Dimensional reduction of fermions in the Gross-Neveu model

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Die Physik ist ein unerschöpfliches
Märchenbuch, in dem das
Unerwartete mit Sicherheit die
Erwartungen übertrifft, wenn man nur
weiterliest.

Valentin Braitenberg

Abstract

We study the Gross-Neveu model in two dimensions first in continuum space-time and then on a lattice with Wilson fermions. In the limit where the lattice spacing goes to zero the same results are obtained for the lattice model as for the continuum model. However, in the lattice model the bare mass has to be fine-tuned in order to restore the chiral symmetry. We then introduce the three-dimensional Gross-Neveu model and show that the two-dimensional model emerges from the three-dimensional model through dimensional reduction either by imposing periodic or domain wall boundary conditions in the three direction.

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Chapter 1

Introduction

Most of us have a good intuition for the phenomena of classical mechanics. We developed them while looking at simple examples in our every day lives. It is, however, difficult to obtain an intuition in quantum field theory as there are so few simple and exactly soluble examples. Besides that, we should not expect the few toy models that exist to have any direct implication to the real world. Still, it is important to study exactly soluble model problems. They are sources of inspiration and provide an optimal playground for learning more about the fascinating world of field theories. The techniques learned while working on toy models may later be helpful for resolving the real hard problems which affect QCD.
In this work we concentrate on the Gross-Neveu model [1], a simple fermionic theory with a scalar-scalar four-fermion interaction. The model was first studied by Gross and Neveu in the large N limit, who found a dynamical mass generation breaking down the chiral symmetry. The model is asymptotically free and thus resembles QCD in these two key points. In the large N limit it is renormalizable both in two and in three dimensions. We are particularly interested in the model from the point of view of dimensional reduction. The two-dimensional model can be obtained through dimensional reduction of the three-dimensional model either with periodical or with domain wall boundary conditions. The motivation to do this is that the same should be possible for QCD. It should result from dimensional reduction of a five-dimensional model with domain wall boundary conditions. This is realized in the D-theory approach to field theory in which classical fields arise from the dimensional reduction of discrete variables [2].
The structure of this work is as follows. In chapter 1, we analyze the Gross-Neveu model in two dimensions. We show that the model has a chiral symmetry and how in the large N limit this symmetry is spontaneously broken [3]. Chapter 2 discusses the two-dimensional massive Gross-Neveu model on a space-time lattice with Wilson fermions [4], [5]. On the lattice we have to fine-tune the bare mass in order to restore the chiral symmetry. This fine-tuning is not a very natural process and besides that, if we were interested in simulations at $N < \infty$ the fine-tuning would be a major problem. Therefore we are looking for another way of constructing the two-dimensional model, namely by dimensional reduction. In chapter 3 we study the three dimensional Gross-Neveu model [6]. We find a symmetry that includes a change of the sign in the third direction. The three-dimensional model

can either be in a broken or an unbroken phase, depending on the strength of the coupling constant. Chapter 4 is devoted to the dimensional reduction of the three-dimensional to the two-dimensional model. We start with the three-dimensional model in the unbroken phase and introduce periodic boundary conditions in the third direction. When the extension of the third direction becomes small in units of the correlation length the two-dimensional model emerges from the three-dimensional one. In chapter 5 the massive Gross-Neveu model in three dimensions and three-dimensional free fermions with domain wall boundary conditions are analyzed. Chapter 6 combines the two approaches from Chapter 5 and deals with the three-dimensional Gross-Neveu model with a dynamical domain wall [7], [8], [9]. The model is now in the broken phase. A potential with the shape of ether a kink or a double kink is postulated and the self consistency of these potentials is shown. When the extension of the third direction becomes small in units of the correlation length, again, the two dimensional model emerges.

Chapter 2

The two-dimensional Gross-Neveu model

The Gross-Neveu model [1] is a renormalizable, asymptotically free two-dimensional toy model which displays chiral symmetry breaking and dynamical mass generation. In Euclidean continuum space-time it is defined by the action

$$S[\bar{\Psi}, \Psi] = \int d^2x \left[\sum_i \bar{\Psi}^i(x)\gamma_\mu \partial_\mu \Psi^i(x) - \frac{G}{2} \left(\sum_i \bar{\Psi}^i(x)\Psi^i(x) \right)^2 \right], \tag{2.1}$$

where Ψ and $\bar{\Psi}$ are independent N-component fermion fields ($i = 1, \ldots, N$) and G denotes the coupling constant. We will generally suppress the flavor indices i and use the notation

$$\begin{aligned} \bar{\Psi}\Psi &= \sum_i \bar{\Psi}^i \Psi^i, \\ \bar{\Psi}\gamma_\mu \partial_\mu \Psi &= \sum_i \bar{\Psi}^i \gamma_\mu \partial_\mu \Psi^i. \end{aligned} \tag{2.2}$$

The two dimensional γ-matrices are defined as $\gamma_1 = \sigma_1, \gamma_2 = \sigma_2, \gamma_3 = \sigma_3$, where σ_i are the Pauli matrices.

2.1 Chiral symmetry

First we investigate the symmetry of the two-dimensional Gross-Neveu model. To do so, the spinors are decomposed into left- and right-handed components

$$\Psi_{R,L} = \frac{1 \pm \gamma_3}{2} \Psi, \tag{2.3}$$

$$\bar{\Psi}_{R,L} = \bar{\Psi} \frac{1 \mp \gamma_3}{2}. \tag{2.4}$$

Inserting the decomposed spinors into the action we obtain

$$S[\bar{\Psi},\Psi] = \int d^2x \left[(\bar{\Psi}_R + \bar{\Psi}_L)\gamma_\mu \partial_\mu (\Psi_R + \Psi_L) - \frac{G}{2}\left((\bar{\Psi}_R + \bar{\Psi}_L)(\Psi_R + \Psi_L) \right)^2 \right]. \tag{2.5}$$

By investigating the terms

$$\begin{aligned} \bar{\Psi}_R \gamma_\mu \partial_\mu \Psi_L &= \bar{\Psi}\frac{1}{2}(1-\gamma_3)\gamma_\mu \partial_\mu \frac{1}{2}(1-\gamma_3)\Psi = \bar{\Psi}\frac{1}{4}(1-\gamma_3)(1+\gamma_3)\gamma_\mu\partial_\mu \Psi = 0, \\ \bar{\Psi}_L \gamma_\mu \partial_\mu \Psi_R &= \bar{\Psi}\frac{1}{2}(1+\gamma_3)\gamma_\mu \partial_\mu \frac{1}{2}(1+\gamma_3)\Psi = \bar{\Psi}\frac{1}{4}(1+\gamma_3)(1-\gamma_3)\gamma_\mu\partial_\mu \Psi = 0, \\ \bar{\Psi}_R \Psi_R &= \bar{\Psi}\frac{1}{2}(1-\gamma_3)\frac{1}{2}(1+\gamma_3)\Psi = 0, \\ \bar{\Psi}_L \Psi_L &= \bar{\Psi}\frac{1}{2}(1+\gamma_3)\frac{1}{2}(1-\gamma_3)\Psi = 0, \end{aligned} \tag{2.6}$$

we can write the action as

$$S[\bar{\Psi},\Psi] = \int d^2x \left[\bar{\Psi}_R \gamma_\mu \partial_\mu \Psi_R + \bar{\Psi}_L \gamma_\mu \partial_\mu \Psi_L - \frac{G}{2}(\bar{\Psi}_R \Psi_L)^2 - \frac{G}{2}(\bar{\Psi}_L \Psi_R)^2 \right]. \tag{2.7}$$

Hence the action is invariant under separate transformations of the right- and left-handed components of the fields.

$$\begin{aligned} &\Psi_L \to s_L \Psi_L, \quad \bar{\Psi}_L \to s_L \bar{\Psi}_L \quad \text{and} \\ &\Psi_R \to s_R \Psi_R, \quad \bar{\Psi}_R \to s_R \bar{\Psi}_R, \end{aligned} \tag{2.8}$$

with $s_L = \pm 1,\ s_R = \pm 1$.

2.2 The large N limit

In order to perform explicit calculations we consider the N-component fermion fields in the large N limit where $g = GN$ is kept fixed and show that the symmetry is spontaneously broken. Note that in the two-dimensional model the coupling constant is dimensionless. It is convenient to linearize (2.1) and replace it by the action

$$S[\bar{\Psi},\Psi,\Phi] = \int d^2x \left[\bar{\Psi}(x)\gamma_\mu \partial_\mu \Psi(x) + \frac{1}{2G}\Phi(x)^2 + \bar{\Psi}(x)\Psi(x)\Phi(x) \right], \tag{2.9}$$

where

$$\Phi(x) = -G\bar{\Psi}(x)\Psi(x). \tag{2.10}$$

The two actions are related to each other by

$$\exp(-S[\bar{\Psi},\Psi]) = \int \mathcal{D}\Phi \exp(-S[\bar{\Psi},\Psi,\Phi]). \tag{2.11}$$

In the large N limit we may restrict $\Phi(x)$ to its constant zero-mode Φ_0. Hence the integration over all $\Phi(x)$ reduces to a simple integration over Φ_0 ($\mathcal{D}\Phi \to d\Phi_0$). We transform the action into momentum space and obtain

$$\begin{aligned} S[\bar\Psi,\Psi,\Phi_0] &= \frac{1}{(2\pi)^2}\int d^2k \int d^2k' \big[\bar\Psi(k')(i\gamma_\mu k_\mu + \Phi_0)\Psi(k)\times \\ &\quad \frac{1}{(2\pi)^2}\int d^2x \ \exp(i(k+k')x)\big] + \frac{L^2}{2G}\Phi_0^2 \\ &= \frac{1}{(2\pi)^2}\int d^2k \ \big[\bar\Psi(-k)(i\gamma_\mu k_\mu + \Phi_0)\Psi(k)\big] + \frac{L^2}{2G}\Phi_0^2, \end{aligned} \tag{2.12}$$

where L^2 is the volume of space-time. The Fourier transformation is defined as

$$\Psi(x) = \frac{1}{(2\pi)^2}\int d^2k \ \Psi(k)\exp(ikx). \tag{2.13}$$

To solve the model we integrate out the fermion fields

$$\int \mathcal{D}\bar\Psi\mathcal{D}\Psi \ \exp(-S[\bar\Psi,\Psi,\Phi_0]) = \exp(-V_{eff}(\Phi_0)L^2), \tag{2.14}$$

where $V_{eff}(\Phi_0)$ is the effective potential. To do so we introduce periodic boundary conditions in space-time, thus obtain a lattice in momentum space ($\int d^2k \to (\frac{2\pi}{L})^2\sum_k$) and perform on each lattice point k the integral $\int d\bar\Psi_k d\Psi_k e^{-\bar\Psi_k A \Psi_k} = \mathrm{Det}(A)$. When we have integrated out the fermion fields we let the lattice spacing go to zero and return to the infinite volume. The integration of fermion fields is explicitly shown in appendix A.

