CONVEXITY AND TOPOLOGY ON MEASURE SPACES

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Abstract. This draft will develop in notes for the Minerva lectures given at Columbia University by the author in October of 2015. At this point, the text is very incomplete and may contain substantial mistakes and inconsistencies. Please do not distribute without permission of the author, and contact via email at `k.kardaras@lse.ac.uk` to report and errors you may have discovered.

Introduction

Discussion. The microeconomic theory of consumer demand starts with the description of the set of available consumption bundles for acting agents—see for example, [MCWG95]. Without taking budget constraints into account for the time being, a natural candidate for consumption space is $\mathbb{R}_+^S$, consisting of all functions $f : S \mapsto \mathbb{R}_+$, where S in an appropriate index set. The modelling freedom in choosing S allows for consideration of rather general scenarios. In the possibly most basic *deterministic* and *static* case, S is the index set J of a collection of a distinct commodities[1] available for consumption. By considering S as the product of the commodities index set J with a (discrete, or continuous) time index set T, a dynamic time component may be appended, reflecting the fact that consumption of goods in different times has different effect on consumer satisfaction.[2] Given an already-existing time component, future uncertainty may be also introduced by considering $S = \Omega \times T \times J$, where Ω is the index set consisting of all possible states-of-nature. This last index set is a representative (but in no way general-encompassing) case that one should keep in mind throughout this introductory discussion. However, for the most part we keep simply the notation S for the index set, and sometimes call the elements $s \in S$ *states.*

Assume initially that S is a finite set, implying the finite-dimensionality of the consumption cone $\mathbb{R}_+^S$. A linear *price* vector $\mu \in \mathbb{R}_+^S$ assigns price $\mu(s) \in \mathbb{R}_+$ on a unit of consumption at state $s \in S$. The price of a consumption bundle $f \in \mathbb{R}_+^S$ under prices $\mu \in \mathbb{R}_+^S$ is given by the "inner product"

$$\langle \mu, f \rangle := \sum_{s \in S} f(s)\mu(s). \tag{IP}$$

For every $\mu \in \mathbb{R}_+^S$, the price functional $\langle \mu, \cdot \rangle : \mathbb{R}_+^S \mapsto \mathbb{R}_+$ is continuous; therefore, the budget set

$$B(\mu) := \left\{ f \in \mathbb{R}_+^S \mid \langle \mu, f \rangle \leq 1 \right\} \tag{B}$$

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[1] Or, in the theory of financial economics, a set of currencies.

[2] The introduction of a time component is important as it allows, for example, for the inclusion of impatience in the modelling of preferences.

of an economic agent with (normalized) unit capital facing prices μ is closed. It is also straightforward that $B(\mu)$ is a convex set for all $\mu \in \mathbb{R}_+^S$. Let us assume that the aforementioned economic agent has preferences over consumption that are numerically represented by a quasi-concave and upper semi-continuous *utility function* $u : \mathbb{R}_+^S \mapsto [-\infty, \infty]$. We wish to ensure that the the following utility maximization problem:

$$\text{find } g \in B(\mu) \text{ such that } u(g) = \sup_{f \in B(\mu)} u(f), \tag{UM}$$

has a solution. Since $\mathbb{R}_+^S$ is finite-dimensional and u is upper semi-continuous, an obvious sufficient[3] condition is that $B(\mu)$ is compact. Given that $B(\mu)$ is already closed, only its boundedness has to be ensured, which is the case if and only if $\mu(s) > 0$ holds for all $s \in S$. The last condition can be restated in terms of *absence of arbitrage*; indeed, $\mu(s) = 0$ for $s \in S$ would imply that the agent's budget set would include arbitrary quantities of bundle $1_{\{s\}}$, since the latter non-zero bundle would have zero price. To recapitulate the previous discussion: continuity of $\langle \mu, \cdot \rangle$ implies that the budget set $B(\mu)$ is closed, and $\mu \in \mathbb{R}_{++}^S$ leads to boundedness of $B(\mu)$; therefore, compactness of $B(\mu)$ for no-arbitrage prices $\mu \in \mathbb{R}_{++}^S$ implies that (UM) has a solution.

The assumption that S is a finite set that was underlying the above discussion may be justified on the grounds of "limited human capability;" however, (mathematical, at the very least) need arises to extend the theory for infinite-state sets. Start with the case where S has countably infinite cardinality—for example, this could be the case in infinite-horizon discrete-time models. With the same interpretation of linear price vectors as previously, the price of a consumption bundle $f \in \mathbb{R}_+^S$ under prices $\mu \in \mathbb{R}_+^S$ is still given by (IP), where the sum is well defined in view of the nonnegativity of the entries, but may take the value ∞. There are at least a couple of ways to deal with the last issue of potential infinite prices, as we explain below.

