* β, but this is not the "natural" starting point we were looking for.

Hence we start from the 3-d *broken* phase and build a brane world to achieve the desired mechanism. A single domain wall with a standard profile has a localized left-handed bound state. The bulk states then establish full self-consistency.

By means of a wall anti-wall pair we can accommodate left- and right-handed fermion components, localized on the wall resp. anti-wall. If their separation β is large, this appears to the $(1+1)$-d observer indeed as a point-like, light Dirac fermion. This world is stable, but consistency requires to fill the bound states up to the threshold for an escape into the 3-direction. Therefore the construction is basically successful, but unfortunately this world does not enjoy any flexibility for physical processes.

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