$$\begin{aligned} Z &= \int \mathcal{D}\bar\Psi\mathcal{D}\Psi d\Phi_0 \exp\left[-\frac{1}{(2\pi)^2}\int d^2k \ \bar\Psi(-k)(i\gamma_\mu k_\mu + \Phi_0)\Psi(k) - \frac{L^2}{2G}\Phi_0^2\right] \\ &\simeq \prod_k \int d\bar\Psi_{-k} d\Psi_k d\Phi_0 \exp\left[-\frac{1}{L^2}\sum_k \left(\bar\Psi_{-k}(i\gamma_\mu k_\mu + \Phi_0)\Psi_k\right) - \frac{L^2}{2G}\Phi_0^2\right] \\ &= \prod_k \int d\Phi_0 (k^2+\Phi_0^2)^N \exp\left[-\frac{L^2}{2G}\Phi_0^2\right] \\ &= \int d\Phi_0 \exp\left[\sum_k \ln(k^2+\Phi_0^2)^N - \frac{L^2}{2G}\Phi_0^2\right] \\ &\simeq \int d\Phi_0 \exp\left[\left(\frac{L}{2\pi}\right)^2 N \int d^2k \ \ln(k^2+\Phi_0^2) - \frac{L^2}{2G}\Phi_0^2\right]. \end{aligned} \tag{2.15}$$

We introduce a momentum cutoff Λ and thus the effective potential reads

$$\begin{aligned} V_{eff}(\Phi_0) &= N\left[-\frac{1}{(2\pi)^2}\int d^2k \ln\left(k^2+\Phi_0^2\right)+\frac{1}{2g}\Phi_0^2\right] \\ &= N\left[-\frac{1}{2\pi}\int_0^\Lambda dk\, k\ln\left(k^2+\Phi_0^2\right)+\frac{1}{2g}\Phi_0^2\right] \\ &= N\left[\frac{\Phi_0^2}{2g}-\frac{1}{4\pi}\left((\Lambda^2+\Phi_0^2)\ln(\Lambda^2+\Phi_0^2)-\Phi_0^2\ln(\Phi_0^2)-\Lambda^2\right)\right]. \end{aligned} \tag{2.16}$$

For large Λ and after adding the convenient constant $\frac{N\Lambda^2}{4\pi}(\ln(\Lambda^2)-1)$, the potential reduces to

$$\begin{aligned} V_{eff}(\Phi_0) &= N\left[\frac{\Phi_0^2}{2g}-\frac{1}{4\pi}\left(\Lambda^2\ln\left(\frac{\Lambda^2+\Phi_0^2}{\Lambda^2}\right)+\Phi_0^2\ln\left(\frac{\Lambda^2+\Phi_0^2}{\Lambda^2}\right)\right)\right] \\ &= N\left[\frac{\Phi_0^2}{2g}+\frac{\Phi_0^2}{4\pi}\left(\ln\left(\frac{\Phi_0^2}{\Lambda^2}\right)-1\right)\right]. \end{aligned} \tag{2.17}$$

Figure 2.1: Effective potential of the Gross-Neveu model.

Starting from (2.16), the minimum of the effective potential is given by

$$\partial_{\Phi_0}V_{eff}(\Phi_0) = -\frac{1}{(2\pi)^2}\int d^2k\frac{2\Phi_0}{k^2+\Phi_0^2}+\frac{\Phi_0}{g}=0, \tag{2.18}$$

and we obtain the so-called mass gap equation

$$\frac{1}{(2\pi)^2}\int d^2k\frac{2}{k^2+\Phi_0^2}=\frac{1}{g}. \tag{2.19}$$

To evaluate the mass gap equation a momentum cutoff Λ is introduced and for $\Lambda\to\infty$ the potential Φ_0 amounts to

$$\Phi_0=\Lambda\frac{1}{\sqrt{\exp\frac{2\pi}{g}-1}}. \tag{2.20}$$

We assume that $g\ll 2\pi$ and then (2.20) reduces to

$$\Phi_0=\Lambda\exp\left(-\frac{\pi}{g}\right). \tag{2.21}$$

As the model displays dynamical mass generation ($\Phi_0\neq 0$), the symmetry is spontaneously broken.

Chapter 3

Lattice Gross-Neveu model

In this chapter we give a brief introduction to lattice fermions [10], [11], [12] and then investigate the Gross-Neveu model on a two dimensional lattice [4], [5].

3.1 Lattice fermions

In the preceding chapter we have studied the continuum action and the partition function $Z = \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi \exp(-S[\bar{\Psi}, \Psi])$. So far the path integrals have not been given a precise mathematical meaning. We do this now by replacing the continuum fermion fields by variables that live on the lattice points x. The transition is performed by making the following substitutions

$$\begin{aligned} \Psi(x) &\rightarrow \Psi_x, \\ \bar{\Psi}(x) &\rightarrow \bar{\Psi}_x, \\ \partial_\mu \Psi(x) &\rightarrow \frac{1}{a}\left(\Psi_{x+\hat{\mu}} - \Psi_{x-\hat{\mu}}\right), \\ \int d^2x &\rightarrow a^2 \sum_x, \end{aligned} \tag{3.1}$$

where a is the lattice spacing.
The action of massive naive lattice fermions without an interaction term is

$$\begin{aligned} S[\bar{\Psi}_x, \Psi_x] &= \frac{a}{2} \sum_{x,\mu} \left(\bar{\Psi}^i_x \gamma_\mu \Psi^i_{x+\hat{\mu}a} - \bar{\Psi}^i_x \gamma_\mu \Psi^i_{x-\hat{\mu}a}\right) + a^2 m_0 \sum_x \bar{\Psi}^i_x \Psi^i_x \\ &= a \sum_{x,y} \bar{\Psi}^i_x K \Psi^i_y, \end{aligned} \tag{3.2}$$

with

$$K = \sum_\mu \gamma_\mu \frac{1}{2}(\delta_{x+\hat{\mu}a,y} - \delta_{x-\hat{\mu}a,y}) + a m_0 \delta_{x,y}, \tag{3.3}$$

and $|\hat{\mu}| = 1$. It is instructive to transform this expression to momentum space, because it allows us to read off the lattice fermion propagator

$$< \bar{\Psi}_x \Psi_y >= K^{-1} = \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^2k}{(2\pi)^2} \frac{-i\sum_\mu \gamma_\mu \sin(k_\mu a) + ma}{\sum_\mu \frac{1}{a^2}\sin(k_\mu a)^2 + m^2} \exp(ik(x-y)). \qquad (3.4)$$

The energies of the lattice fermions show up as poles in the propagator (3.4). For small m the denominator (3.4) goes to zero not only at $k_\mu = 0$, but also at the corners of the Brillouin zone, where $k_\mu = \pi/a$. Therefore the spectrum has 3 (resp. $2^d - 1$ in a d dimensional theory) extra fermions that are absent in the continuum theory. These extra states do not disappear in the continuum limit, thus the naive lattice fermion action (3.2) does not lead to the correct continuum theory. This is the so-called fermion doubling problem. It has been shown by Nielsen and Ninomiya [13] that one cannot solve this problem without breaking the chiral symmetry in the limit $m \to 0$.
One proposal how to deal with lattice fermions was originally made by Wilson [14]. The action in (3.2) is modified in such a way that the zeros at the edges of the Brillouin zone in the denominator of (3.4) are lifted by an amount proportional to the inverse lattice spacing. The so-called Wilson term thus eliminates the doublers. The price one has to pay to eliminate the doublers and hopefully to ensure the correct continuum limit is the explicit breaking of the chiral symmetry.

3.2 Gross-Neveu model on a two-dimensional lattice

We calculate the effective potential and the mass gap equation. It is shown that the effective potential becomes chirally symmetric in the continuum limit if the coupling constant is adjusted. On the lattice the action of the massive Gross-Neveu model with the Wilson term looks as follows

$$\begin{aligned} S[\bar{\Psi},\Psi] &= \frac{a}{2}\sum_{x,\mu}\left(\bar{\Psi}^i_x\gamma_\mu\Psi^i_{x+\hat{\mu}a} - \bar{\Psi}^i_x\gamma_\mu\Psi^i_{x-\hat{\mu}a}\right) + a^2 m_0 \sum_x \bar{\Psi}^i_x\Psi^i_x \\ &+ r\frac{a}{2}\sum_{x,\mu}\left(2\bar{\Psi}^i_x\Psi^i_x - \bar{\Psi}^i_x\Psi^i_{x+\hat{\mu}a} - \bar{\Psi}^i_x\Psi^i_{x-\hat{\mu}a}\right) - a^2\frac{G}{2}\sum_x\left(\bar{\Psi}^i_x\Psi^i_x\right)^2. \end{aligned} \qquad (3.5)$$

We linearize the action by defining $\sigma(x) = m_0 - G\bar{\Psi}_x\Psi_x$, and promptly restrict $\sigma(x)$ to its zero-mode σ. Furthermore we set the Wilson parameter $r = 1$.

$$\begin{aligned} S[\bar{\Psi},\Psi,\sigma] &= \frac{a}{2}\sum_{x,\mu}\left(\bar{\Psi}^i_x\gamma_\mu\Psi^i_{x+\hat{\mu}a} - \bar{\Psi}^i_x\gamma_\mu\Psi^i_{x-\hat{\mu}a}\right) + a^2\sigma\sum_x \bar{\Psi}^i_x\Psi^i_x \\ &+ \frac{a}{2}\sum_{x,\mu}\left(2\bar{\Psi}^i_x\Psi^i_x - \bar{\Psi}^i_x\Psi^i_{x+\hat{\mu}a} - \bar{\Psi}^i_x\Psi^i_{x-\hat{\mu}a}\right) - \frac{a^2}{2G}(\sigma - m_0)^2. \end{aligned} \qquad (3.6)$$