Restricting the sets of possible consumption bundles and price vectors may result in finite prices. For example, this can be achieved through weighted ℓ^p-ℓ^q duality, where $p \in [1, \infty]$ and $q \in [1, \infty]$ are such that $1/p + 1/q = 1$; such restriction would also offer a natural induced pair of topologies, under which the dual pairing $\langle \cdot, \cdot \rangle$ would be continuous. Its convex-analytic usefulness notwithstanding, as the tools needed to attack convex problems are already developed and available, this first way of dealing with the potentiality of infinite prices is rather ad-hoc: there is typically no natural way to choose the appropriate weights and power in forming the relevant restricted weighted ℓ^p-ℓ^q spaces. Even more to the point, different such choices can change the structure and topology of the space substantially, therefore lacking a reasonable *robustness* requirement.

An alternative way to go about is to keep $\mathbb{R}_+^S$ as the space of consumption bundles and price vectors, and simply accept the possibility $\langle \mu, f \rangle = \infty$, with the plain interpretation that the consumption bundle $f \in \mathbb{R}_+^S$ under prices $\mu \in \mathbb{R}_+^S$ is never attainable, irrespective of the level of the capital-at-hand. Furthermore, in order to keep a "coordinate-free" structure, a natural analytic structure on $\mathbb{R}_+^S$ is the product topology. This choice of topology, while consistent with

[3]And, in this finite-dimensional case, also "morally" necessary.

the finite-dimensional case, results in yet another peculiarity: the functional $\langle \mu, \cdot \rangle : \mathbb{R}^S_+ \mapsto [0, \infty]$ is continuous only when $\mu(s) = 0$ for all but a potentially *finite* number of $s \in S$. However, this precludes the use of "no-arbitrage" pricing rules, which would require that $\mu(s) > 0$ for *all* $s \in S$. As the use of no-arbitrage pricing functionals in unavoidable[4] in the theory of Financial Economics, we are led to also give up on the continuity requirement for $\langle \cdot, \cdot \rangle$. However, not all is lost: it is rather straightforward to show that $\langle \mu, \cdot \rangle : \mathbb{R}^S_+ \mapsto [0, \infty]$ is always *lower semicontinuous* (for the considered product topology), implying that the budget set of (B) is still closed. If $\mu \in \mathbb{R}^S_{++}$, it is easily seen that (B) is actually compact (always, in the product topology). In turn, the previous imply that the utility maximization problem (UM) has a solution whenever $\mu \in \mathbb{R}^S_{++}$ and $u : \mathbb{R}^S_+ \mapsto [-\infty, \infty]$ is quasi-concave and upper semi-continuous in the product topology.

When S is uncountably infinite, the cone $\mathbb{R}^S_+$ is typically too large (and its product topology too strong) to be useful. For example, while $\langle \mu, f \rangle$ in (IP) for $f \in \mathbb{R}^S_+$ and $\mu \in \mathbb{R}^S_+$ is still well defined[5], for no-arbitrage prices $\mu \in \mathbb{R}^S_{++}$, a necessary condition for $\langle \mu, f \rangle < \infty$ is that $f(s)$ is non-zero for an at most countable number of $s \in S$. This is rarely satisfactory in practice; for example, in the single-commodity case $S = \Omega \times \mathbb{R}_+$, where the time index component $\mathbb{R}_+$ models continuous time and there is a continuum of states-of-nature in Ω, elements $f \in \mathbb{R}^S_+$ typically represent consumption *rates* with respect to a Lebesgue clock, and a probabilistic structure on Ω identifies random consumption-rate streams that agree in the almost sure sense. One is then led to consider a measure space structure $(S, \mathcal{S}, \sigma)$, where σ is a sigma-finite measure on the measurable space $(S, \mathcal{S})$ and $\mathcal{S}$ is a sigma-algebra on S, and instead of $\mathbb{R}^S_+$ one considers its quotient $\mathbb{L}^0_+$ of nonnegative measurable functions modulo σ-a.e. equivalence. For $f \in \mathbb{L}^0_+$ and $\mu \in \mathbb{L}^0_+$, the definition fo the "inner product" becomes $\langle \mu, f \rangle := \int f \mu \mathrm{d}\sigma$. The whole set-up is is a proper generalization of the case where S is at most countable, whereby $\mathcal{S}$ would be the powerset of S and σ the counting measure on $(S, \mathcal{S})$. By considering on $\mathbb{L}^0_+$ the topology of convergence in probability, the functionals $\langle \mu, \cdot \rangle$ become lower semicontinuous for all $\mu \in \mathbb{L}^0_+$. In general, when μ is a no-arbitrage pricing functional, i.e., when $\mu > 0$ holds σ-a.e., the convex and closed budget set $B(\mu)$ of (B) will fail to be compact; however, it will be shown that it still has a "convex compactness" property in $\mathbb{L}^0_+$ which will result in solutions to utility maximization problems like (UM).