The partition function is given by

$$Z = \int \mathcal{D}\bar{\Psi}_x\mathcal{D}\Psi_y d\sigma \exp\left(-a\sum_{x,y}\bar{\Psi}_x K\Psi_y + a^2\frac{1}{2G}(\sigma - m_0)^2\right), \qquad (3.7)$$

with

$$\begin{aligned}
K &= \frac{1}{2}\sum_{\mu}\left[\gamma_\mu(\delta_{x+\hat{\mu}a,y} - \delta_{x-\hat{\mu}a,y}) + 2\delta_{x,y} - \delta_{x+\hat{\mu}a,y} - \delta_{x-\hat{\mu}a,y}\right] + a\sigma\delta_{x,y} \\
&= \frac{a^2}{2}\sum_{\mu}\left[\gamma_\mu \int \frac{d^2k}{(2\pi)^2}\exp(ik(x-y))(\exp(ik_\mu a) - \exp(-ik_\mu a))\right. \\
&\left. + \int \frac{d^2k}{(2\pi)^2}\exp(ik(x-y))(2 - (\exp(ik_\mu a) - \exp(-ik_\mu a)) + a\sigma)\right] \\
&= a^2 \int \frac{d^2k}{(2\pi)^2}\exp(ik(x-y))\left[i\gamma_\mu \sin k_\mu a + \sum_\mu (1-\cos k_\mu a) + a\sigma\right].
\end{aligned} \tag{3.8}$$

Here we introduced the notation $k\hat{\mu} = k_\mu$. Inserting (3.8) into (3.7), replacing the sums over x and y by integrals and going to momentum space, we obtain

$$\begin{aligned}
Z = &\prod_{x,y}\int d\bar{\Psi}_x d\Psi_y d\sigma \exp\left[-\frac{1}{a}\int d^2x \int d^2y \int \frac{d^2k}{(2\pi)^2}\int \frac{d^2k'}{(2\pi)^2}\int \frac{d^2k''}{(2\pi)^2}\times\right. \\
&\exp(ix(k'+k))\exp(iy(k''-k))\Psi_{k'}\left(i\gamma_\mu \sin k_\mu a + \sum_\mu (1-\cos k_\mu a) + a\sigma\right)\Psi_{k''} \\
&\left.+\frac{1}{2G}(\sigma - m_0)^2\right] \\
= &\prod_k \int d\bar{\Psi}_{-k} d\Psi_k d\sigma \exp\left[-\frac{1}{a}\int \frac{d^2k}{(2\pi)^2}\bar{\Psi}_{-k}\left(i\gamma_\mu \sin k_\mu a + \sum_\mu (1-\cos k_\mu a) + a\sigma\right)\Psi_k\right. \\
&\left.+\frac{1}{2G}(\sigma - m_0)^2\right].
\end{aligned} \tag{3.9}$$

Then we integrate out the fermion fields and obtain

$$\begin{aligned}
Z = \exp&\left[NL^2 \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^2k}{(2\pi)^2}\mathrm{Det}\left(\ln(i\gamma_\mu \sin k_\mu a + \sum_\mu (1-\cos k_\mu a) + a\sigma)\right)\right. \\
&\left.+\frac{L^2}{2G}(\sigma - m_0)^2\right].
\end{aligned} \tag{3.10}$$

Thus the effective potential is given by

$$\begin{aligned}
V_{eff} = &-N\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^2k}{(2\pi)^2}\ln\left(\sum_\mu \sin^2 k_\mu a + (a\sigma + \sum_\mu (1-\cos k_\mu a))^2\right) \\
&+\frac{N}{2g}(\sigma - m_0)^2.
\end{aligned} \tag{3.11}$$

Varying the effective potential, we obtain the bare gap equation

$$\frac{\partial V_{eff}}{\partial\sigma} = -N\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^2k}{(2\pi)^2}\frac{2(\sigma+\sum_\mu(1-\cos k_\mu))}{\sum_\mu \sin^2 k_\mu a + (a\sigma^2+\sum_\mu(1-\cos k_\mu a))^2} + \frac{N}{g}(\sigma - m_0) = 0, \tag{3.12}$$

$$\frac{\sigma-m_0}{2g} = \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^2k}{(2\pi)^2}\frac{\sigma+\sum_\mu(1-\cos k_\mu)}{\sum_\mu \sin^2 k_\mu a + (a\sigma^2+\sum_\mu(1-\cos k_\mu a))^2}. \tag{3.13}$$

In order to evaluate the effective potential we rewrite it as follows:

$$\frac{V_{eff}}{N} = A - I, \tag{3.14}$$

where

$$A = \frac{1}{2g}(\sigma-m_0)^2, \tag{3.15}$$

$$I = \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^2k}{(2\pi)^2}\left(\ln\Delta + \ln(1+\frac{\varepsilon}{\Delta})\right), \tag{3.16}$$

$$\Delta = \sum_\mu \sin^2 k_\mu a + (\sum_\mu(1-\cos k_\mu a))^2 + \sigma^2 a^2, \tag{3.17}$$

$$\varepsilon = 2a\sigma\sum_\mu(1-\cos k_\mu a). \tag{3.18}$$

We then expand the integrand into a power series of ε

$$I = I_0 + I_1 + I_2 + \cdots, \tag{3.19}$$

where

$$I_0 = \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^2k}{(2\pi)^2}\ln\Delta, \qquad I_n = -\frac{(-1)^n}{n}\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^2k}{(2\pi)^2}\frac{\varepsilon^n}{\Delta^n},\ n = 1,2,3,\ldots. \tag{3.20}$$

It can be shown that I_1 reduces to a linear term in σ and I_2 reduces to a quadratic term in σ, while $I_n(n\geq 3)$ vanishes in the limit $a\to 0$. This is seen by rewriting (3.20), using a rescaled variable $p_\mu = k_\mu a$.

$$I_n = -\frac{(-1)^n}{n}(2\sigma)^n a^{n-2}\int_{-\pi}^{\pi}\frac{d^2p}{(2\pi)^2}\frac{(\sum_\mu(1-\cos p_\mu))^n}{(\sum_\mu \sin^2 p_\mu + (\sum_\mu(1-\cos p_\mu))^2 + a^2\sigma^2)^n} \tag{3.21}$$

These integrals are well defined in the limit $a \to 0$.

$$\begin{aligned} I_1 &= \frac{2\sigma}{a} c_1, \\ I_2 &= -2\sigma^2 c_2, \\ I_n &= 0, (n \geq 3), \end{aligned} \tag{3.22}$$

with

$$\begin{aligned} c_1 &= \int_{-\pi}^{\pi} \frac{d^2k}{(2\pi)^2} \frac{\sum_\mu (1-\cos p_\mu)}{\sum_\mu \sin^2 p_\mu + (\sum_\mu (1-\cos p_\mu))^2} = 0.3849, \\ c_2 &= \int_{-\pi}^{\pi} \frac{d^2k}{(2\pi)^2} \frac{(\sum_\mu (1-\cos p_\mu))^2}{(\sum_\mu \sin^2 p_\mu + (\sum_\mu (1-\cos p_\mu))^2)^2} = 0.1548. \end{aligned} \tag{3.23}$$

Next I_0 can be written in the integral representation

$$\begin{aligned} I_0 &= \int_0^{\sigma^2} d\rho F(\rho), \\ F(\rho) &= \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^2k}{(2\pi)^2} \frac{a^2}{\sum_\mu \sin^2 k_\mu a + (\sum_\mu (1-\cos k_\mu a))^2 + \rho a^2}. \end{aligned} \tag{3.24}$$

In the limit $a \to 0$ we have

$$F(\rho) = \int_{-\infty}^{\infty} \frac{d^2k}{(2\pi)^2} \frac{1}{\sum_\mu k_\mu^2 + \rho} + c_0, \tag{3.25}$$

with

$$c_0 = \int_{-\pi}^{\pi} \frac{d^2p}{(2\pi)^2} \frac{(\sum_\mu (p_\mu - \sin p_\mu))^2 - (\sum_\mu (1-\cos p_\mu))^2}{(\sum_\mu \sin^2 p_\mu + (\sum_\mu (1-\cos p_\mu))^2)(\sum_\mu p_\mu^2)} = 0.427. \tag{3.26}$$

We can check (3.26) by calculating

$$\begin{aligned} &\lim_{a\to 0} \left(F(\rho) - \int_{-\frac{a}{\pi}}^{\frac{a}{\pi}} \frac{d^2k}{(2\pi)^2} \frac{1}{\sum_\mu k_\mu^2 + \rho} \right) = \\ &\lim_{a\to 0} \int_{-\pi}^{\pi} \frac{d^2p}{(2\pi)^2} \frac{(\sum_\mu (p_\mu - \sin p_\mu))^2 - (\sum_\mu (1-\cos p_\mu))^2}{(\sum_\mu \sin^2 p_\mu + (\sum_\mu (1-\cos p_\mu))^2 + \rho a^2)(\sum_\mu p_\mu^2) + \rho a^2} = c_0. \end{aligned} \tag{3.27}$$

Then $F(\rho)$ and I_0 are computed

$$F(\rho) = \frac{1}{4\pi} \ln(\frac{\pi^2}{a^2\rho} + 1) + c_0 \approx -\frac{1}{4\pi} \ln(a^2\rho) + \hat{c}_0, \tag{3.28}$$

$$I_0 = -\frac{1}{4\pi} \sigma^2 \ln(\frac{a^2\sigma^2}{e}) + \hat{c}_0 \sigma^2. \tag{3.29}$$

Now we are ready to compute V_{eff} in the limit $a \to 0$

$$\frac{V_{eff}}{N} = \frac{1}{2g}(\sigma - m_0)^2 - \frac{2\sigma}{a}c_1 + 2\sigma^2 c_2 + \frac{1}{4\pi}\sigma^2 \ln\frac{a^2\sigma^2}{e} - \hat{c}_0\sigma^2. \tag{3.30}$$

Since we are interested in a renormalized theory with chiral symmetry, we map the lattice spacing to the continuum cutoff

$$2c_2 + \frac{1}{4\pi}\ln a^2 - \hat{c}_0 = \frac{1}{4\pi}\ln\left(\frac{1}{\Lambda^2}\right), \tag{3.31}$$

and have to fine-tune the bare mass

$$m_0 = -\frac{2c_1 g}{a}. \tag{3.32}$$

Finally we get

$$\frac{V_{eff}}{N} = \frac{1}{2g}\sigma^2 + \frac{\sigma^2}{4\pi}\ln(\frac{\sigma^2}{\Lambda} - 1). \tag{3.33}$$

This is exactly the formula we obtained in the continuum model.

In this analytic calculation in the large N limit the fine-tuning of m_0 was not such a big problem. However, it is not very natural that the chiral symmetry is only restored after the fine-tuning of the mass m_0. Besides that, if one is interested in monte-carlo simulations at $N < \infty$, fine-tuning is a mayor problem. Therefore we are interested in finding another way of constructing the two-dimensional Gross-Neveu model.

Chapter 4

The three-dimensional Gross-Neveu model

In this chapter we perform similar calculations as in the first one, although now in three dimensions. One mayor difference between the two- and the three-dimensional model is that the three-dimensional model can be in a broken or an unbroken phase, depending on the strength of the coupling constant, whereas the two-dimensional model is always in the broken phase. The action of the Gross-Neveu model in three dimensions is given by

$$S[\bar{\Psi}, \Psi] = \int d^3x \left[\bar{\Psi}(x)\gamma_\mu \partial_\mu \Psi(x) - \frac{G^{(3)}}{2} \left(\bar{\Psi}(x)\Psi(x) \right)^2 \right]. \tag{4.1}$$

Note that $G^{(3)}$ has the dimension m^{-1} whereas the two dimensional coupling constant G was dimensionless.

4.1 Symmetry

To investigate the symmetry the spinors are, like in the two dimensional theory, decomposed into left- and right-handed components

$$\begin{aligned} \Psi_{R,L} &= \frac{1 \pm \gamma_3}{2} \Psi, \\ \bar{\Psi}_{R,L} &= \bar{\Psi} \frac{1 \mp \gamma_3}{2}. \end{aligned} \tag{4.2}$$

Inserting the decomposed spinors into the action we obtain

$$S[\bar{\Psi}, \Psi] = \int d^3x (\bar{\Psi}_R + \bar{\Psi}_L)\gamma_\mu \partial_\mu (\Psi_R + \Psi_L) - \frac{G^{(3)}}{2} ((\bar{\Psi}_R + \bar{\Psi}_L)(\Psi_R + \Psi_L))^2. \tag{4.3}$$

After investigating the following terms

$$\begin{aligned}\bar{\Psi}_R\gamma_\mu\partial_\mu\Psi_L &= \bar{\Psi}\frac{1}{2}(1-\gamma_3)\gamma_\mu\partial_\mu\frac{1}{2}(1-\gamma_3)\Psi = \bar{\Psi}\frac{1}{2}(1-\gamma_3)\gamma_3\partial_3\Psi,\\ \bar{\Psi}_L\gamma_\mu\partial_\mu\Psi_R &= \bar{\Psi}\frac{1}{2}(1+\gamma_3)\gamma_\mu\partial_\mu\frac{1}{2}(1+\gamma_3)\Psi = \bar{\Psi}\frac{1}{2}(1+\gamma_3)\gamma_3\partial_3\Psi,\\ \bar{\Psi}_R\Psi_R &= \bar{\Psi}\frac{1}{2}(1-\gamma_3)\frac{1}{2}(1+\gamma_3)\Psi = 0,\\ \bar{\Psi}_L\Psi_L &= \bar{\Psi}\frac{1}{2}(1+\gamma_3)\frac{1}{2}(1-\gamma_3)\Psi = 0,\end{aligned} \tag{4.4}$$

we can write the action as

$$\begin{aligned}S[\bar{\Psi},\Psi] = \int d^3x\Big[&\bar{\Psi}_R\gamma_\mu\partial_\mu\Psi_R + \bar{\Psi}_L\gamma_\mu\partial_\mu\Psi_L - \frac{G^{(3)}}{2}(\bar{\Psi}_R\Psi_L)^2 - \frac{G^{(3)}}{2}(\bar{\Psi}_L\Psi_R)^2\\ &+ \bar{\Psi}_R\gamma_3\partial_3\Psi_L + \bar{\Psi}_L\gamma_3\partial_3\Psi_R\Big].\end{aligned} \tag{4.5}$$

In contrast to the the two dimensional model we do not have a chiral symmetry but a symmetry that includes a change of the sign in the third direction. The action is invariant under the transformations

$$\begin{aligned}&\Psi_L(x_1,x_2,x_3) \to s_L\Psi_L(x_1,x_2,-x_3), \quad \bar{\Psi}_L(x_1,x_2,x_3) \to s_L\bar{\Psi}_L(x_1,x_2,-x_3) \quad \text{and}\\ &\Psi_R(x_1,x_2,x_3) \to s_R\Psi_R(x_1,x_2,-x_3), \quad \bar{\Psi}_R(x_1,x_2,x_3) \to s_R\bar{\Psi}_R(x_1,x_2,-x_3),\end{aligned} \tag{4.6}$$

with $s_R = \pm 1,\ s_L = \pm 1$.

4.2 The Large N Limit

In order to perform explicit calculations we consider the fields in the large N limit where $g^{(3)} = G^{(3)}N$ is kept fixed. In an analogous manner as in the two dimensional model, we replace (4.1) by the action

$$S[\bar{\Psi},\Psi,\Phi] = \int d^3x\left[\bar{\Psi}(x)\gamma_\mu\partial_\mu\Psi(x) + \frac{1}{2G^{(3)}}\Phi^2(x) + \bar{\Psi}(x)\Psi(x)\Phi(x)\right], \tag{4.7}$$

where

$$\Phi(x) = -G^{(3)}\bar{\Psi}(x)\Psi(x). \tag{4.8}$$

The two actions are connected by

$$\exp(-S[\bar{\Psi},\Psi]) = \int \mathcal{D}\Phi\,\exp(-S[\bar{\Psi},\Psi,\Phi]). \tag{4.9}$$

In the large N limit we may restrict $\Phi(x)$ to its zero-mode Φ_0. Doing so the action in momentum space reads

$$\begin{aligned} S[\bar{\Psi}, \Psi, \Phi_0] =& \frac{1}{(2\pi)^3} \int d^3k \int d^3k' \bar{\Psi}(k')(i\gamma_\mu k_\mu + \Phi_0)\Psi(k) \times \\ & \frac{1}{(2\pi)^3} \int d^3x \exp(i(k+k')x) + \frac{L^3}{2G^{(3)}}\Phi_0^2 \\ =& \frac{1}{(2\pi)^3} \int d^3k \bar{\Psi}(-k)(i\gamma_\mu k_\mu + \Phi_0)\Psi(k) + \frac{L^3}{2G^{(3)}}\Phi_0^2, \end{aligned} \tag{4.10}$$

where L^3 is the volume of space-time. To solve the model we again integrate out the fermion fields

$$\int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi \ \exp(-S[\bar{\Psi}, \Psi, \Phi_0]) = \exp(-V_{eff}(\Phi_0)L^3). \tag{4.11}$$

As in the two-dimensional model we introduce periodic boundary conditions in coordinate space-time, thus discretize the momentum space ($\int d^3k \to (\frac{2\pi}{L})^3 \sum_k$) and perform on each lattice point k the integral $\int d\bar{\Psi}_k d\Psi_k e^{-\bar{\Psi}_k A \Psi_k} = \mathrm{Det}(A)$. When we have integrated out the fermion fields we let the lattice spacing go to zero and return to the infinite volume.

$$\begin{aligned} Z &= \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi d\Phi_0 \exp\left[-\frac{1}{(2\pi)^3}\int d^3k \bar{\Psi}(-k)(i\gamma_\mu k_\mu + \Phi_0)\Psi(k) - \frac{L^3}{2G^{(3)}}\Phi_0^2\right] \\ &\simeq \prod_k \int d\bar{\Psi}_k d\Psi_k d\Phi_0 \exp\left[-\frac{1}{L^3}\sum_k \left(\bar{\Psi}_{-k}(i\gamma_\mu k_\mu + \Phi_0)\Psi_k\right) - \frac{L^3}{2G^{(3)}}\Phi_0^2\right] \\ &= \prod_k \int d\Phi_0 (k^2 + \Phi_0^2)^N \exp\left[-\frac{L^3}{2G^{(3)}}\Phi_0^2\right] \\ &= \int d\Phi_0 \exp\left[\sum_k \ln(k^2+\Phi_0^2)^N - \frac{L^3}{2G^{(3)}}\Phi_0^2\right] \\ &\simeq \int d\Phi_0 \exp\left[\left(\frac{L}{2\pi}\right)^3 N \int d^3k \ln(k^2+\Phi_0^2) - \frac{L^3}{2G^{(3)}}\Phi_0^2\right]. \end{aligned} \tag{4.12}$$

We then introduce a momentum cutoff Λ and with $g^{(3)} = G^{(3)}N$ the effective potential reads

$$\begin{aligned} V_{eff}(\Phi_0) &= N\left[-\frac{1}{(2\pi)^3}\int d^3k \ln\left(k^2+\Phi_0^2\right) + \frac{1}{2g^{(3)}}\Phi_0^2\right] \\ &= N\left[-\frac{4\pi}{(2\pi)^3}\int_0^\Lambda dk\, k^2 \ln\left(k^2+\Phi_0^2\right) + \frac{1}{2g^{(3)}}\Phi_0^2\right] \\ &= N\left[\frac{\Phi_0^2}{2g^{(3)}} - \frac{1}{6\pi^2}\left(\Lambda^3\ln(\Lambda^2+\Phi_0^2) - \frac{2}{3}\Lambda^3 + 2\Phi_0^2\Lambda - 2\Phi_0^3\arctan\left(\frac{\Lambda}{\Phi_0}\right)\right)\right]. \end{aligned} \tag{4.13}$$

For large Λ and with the convenient additive constant $\frac{N\Lambda^3}{6\pi^2}(\ln\Lambda^2 - \frac{2}{3})$ the potential reduces to

$$\begin{aligned}V_{eff}(\Phi_0) =& N\left[\frac{\Phi_0^2}{2g^{(3)}} - \frac{1}{6\pi^2}\left(\Lambda^3\ln\left(\frac{\Lambda^2+\Phi_0^2}{\Lambda^2}\right) + 2\Phi_0^2\Lambda + \pi|\Phi_0|^3\right)\right] \\ =& N\left[\frac{|\Phi_0|^3}{6\pi} + \left(\frac{1}{2g^{(3)}} - \frac{\Lambda}{2\pi^2}\right)\Phi_0^2\right].\end{aligned} \tag{4.14}$$

Starting from (4.13), the minimum of the effective potential is given by

$$\partial_{\Phi_0} V_{eff}(\Phi_0) = -\frac{1}{(2\pi)^3}\int d^3k \frac{2\Phi_0}{k^2+\Phi_0^2} + \frac{\Phi_0}{g^{(3)}} = 0, \tag{4.15}$$

and we obtain the mass gap equation

$$\frac{1}{(2\pi)^3}\int d^3k \frac{2}{k^2+\Phi_0^2} = \frac{1}{g^{(3)}}. \tag{4.16}$$

To evaluate the mass gap equation a momentum cutoff Λ is introduced and with $\Lambda \to \infty$ we obtain

$$\begin{aligned}\frac{1}{g^{(3)}} =& \frac{1}{(2\pi)^3}\int d^3k \frac{2}{k^2+\Phi_0^2} \\ =& \frac{1}{\pi^2}\int_0^\Lambda dk\, k^2 \frac{1}{k^2+\Phi_0^2} \\ =& \frac{1}{\pi^2}\left(\Lambda - \Phi\arctan\left(\frac{\Lambda}{\Phi}\right)\right) \\ \approx& \frac{1}{\pi^2}(\Lambda - \frac{\pi}{2}|\Phi|).\end{aligned} \tag{4.17}$$

We introduce the critical coupling constant $g^{(c)} = \pi^2/\Lambda$. If $1/g^{(3)} > 1/g^{(c)}$ the minimum of the potential is at $\Phi_0 = 0$, the model is in the unbroken phase. If $1/g^{(3)} < 1/g^{(c)}$ the minimum of the potential is at $|\Phi_0| = 2\pi\left(1/g^{(c)} - 1/g^{(3)}\right)$, the model is in the broken phase.

Figure 4.1: Effective potential of the 3D Gross-Neveu model in the unbroken phase.

Figure 4.2: Effective potential of the 3D Gross-Neveu model in the broken phase.

Chapter 5

Dimensional reduction

We start in the symmetric phase of the three-dimensional Gross-Neveu model and introduce periodic boundary conditions in the third direction ($x_3 : 0 \ldots \beta$). It is shown that in the limit $\beta \to \infty$ the two-dimensional model emerges from the three-dimensional one.

5.1 From the three- to the two-dimensional model

The three-dimensional mass gap equation with periodic boundary conditions in the third direction becomes

$$\frac{2}{(2\pi)^3} \int d^2k \frac{2\pi}{\beta} \sum_n \frac{1}{k_1^2 + k_2^2 + \Phi_0^2 + (2\pi n/\beta)^2} = \frac{1}{g^{(3)}}. \tag{5.1}$$

To evaluate the sum, we use the Poisson Formula

$$\sum_{n=-\infty}^{\infty} \phi(2\pi n) = \frac{1}{2\pi} \sum_{\nu=-\infty}^{\infty} \int d\tau \phi(\tau) \exp(-i\nu\tau), \tag{5.2}$$

and we obtain

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{1}{k_1^2 + k_2^2 + \Phi_0^2 + \left(\frac{2\pi n}{\beta}\right)^2} &= \frac{1}{2\pi} \sum_{\nu=-\infty}^{\infty} \int_{-\infty}^{\infty} d\tau \frac{1}{k_1^2 + k_2^2 + \Phi_0^2 + \left(\frac{\tau}{\beta}\right)^2} \exp(-i\nu\tau) \\ &= \frac{\beta}{2E} \coth\left(\frac{E\beta}{2}\right), \end{aligned} \tag{5.3}$$

where $E^2 = k_1^2 + k_2^2 + \Phi_0^2$.
The mass gap equation then reads

$$\frac{1}{(2\pi)^2} \int d^2k \frac{1}{E} \coth\left(\frac{E\beta}{2}\right) = \frac{1}{g^{(3)}}. \tag{5.4}$$

We introduce a spherical cutoff Λ and perform the integral

$$\begin{aligned} &\frac{1}{2\pi}\int_0^\Lambda dk \frac{k}{\sqrt{k^2+\Phi_0^2}} \coth\left(\frac{\sqrt{k^2+\Phi_0^2}}{2}\beta\right) \\ &= \frac{1}{\pi\beta}\left[\ln(\sinh(\frac{\beta}{2}\sqrt{\Lambda^2+\Phi_0^2})) - \ln(\sinh(\frac{1}{2}|\Phi_0|\beta))\right] = \frac{1}{g^{(3)}}. \end{aligned} \tag{5.5}$$

For large Λ we obtain

$$\ln\sinh(\frac{1}{2}|\Phi_0|\beta) = \pi\beta\left(\frac{1}{g^{(c)}} - \frac{1}{g^{(3)}}\right), \tag{5.6}$$

where $\frac{1}{g^{(c)}} = \frac{\Lambda}{2\pi}$.
Starting from the symmetric phase and taking the limit $\beta \to \infty$, leads to

$$\frac{1}{2}|\Phi_0|\beta = e^{-\left(\frac{1}{g^{(3)}} - \frac{1}{g^{(c)}}\right)\pi\beta}. \tag{5.7}$$

We relate the three- to the two-dimensional coupling constant $1/g = \beta(1/g^{(3)} - 1/g^{(c)})$. Now we have a hierarchy of different scales with the correlation length $\xi = 1/\Phi_0$ being the longest, β playing the role of the inverse two-dimensional cutoff in the middle, and the three-dimensional cutoff Λ being the shortest. In units of the correlation length $\xi = 1/\Phi_0$ the extent of the third dimension β becomes small and the two dimensional model emerges.

Figure 5.1: $\Phi_0(\frac{1}{g^{(3)}})$ for different β.

Chapter 6

Massive fermions

In this chapter we consider massive fermions. First we introduce the massive Gross-Neveu model at infinite volume. Then we consider free, massive fermions with domain wall boundary conditions in the third direction s.

6.1 The massive three-dimensional Gross-Neveu model

In the massive Gross-Neveu model an explicit mass term m is added in the action. The linearized action then reads

$$S[\bar{\Psi}, \Psi, \Phi_0] = \int d^3x \bar{\Psi}(x)(\gamma_\mu \partial_\mu + m)\Psi(x) + \frac{1}{2G^{(3)}}\Phi_0^2 + \bar{\Psi}(x)\Psi(x)\Phi_0. \tag{6.1}$$

Just as in the massless model we go into momentum space, integrate out the fermion fields and obtain the effective potential

$$\begin{aligned} V_{eff}(\Phi_0) =& N\left[-\frac{1}{(2\pi)^3}\int d^3k \ln\left(k^2 + (\Phi_0+m)^2\right) + \frac{1}{2g^{(3)}}\Phi_0^2\right] \\ =& N\left[\frac{\Phi_0^2}{2g^{(3)}} - \frac{1}{6\pi^2}\left(\Lambda^3 \ln(\Lambda^2 + (\Phi_0+m)^2) - \frac{2}{3}\Lambda^3\right.\right. \\ & \left.\left. +2(\Phi_0+M)^2\Lambda - 2(\Phi_0+m)^3 \arctan\left(\frac{\Lambda}{\Phi_0+m}\right)\right)\right]. \end{aligned} \tag{6.2}$$

For large Λ and after adding the convenient integration constant $\frac{N\Lambda^3}{6\pi^2}(\ln(\Lambda^2) - \frac{2}{3})$, the potential reduces to

$$V_{eff} = N\left[\frac{|\Phi_0+m|^3}{6\pi} + \frac{\Phi_0^2}{2g^{(3)}} - \frac{\Lambda}{2\pi^2}(\Phi_0+m)^2\right]. \tag{6.3}$$

Starting from (6.2), the mass gap equation reads

$$\begin{aligned}\frac{\Phi_0}{g^{(3)}} &= \frac{1}{(2\pi)^3}\int d^3k \frac{2(\Phi_0+m)}{k^2+(\Phi_0+m)^2} \\ &= \frac{1}{\pi^2}\int_0^\Lambda dk\, k^2 \frac{\Phi_0+m}{k^2+(\Phi_0+m)^2} \\ &= \frac{\Phi_0+m}{\pi^2}\left(\Lambda-(\Phi_0+m)\arctan\left(\frac{\Lambda}{\Phi_0+m}\right)\right).\end{aligned} \tag{6.4}$$

For large Λ we obtain

$$\Phi_0 = -m - \frac{\pi}{g^{(3)}} + \frac{\Lambda}{\pi} \pm \sqrt{\left(\frac{\Lambda}{\pi}-\frac{\pi}{g}\right)^2 + \frac{2m\pi}{g}}. \tag{6.5}$$

6.2 Free massive fermions with domain wall boundary conditions

We consider free, massive fermions and introduce domain wall boundary conditions in the third direction s with a domain wall at $s=0$ and an anti domain wall at $s=\beta$. When solving the Dirac equation the left-handed part of the zero mode shall live on the domain wall and the right-handed part on the anti domain wall. The action is given by

$$S = \int dtdx \int_0^\beta ds \left[\bar{\Psi}\left(i\gamma^\mu\partial_\mu + m\right)\Psi\right], \tag{6.6}$$

the corresponding Dirac equation is

$$\left(i\gamma^\mu\partial_\mu + m\right)\Psi = 0. \tag{6.7}$$

From here we work in Minkowski space and use the following representation of the gamma matrices

$$\gamma^0 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \gamma^2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{6.8}$$

This leads to

$$\begin{pmatrix} \partial_s + m & i\partial_t + i\partial_x \\ i\partial_t - i\partial_x & -\partial_s + m \end{pmatrix}\Psi = 0. \tag{6.9}$$

Using the ansatz: $\Psi = e^{i(Et+k_1x)}\psi$, we get

$$\begin{pmatrix} -\partial_s + m & E-k_1 \\ E+k_1 & \partial_s + m \end{pmatrix}\begin{pmatrix} \Psi_R \\ \Psi_L \end{pmatrix} = 0. \tag{6.10}$$

The equations for Ψ_R and Ψ_L thus are

$$\partial_s^2\Psi_{R,L} = (m^2 + k_1^2 - E^2)\Psi_{R,L}. \tag{6.11}$$

We impose domain wall boundary conditions

$$\begin{aligned} \Psi_R(s=0) &= 0, \\ \Psi_L(s=\beta) &= 0, \end{aligned} \tag{6.12}$$

where β is the distance between the two domain walls and make the following ansatz for the wave functions ψ_R

$$\begin{aligned} \psi_R &= A\sinh\left(s\sqrt{m^2+k_1^2-E^2}\right), \quad m^2+k_1^2 > E^2 \\ \psi_R &= B\sin\left(s\sqrt{E^2-m^2-k_1^2}\right), \quad m^2+k_1^2 < E^2. \end{aligned} \tag{6.13}$$

For ψ_L we obtain

$$\begin{aligned} \psi_L =& \frac{A}{E-k_1}\left[-\sqrt{m^2+k_1^2-E^2}\cosh\left(s\sqrt{m^2+k_1^2-E^2}\right)\right. \\ &\left.+m\sinh\left(s\sqrt{m^2+k_1^2-E^2}\right)\right], \qquad m^2+k_1^2 > E^2, \\ \psi_L =& \frac{B}{E-k_1}\left[-\sqrt{E^2-m^2-k_1^2}\cos\left(s\sqrt{E^2-m^2-k_1^2}\right)\right. \\ &\left.+m\sin\left(s\sqrt{E^2-m^2-k_1^2}\right)\right], \qquad m^2+k_1^2 < E^2. \end{aligned} \tag{6.14}$$

With $\psi_L(s=\beta)=0$ we get

$$\begin{aligned} \sqrt{m^2+k_1^2-E^2} &= m\tanh\left(\beta\sqrt{m^2+k_1^2-E^2}\right), \\ m^2+k_1^2 &> E^2, \\ \sqrt{E^2-m^2-k_1^2} &= m\tan\left(\beta\sqrt{E^2-m^2-k_1^2}\right), \\ m^2+k_1^2 &< E^2. \end{aligned} \tag{6.15}$$

In the case of $m^2+k_1^2 > E^2$ and in the limit ($\beta \to \infty$) we obtain $\tanh(\beta\sqrt{m^2+k_1^2-E^2}) = 1$ and thus $E^2 = k_1^2$.

Chapter 7

Gross-Neveu model with a dynamical domain wall

In this chapter we study the Gross-Neveu model with a dynamical domain wall. First we assume that this model can also be regarded as a free fermion model with a space-dependent mass term. The mass term m(s) plays the role of $\bar{\Psi}\Psi$ and shall be determined in a self-consistent way. It has been shown by Dashen et al. [15] that such a mass term can have the shape of a kink or a double kink. We will start with the results from their work and verify the self-consistency of the solutions, namely show that when solving the corresponding Dirac equation and summing over all occupied states one indeed gets back the original term [7], [8], [9]. The mass term can also be interpreted as a space dependent potential.

7.1 Gross-Neveu model with a kink

The action of the model is given by

$$S = \int dtdxds \left[\bar{\Psi}\left(i\gamma^{\mu}\partial_{\mu} - m(s)\right)\Psi\right]. \tag{7.1}$$

We want $m(s)$ to have a domain wall shape and choose $m(s) = m_0 \tanh(m_0 s)$. Hence we have to solve the following Dirac equation

$$\left(i\gamma^{\mu}\partial_{\mu} - m(s)\right)\Psi = 0. \tag{7.2}$$

Using the same representation of the γ matrices as in section 6.2 leads to

$$\begin{pmatrix} -\partial_s - m(s) & -i\partial_t + i\partial_x \\ -i\partial_t - i\partial_x & +\partial_s - m(s) \end{pmatrix} \Psi = 0. \tag{7.3}$$

With the ansatz: $\Psi = e^{i(Et+k_1 x)}\psi$, we get

$$\begin{pmatrix} -\partial_s - m(s) & E - k_1 \\ E + k_1 & \partial_s - m(s) \end{pmatrix} \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} = 0. \tag{7.4}$$

These two equations can be decoupled and converted into two second order differential equations. The equations for ψ_R and ψ_L are

$$\left(-\partial_s^2 \mp \partial_s m(s) + m(s)^2 - E^2 + k_1^2\right)\psi_{R,L} = 0. \tag{7.5}$$

Inserting $m(s)$ in equation (7.5) and with $E^2 = k_1^2 + k_2^2 + m_0^2$, we obtain the following equations

$$\begin{aligned} \left(\partial_s^2 + 2m_0^2(1-\tanh^2(m_0 s)) + k_2^2\right)\psi_R &= 0, \\ \left(\partial_s^2 + k_2^2\right)\psi_L &= 0. \end{aligned} \tag{7.6}$$

The solutions of this equation are one discrete state

$$\psi_0 = \begin{pmatrix} \frac{1}{\cosh(m_0 s)} \\ 0 \end{pmatrix}, \tag{7.7}$$

with the eigenvalues $E = \pm k_1$ and the continuum states

$$\psi_k^{\pm} = \begin{pmatrix} (ik_2 + m(s))e^{ik_2 s} \\ \pm(k_1 - E)e^{ik_2 s} \end{pmatrix}, \tag{7.8}$$

with the eigenvalues $E = \pm\sqrt{k_1^2 + k_2^2 + m_0^2}$. These states form a complete set of states though they are not yet orthonormal. We can take linear combinations and form a orthonormal basis for the theory which we find to be

$$\begin{aligned} \Psi_{k,odd} &= \left(\frac{1}{8\pi^3}\frac{1}{E(E-k_1)}\right)^{\frac{1}{2}} \begin{pmatrix} k_2\cos(k_2 s) - m(s)\sin(k_2 s) \\ (-E+k_1)\sin(k_2 s) \end{pmatrix} e^{i(Et+k_1 x)}, \\ \Psi_{k,even} &= \left(\frac{1}{8\pi^3}\frac{1}{E(E-k_1)}\right)^{\frac{1}{2}} \begin{pmatrix} -k_2\sin(k_2 s) - m(s)\cos(k_2 s) \\ (-E+k_1)\cos(k_2 s) \end{pmatrix} e^{i(Et+k_1 x)}, \\ \Psi_{k,0} &= \left(\frac{1}{4\pi^2}\frac{m_0}{2}\right)^{\frac{1}{2}} \begin{pmatrix} \mathrm{sech}(m_0 s) \\ 0 \end{pmatrix} e^{i(Et+k_1 x)}. \end{aligned} \tag{7.9}$$

The subscript k, α characterizes the different states. k refers to (E, k_1, k_2) and α refers to odd, even or zero. The allowed range of k is given by $-\infty < E, k_1 < +\infty$ and $0 \leq k_2 < \infty$. The asymmetry in the range for k_2 arises because we have made linear combinations of $+k_2$ and $-k_2$ to form the odd and even states. The orthogonality can be tested by showing $\int ds\, dx\, dt\, \Psi_{k,\alpha}^\dagger \Psi_{k',\alpha'} = \delta_{\alpha,\alpha'}\delta^3(k-k')$ when α and α' are not zero. If both α and α' are zero the three dimensional delta function is replaced by a two dimensional delta function in E and k_1. In appendix B the integral is explicitly calculated for all possible combinations of α and α'. Having solved these equations, we want to verify the self-consistency of the solutions and get

$$\begin{aligned} \bar{\Psi}_{k,0}\Psi_{k,0} &= 0, \\ \bar{\Psi}_{k,odd}\Psi_{k,odd} &= \frac{1}{4\pi^3 E}(k_2 \sin k_2 s \cos k_2 s - \sin^2 k_2 s\, m_0 \tanh m_0 s), \\ \bar{\Psi}_{k,even}\Psi_{k,even} &= \frac{1}{4\pi^3 E}(-k_2 \sin k_2 s \cos k_2 s - \cos^2 k_2 s\, m_0 \tanh m_0 s), \end{aligned} \tag{7.10}$$

and

$$\sum_{\alpha} \bar{\Psi}_{k,\alpha} \Psi_{k,\alpha} = \bar{\Psi}_k \Psi_k = -\frac{m_0}{4\pi^3 E(k)} \tanh m_0 s. \tag{7.11}$$

Summing over the filled Dirac sea, the self-consistency condition leads to

$$\frac{NG^{(3)} m_0}{4\pi^3} \tanh m_0 s \int_{-\Lambda/2}^{\Lambda/2} \frac{d^2k}{(2\pi)^2} \frac{1}{E(k)} = m_0 \tanh m_0 s, \tag{7.12}$$

where the factor N comes from the fact that each negative state is filled with N fermions. To get back the original potential we thus must require

$$\begin{aligned} \frac{NG^{(3)}}{4\pi^3} \int_{-\Lambda/2}^{\Lambda/2} \frac{d^2k}{(2\pi)^2} \frac{1}{\sqrt{k^2+m_0^2}} &= \\ \frac{NG^{(3)}}{8\pi^4} \int_0^{\Lambda'} dk \frac{k}{\sqrt{k^2+m_0^2}} &= \\ \frac{NG^{(3)}}{8\pi^4} (\sqrt{\Lambda'^2+m_0^2} - m_0) = 1. \end{aligned} \tag{7.13}$$

7.2 Gross-Neveu model with a double kink

Again we consider the action

$$S = \int dt dx ds \left[\bar{\Psi} \left(i\gamma^\mu \partial_\mu - m(s) \right) \Psi \right]. \tag{7.14}$$

Now the mass term has the form $m(s) = m_0(1 + y(\tanh(\xi_-) - \tanh(\xi_+)))$, with $y = \sin\left(\frac{\pi}{2}\frac{n}{N}\right), \xi_\pm = y m_0 s \pm c_0, c_0 = \frac{1}{2}\text{arctanh}(y)$. N is the number of flavors and n is the occupation number of the discrete positive state. To solve the Dirac equation

$$\left(i\gamma^\mu \partial_\mu - m(s) \right) \Psi = 0, \tag{7.15}$$

we start in the same way as in the preceding section and obtain

$$\begin{pmatrix} -\partial_s - m(s) & -i\partial_t + i\partial_x \\ -i\partial_t - i\partial_x & \partial_s - m(s) \end{pmatrix} \Psi = 0. \tag{7.16}$$

With the ansatz: $\Psi = e^{i(Et+k_1 x)}\psi$, we get

$$\begin{pmatrix} -\partial_s - m(s) & E - k_1 \\ E + k_1 & \partial_s - m(s) \end{pmatrix} \begin{pmatrix} \Psi_R \\ \Psi_L \end{pmatrix} = 0. \tag{7.17}$$

The equations for Ψ_R and Ψ_L thus are

$$\left(-\partial_s^2 \mp \partial_s m(s) + m(s)^2 - E^2 + k_1^2 \right) \Psi_{R,L} = 0. \tag{7.18}$$

We subtract on both sides of (7.18) the term $m_0^2\Psi$, to make the potential in this Schrödinger like equation vanish asymptotically and use $E^2 = m_0^2 + k_1^2 + k_2^2$. This leads to

$$\left(\partial_s^2 \pm \partial_s m(s) - m(s)^2 + m_0^2\right)\Psi_{R,L} = -k_2^2\Psi_{R,L}. \tag{7.19}$$

Now we insert the explicit form of $m(s)$ into (7.19) and evaluate the expression term by term

$$\begin{aligned}\frac{\partial m(s)}{\partial s} &= m_0^2 y^2\left(\tanh^2(\xi_+) - \tanh^2(\xi_-)\right)\\ &= m_0^2 y^2\left(\frac{1}{\cosh^2(\xi_-)} - \frac{1}{\cosh^2(\xi_+)}\right)\end{aligned} \tag{7.20}$$

and

$$\begin{aligned}m(s)^2 - m_0^2 &= (m(s) - m_0)(m(s) + m_0)\\ &= m_0^2 y(\tanh(\xi_-) - \tanh(\xi_+))(2 + y(\tanh(\xi_-) - \tanh(\xi_+)))\\ &= m_0^2 y^2(\tanh^2(\xi_-) + \tanh^2(\xi_+))\\ &+ 2m_0^2 y(\tanh(\xi_-) - \tanh(\xi_+) - y\tanh(\xi_-)\tanh(\xi_+)).\end{aligned} \tag{7.21}$$

Using (B.8) one finds

$$\begin{aligned}m(s)^2 - m_0^2 &= m_0^2 y^2\left(\tanh(\xi_-)^2 + \tanh(\xi_+)^2 - 2\right)\\ &= -m_0^2 y^2\left(\frac{1}{\cosh^2(\xi_-)} + \frac{1}{\cosh^2(\xi_+)}\right).\end{aligned} \tag{7.22}$$

Combining (7.20) and (7.22) we get

$$\pm\frac{\partial m(s)}{\partial s} - m(s)^2 + m_0^2 = 2m_0^2 y^2\frac{1}{\cosh^2(\xi_\mp)}. \tag{7.23}$$

Inserting (7.23) into (7.19) we finally obtain

$$\left(\frac{\partial^2}{\partial s^2} + \frac{2y^2 m_0^2}{\cosh^2(\xi_\mp)}\right)\Psi_{R,L} = -k_2^2\Psi_{R,L}. \tag{7.24}$$

The solutions of (7.24) are two discrete states $\psi_0^\pm$ and two continua of states $\psi_k^\pm$. The wave functions of the discrete states are given by

$$\psi_0^\pm = \frac{\sqrt{y m_0}}{2}\begin{pmatrix}\frac{1}{\cosh(\xi_-)}\\ \mp\frac{1}{\cosh(\xi_+)}\end{pmatrix} \tag{7.25}$$

with the eigenvalues

$$E_0^\pm = \pm\sqrt{m_0^2(1-y^2) + k_1^2}. \tag{7.26}$$

The continuum states are

$$\psi_k^{\pm} = \frac{1}{E(k)(ik_2 + ym_0)} \begin{pmatrix} (ik_2 - m_0)(ik_2 - ym_0 \tanh(\xi_-)) \\ \pm E(k)(ik_2 - ym_0 \tanh(\xi_+)) \end{pmatrix} e^{ik_2 s}, \tag{7.27}$$

with the eigenvalues $E^2 = k_1^2 + k_2^2 + m_0^2$.
Now we verify the self-consistency of the solutions. Summing $\bar{\Psi}\Psi$ over all occupied states we want to get $m(s)$ back. The negative states are fully occupied whereas the discrete positive state has n fermions and the continuous positive state is empty. Hence the following condition has to be fulfilled

$$S(s) \doteq -G^{(3)} N \left(\sum_\alpha \bar{\Psi}_\alpha^- \Psi_\alpha^- + \frac{n}{N} \bar{\Psi}_0^+ \Psi_0^+ \right) = m(s). \tag{7.28}$$

For the discrete states we obtain

$$\begin{aligned} \bar{\Psi}_0^{\pm} \Psi_0^{\pm} &= \mp \frac{ym_0}{4} \frac{2}{\cosh(\xi_+)\cosh(\xi_-)} \\ &= \mp \frac{\sqrt{1-y^2} m_0}{2} (\tanh(\xi_+) - \tanh(\xi_-)). \end{aligned} \tag{7.29}$$

The negative energy state Ψ_0^- is fully occupied, whereas the positive state Ψ_0^+ contains n fermions.

$$S(s)_{discrete} = G^{(3)} (N-n) \sqrt{1-y^2} \frac{m_0}{2} (\tanh(\xi_+) - \tanh(\xi_-)) \tag{7.30}$$

The continuum states are

$$\begin{aligned} \bar{\Psi}_k^{\pm} \Psi_k^{\pm} &= \frac{1}{2E(k)^2(k_2^2 + y^2 m^2)} \left[E(k)(-ik_2 - ym_0 \tanh(\xi_+))(ik_2 - m_0)(ik_2 - ym_0 \tanh(\xi_-)) \right. \\ &\quad \left. + (-ik_2 - m_0)(-ik_2 - ym_0 \tanh(\xi_-)) E(k)(ik_2 - ym_0 \tanh(\xi_+)) \right] \\ &= \frac{m_0}{2E(k)(k_2^2 + y^2 m_0^2)} \\ &\quad \times \left[-2k_2^2 y (\tanh(\xi_-) - \tanh(\xi_+)) - 2k_2^2 - 2y^2 m_0^2 (\tanh(\xi_-) - \tanh(\xi_+)) \right]. \end{aligned} \tag{7.31}$$

Using (B.8) this reduces to

$$\begin{aligned} &\frac{m_0}{E(k)(k_2^2 + y^2 m_0^2)} \left[-(k_2^2 + y^2 m_0^2) - (k_2^2 y + m_0^2 y)(\tanh(\xi_-) - \tanh(\xi_+)) \right] \\ &= -\frac{m_0}{E(k)} \left[1 + \frac{y(k_2^2 + m_0^2)}{k_2^2 + y^2 m_0^2} (\tanh(\xi_-) - \tanh(\xi_+)) \right]. \end{aligned} \tag{7.32}$$

Performing the integral over all negative continuum states we obtain

$$S(s)_{cont} = NG^{(3)} m_0 \int_{-\Lambda/2}^{\Lambda/2} \frac{d^2k}{(2\pi)^2} \frac{1}{E(k)} \left[1 + y \left(1 + \frac{m_0^2 - m_0^2 y^2}{k_2^2 + y^2 m_0^2} \right) (\tanh(\xi_-) - \tanh(\xi_+)) \right]. \tag{7.33}$$

Now $S(s)$ is split into two parts $S(s)_{cont} = \tilde{S}(s)_{cont} + \delta S(s)_{cont}$. Where

$$\tilde{S}(s)_{cont} = NG^{(3)}m_0\left[1 + y(\tanh(\xi_-) - \tanh(\xi_+))\right]\int_{-\Lambda/2}^{\Lambda/2}\frac{d^2k}{(2\pi)^2}\frac{1}{E(k)} \tag{7.34}$$

and

$$\delta S(s)_{cont} = NG^{(3)}m_0^3(1-y^2)(\tanh(\xi_-) - \tanh(\xi_+))\int_{-\Lambda/2}^{\Lambda/2}\frac{d^2k}{(2\pi)^2}\frac{1}{E(k)(k_2^2+m_0^2y^2)}. \tag{7.35}$$

$\delta S(s)_{cont}$ can be evaluated in the limit $(\Lambda \to \infty)$. With (B.12) we get

$$\begin{aligned}\delta S(s)_{cont} = &\frac{NG^{(3)}m_0}{\pi}\sqrt{1-y^2}(\tanh(\xi_-) - \tanh(\xi_+))\int dk_1\frac{1}{m_0 y}\frac{1}{m_0^2(1-y^2)+k_1^2}\\ &\times\left(\pi - 2\mathrm{arcsin}\left(\frac{m_0 y}{\sqrt{k_1^2+m_0^2}}\right)\right).\end{aligned} \tag{7.36}$$

We now fill all discrete negative states. These have energy $E_0 = -\sqrt{m_0^2(1-y^2)+k_1^2}$ and exactly cancel the contribution π in (7.36). In order to cancel the arcsin-term in (7.36) as well, we need to fill some positive energy states. In order to stabilize the configuration, we occupy all positive states localized on the wall (with energy $E_0 = \sqrt{m_0^2(1-y^2)+k_1^2}$) up to some Fermi momentum k_F. The cancellation condition which determines the value of k_F takes the form

$$\int_{-k_F}^{k_F}\frac{dk_1}{\sqrt{k_1^2+m_0^2(1-y^2)}} = \int_{-\infty}^{\infty}\frac{dk_1}{\sqrt{m_0^2(1-y^2)+k_1^2}}\frac{2}{\pi}\mathrm{arcsin}(\frac{ym_0}{\sqrt{k_1^2+m_0^2}}) = \log\sqrt{\frac{1+y}{1-y}} \tag{7.37}$$

and hence,

$$k_F \to ym_0. \tag{7.38}$$

The energy of the particles on the Fermi-surface

$$\sqrt{k_F^2+m_0^2(1-y^2)} = \sqrt{y^2m_0^2+(1-y^2)m_0^2} = m_0 \tag{7.39}$$

is therefore equal to the lowest energy m_0 of the states propagating in the (2+1)-d bulk of the extra dimension. Any fermion that is added on the wall has enough energy to escape into the extra dimension.

Finally we want to dimensionally reduce this model to the two-dimensional one and thus further analyze the zero mode. We define the distance between the two kinks as $\beta = \mathrm{arctanh}y/ym_0$. For large β we get $y \approx 1$ and thus $y = \tanh\beta m_0 \approx 1 - 2e^{-2\beta m_0}$. Inserting this in (7.26) we obtain

$$E^2 = 4m_0^2e^{-2\beta m_0} + k_1^2, \tag{7.40}$$

and finally

$$\mu = \sqrt{E^2 - k_1^2} = 2m_0e^{-\beta m_0}. \tag{7.41}$$

In units of the correlation length $\xi = 1/\mu$ the third dimension becomes small and the two-dimensional model emerges.

Conclusion and outlook

This chapter summarizes the main results and gives an outlook for possible further studies. It has been shown that the two-dimensional continuum Gross-Neveu model has a chiral symmetry that is spontaneously broken. When the model is considered on a lattice the bare mass has to be fine-tuned in order to preserve the chiral symmetry. The three-dimensional model can be in a broken or an unbroken phase. We showed two ways of reducing the three-dimensional to the two dimensional model. One can ether start in the unbroken phase of the three-dimensional model and impose periodic boundary conditions or start in the broken phase and impose domain wall boundary conditions.
The following studies are suggested to be worked out in further studies: It would be interesting to study the dimensional reduction in the Gross-Neveu model at finite N and check whether the right dynamics are generated. This could be done through numerical calculations such as Monte Carlo simulations.
Dimensional reduction is a generic phenomenon that occurs in a variety of models. Similar calculations as the ones done here in the Gross-Neveu model should be possible eg. in the O(3) model. The final goal would be to apply the techniques learned while working on toy models to QCD. It should be possible obtain QCD through dimensional reduction of a five-dimensional model. Such a to dimensional reduction approach to QCD is realized in the D-theory where classical fields arise from the dimensional reduction of discrete variables.

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Appendix A

Grassmann algebras

Fermion fields are described by anticommuting (Grassmann) variables. Hence, let us have a look at the so-called Grassmann algebra.
The generators of a n-dimensional Grassmann algebra are anticommuting classical variables η_i with $(i = 1 \dots n)$,

$$\{\eta_i, \eta_j\} = 0. \tag{A.1}$$

A general element of a Grassmann algebra is defined as a power series of the generators. Since $\eta_i^2 = 0$, the power series has only a finite number of elements.

$$f(\eta) = f_0 + \sum_i f_i \eta_i + \sum_{i,j} f_{ij} \eta_i \eta_j + \sum_{i,j,k} f_{ijk} \eta_i \eta_j \eta_k + \cdots + f_{12\dots N} \eta_1 \eta_2 \dots \eta_N. \tag{A.2}$$

The $f_{ij\dots l}$ are ordinary complex numbers, which are antisymmetric in $i, j, \dots, l$. The integration rules for Grassmann variables are

$$\int d\eta_i = 0, \int d\eta_i \eta_i = 1, \int d\eta_i d\eta_j \eta_i \eta_j = -1, \tag{A.3}$$

and the differentiation is defined by

$$\frac{\partial}{\partial \eta_i} 1 = 0, \ \frac{\partial}{\partial \eta_i} \eta_i = 1, \ \frac{\partial}{\partial \eta_i} \frac{\partial}{\partial \eta_j} \eta_i \eta_j = -1. \tag{A.4}$$

Notice that, because of the peculiar definition of Grassmann variables the integration over η_i is equivalent to partial differentiation with respect to this variable. The Grassmann algebra we use to define fermion fields is generated by the Grassmann numbers

$$\Psi = \begin{pmatrix} \Psi^1 \\ \Psi^2 \\ \vdots \\ \Psi^N \end{pmatrix}, \quad \bar{\Psi} = \left(\bar{\Psi}^1, \bar{\Psi}^2, \dots, \bar{\Psi}^N \right), \tag{A.5}$$

which are completely independent. We show that

$$\int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi \exp(-\bar{\Psi}A\Psi) = \sum_{i=1}^{N} \int d\bar{\Psi}^i d\Psi^i \exp(-\bar{\Psi}^i A\Psi^i) = \mathrm{Det}(A), \tag{A.6}$$

where A is a Hermitian $N \times N$ matrix. Using the power series expansion of the exponential function and the properties of the Grassmann numbers we get

$$\begin{aligned} \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi \exp(-\bar{\Psi}\Psi) &= \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi \frac{1}{N!}(-\bar{\Psi}\Psi)^N \\ &= \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi \frac{(-1)^N}{N!} N! \left(\bar{\Psi}^1\Psi^1 \cdot \ldots \cdot \bar{\Psi}^N\Psi^N\right) \\ &= (-1)^N \cdot (-1)^{\left(\frac{2N(2N-1)}{2}\right)} = 1. \end{aligned} \tag{A.7}$$

Then the variables are transformed as

$$\Psi = B\Phi, \bar{\Psi} = \bar{\Phi}C, \tag{A.8}$$

where B and C are the transformation matrices. Thus we get

$$\begin{aligned} \prod_i \Psi^i &= \mathrm{Det}(B) \prod_i \Phi^i, \\ \prod_i \bar{\Psi}^i &= \mathrm{Det}(C) \prod_i \bar{\Phi}^i. \end{aligned} \tag{A.9}$$

To preserve the integration rules $\int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi\, \bar{\Psi}\Psi = \int \mathcal{D}\bar{\Phi}\mathcal{D}\Phi\, \bar{\Phi}\Phi$ we require

$$\begin{aligned} \prod_i d\Psi^i &= \frac{1}{\mathrm{Det}(B)} \prod_i d\Phi^i, \\ \prod_i d\bar{\Psi}^i &= \frac{1}{\mathrm{Det}(C)} \prod_i d\bar{\Phi}^i. \end{aligned} \tag{A.10}$$

Substituting (A.8) and (A.10) into (A.7) leads to

$$\frac{1}{\mathrm{Det}(B)\mathrm{Det}(C)} \int \mathcal{D}\bar{\Phi}\mathcal{D}\Phi \exp(-\bar{\Phi}CB\Phi) = 1. \tag{A.11}$$

With $A = CB$ and after renaming the variables we finally get (A.6).

Appendix B

Useful formulae

B.1 Orthonormalization

Here we explicitly show that the functions introdiced in Section 6.1 are orthonormal.

$$\begin{aligned}\int dtdxds\Psi^{\dagger}_{k,odd}\Psi_{k',even} = 0,\\ \int dtdxds\Psi^{\dagger}_{k,even}\Psi_{k',0} = 0,\\ \int dtdxds\Psi^{\dagger}_{k,odd}\Psi_{k',0} = 0,\end{aligned} \tag{B.1}$$

$$\begin{aligned}\int dtdxds\Psi^{\dagger}_{k,0}\Psi_{k',0} &= \frac{m_0}{8\pi^2}\int dx\,dt\,ds\frac{1}{\cosh^2(m_0 s)}e^{-i(Et+k_1x)}e^{i(E't+k'_1x)}\\ &= \frac{1}{4\pi^2}\int dx\,dte^{-i(E-E')t}e^{-i(k_1-k'_1)}\\ &= \delta(E-E')\delta(k_1-k'_1),\end{aligned} \tag{B.2}$$

$$
\begin{aligned}
\int dtdxds\Psi^{\dagger}_{k,odd}\Psi_{k',odd} &= \frac{1}{8\pi^3}\frac{1}{E-k_1}\int dx\,dt\,ds\,\{[(k_2\cos k_2 s - m_0\tanh m_0 s\sin k_2 s) \\
&\times\ (k'_2\cos k'_2 s - m_0\tanh m_0 s\sin k'_2 s) \\
&+\ (-E+k_1)(-E'+k'_1)\sin k_2 s\sin k'_2 s]e^{-i(E-E')t}e^{-i(k_1-k'_1)}\} \\
&= \frac{4\pi^2}{8\pi^3}\frac{1}{E-k_1}4\Big\{\pi k_2 k'_2(\delta(k_2-k'_2)+\delta(k_2+k'_2)) \\
&+\ \pi(-E+k_1)^2(\delta(k_2-k'_2)-\delta(k_2+k'_2)) \\
&+\ \int ds\Big[-m_0\tanh m_0 s\frac{d}{ds}(\sin k_2 s\sin k'_2 s) \\
&+\ m_0^2\tanh^2 m_0 s\sin k_2 s\sin k'_2 s\Big]\Big\}\delta(E-E')\delta(k_1-k'_1) \\
&= \frac{1}{2\pi}\frac{1}{E-k_1}\{2\pi(E^2-Ek_1)(\delta(k_2-k'_2)-\delta(k_2+k'_2)) \\
&-\ 2m_0\tanh m_0 s\sin k_2 s\sin k'_2 s|_0^\infty\}\delta(E-E')\delta(k_1-k'_1) \\
&= \delta(E-E')\delta(k_1-k'_1)\delta(k_2-k'_2),
\end{aligned}
\tag{B.3}
$$

$$
\begin{aligned}
\int dtdxds\Psi^{\dagger}_{k,even}\Psi_{k',even} &= \frac{1}{8\pi^3}\frac{1}{E-k_1}\int dx\,dt\,ds\{[(k_2\sin k_2 + m_0\tanh m_0 s\cos k_2 s) \\
&\times\ (k'_2\sin k'_2 s + m_0\tanh m_0 s\cos k'_2 s) \\
&+\ (-E+k_1)(-E+k_1)\cos k_2 s\cos k'_2 s]e^{-i(E-E')t}e^{-i(k_1-k'_1)}\} \\
&= \frac{4\pi}{8\pi^3}\frac{1}{E-k_1}\Big\{\pi k_2 k'_2(\delta(k_2-k'_2)+\delta(k_2+k'_2)) \\
&+\ \pi(-E+k_1)^2(\delta(k_2-k'_2)-\delta(k_2+k'_2)) \\
&+\ \int ds\Big[-m_0\tanh m_0 s\frac{d}{ds}(\cos k_2 s\cos k'_2 s) \\
&+\ m_0^2\tanh^2 m_0 s\cos k_2 s\cos k'_2 s\Big]\Big\}\delta(E-E')\delta(k_1-k'_1) \\
&= \frac{1}{2\pi}\frac{1}{E-k_1}\{2\pi(E^2-Ek_1)(\delta(k_2-k'_2)-\delta(k_2+k'_2)) \\
&-\ 2m_0\tanh m_0 s\cos k_2 s\cos k'_2 s|_0^\infty\}\delta(E-E')\delta(k_1-k'_1) \\
&= \delta(E-E')\delta(k_1-k'_1)\delta(k_2-k'_2).
\end{aligned}
\tag{B.4}
$$

To evaluate the preceding integrals we have replaced when required the trigonometric functions by equivalent functions that are well defined at infinity

$$\begin{aligned} \cos(ks) \to \cos_\epsilon(ks) &= \lim_{\epsilon\to 0} \frac{e^{iks}+e^{-iks}}{2} e^{-|s|\frac{\epsilon}{2}}, \\ \sin(ks) \to \sin_\epsilon(ks) &= \lim_{\epsilon\to 0} \frac{e^{iks}-e^{-iks}}{2} e^{-|s|\frac{\epsilon}{2}}. \end{aligned} \tag{B.5}$$

B.2 Identities

We collect some useful formulas which are needed in order to derive the results of section 6.2.

$$\begin{aligned} \xi_\pm &= ymx \pm \frac{1}{2}\text{arctanh}(y), \\ \xi_+ - \xi_- &= \text{arctanh}(y). \end{aligned} \tag{B.6}$$

The addition theorem for tanh reads

$$\tanh(\xi_+ - \xi_-) = \frac{\tanh(\xi_+) - \tanh(\xi_-)}{1 - \tanh(\xi_+)\tanh(\xi_-)}. \tag{B.7}$$

(B.6) and (B.7) yield the formula

$$y(1 - \tanh(\xi_+)\tanh(\xi_-)) = \tanh(\xi_+) - \tanh(\xi_-). \tag{B.8}$$

Furthermore, the relation

$$\tanh(\xi_+) - \tanh(\xi_-) = \frac{\sinh(\xi_+ - \xi_-)}{\cosh(\xi_+)\cosh(\xi_-)} \tag{B.9}$$

together with

$$\sinh(\text{arctanh}(y)) = \frac{y}{\sqrt{1-y^2}} \tag{B.10}$$

yields

$$\tanh(\xi_+) - \tanh(\xi_-) = \frac{y}{\sqrt{1-y^2}} \frac{1}{\cosh(\xi_+)\cosh(\xi_-)} \tag{B.11}$$

Finally we evaluate the integral

$$I_1 = \int_{-\infty}^{\infty} dk \frac{1}{E(k)(k^2+y^2m^2)},\ E(k) = \sqrt{m^2+k^2}. \tag{B.12}$$

To do so we substitute $k = m\sinh(z/2)$ and get

$$\begin{aligned} I_1 &= \int_{-\infty}^{\infty} dz \frac{1}{2m^2(\sinh^2(\frac{z}{2}) + y^2)} \\ &= \frac{1}{m^2} \int_{-\infty}^{\infty} dz \frac{1}{2y^2 - 1 + \cosh(z)} \\ &= \frac{2}{m^2 y\sqrt{1-y^2}} \arctan\left(\frac{\sqrt{1-y^2}}{y}\right) \\ &= \frac{2}{m^2 y\sqrt{1-y^2}} (\pi/2 - \arcsin(y)). \end{aligned} \tag{B.13}$$

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