The aforementioned topology on $\mathbb{L}^0_+$ lacks an essential feature (namely, local convexity) that allows the use of traditional[6] convex-analytic tools. Nevertheless, appropriate versions of such results can be proved, typically by use of "bare-hands" approaches. One upside is that no prerequisites from Functional Analysis are needed; basic knowledge of Measure theory and Real Analysis will be enough in understanding the development of the theory. The purpose of these notes is to present convex-analytic results pertaining to $\mathbb{L}^0_+$, especially with connections to the theory of Mathematical Economics and Finance, and explore the strength and limitations of the methods involved.

[4]For example, when no-arbitrage pricing functionals are involved, problem (UM) fails to have a solution in general.

[5]The sum in (IP) is defined as the supremum over all sums for finite subsets of S.

[6]For example, the Hahn-Banach theorem requires local convexity from the space involved.

Notation. Throughout these notes, we fix a measurable space $(S, \mathcal{S})$, where $\mathcal{S}$ is a sigma-field over S. On $\mathcal{S}$ we shall consider a *sigma-finite* "baseline" measure[7] σ. We shall write "a.s." to mean σ-almost-surely. The whole development depends on σ only through its null sets; in other words, only through the equivalence class (of measures) it generates. We use "$\ll$" and "$\sim$" to denote absolute continuity and equivalence of measures, respectively.

Two (Borel) measurable functions $f : S \mapsto \mathbb{R}$ and $g : S \mapsto \mathbb{R}$ are *equivalent* if $f = g$ holds in the a.s. sense. We then define $\mathbb{L}^0 \equiv \mathbb{L}^0(S, \mathcal{S}, \sigma)$ to consist of all equivalence classes of measurable functions. We follow the usual practice of not distinguishing between a measurable function and the equivalence classes in $\mathbb{L}^0$ that it generates. Consequently, all relationships involving measurable functions will be understood in the σ-a.s. sense.

It is immediate that $\mathbb{L}^0$ is a vector space, as well as an algebra with the usual function multiplication. It also has a natural *order structure* given by the corresponding order in $\mathbb{R}$. The nonnegative orthant of $\mathbb{L}^0$ will be denoted by $\mathbb{L}^0_+$; in other words, $\mathbb{L}^0_+$ contains all $f \in \mathbb{L}^0$ such that $f \geq 0$. Furthermore, we shall be using $\mathbb{L}^0_{++}$ to denote the set of all $f \in \mathbb{L}^0$ such that $f > 0$; note that $\mathbb{L}^0_{++}$ contains all *strictly positive* measurable functions. Sometimes we shall just write $f > 0$ instead of $f \in \mathbb{L}^0_{++}$; one should be careful not to confuse the statement $f > 0$ as meaning $f \in \mathbb{L}^0_+ \setminus \{0\}$.

The class of all *probabilities* on $(S, \mathcal{S})$ that are *equivalent* to σ will be denoted by Π. We reserve the symbols $\mathbb{P}$, $\mathbb{Q}$, etc., to denote elements (probabilities) in Π, and use the familiar notation $\mathbb{E}_{\mathbb{P}}[\cdot]$, $\mathbb{E}_{\mathbb{Q}}[\cdot]$ to denote expectation on $\mathbb{L}^0_+$ (which is always defined, but may take the value ∞).

Exercise 0.1. Let $\mathbb{P}$ and $\mathbb{Q}$ be any probabilities on $(S, \mathcal{S})$ with $\mathbb{Q} \ll \mathbb{P}$. Then, for every $\epsilon > 0$ there exists $\delta \equiv \delta(\epsilon) > 0$ such that *for all* $A \in \mathcal{S}$ *with* $\mathbb{P}[A] < \delta$ *we have* $\mathbb{Q}[A] < \epsilon$.

1. Topology

1.1. **Metric.** Fix a probability $\mathbb{P} \in \Pi$, and consider the functional

$$\mathbb{L}^0 \times \mathbb{L}^0 \ni (f, g) \mapsto d_{\mathbb{P}}(f, g) := \mathbb{E}_{\mathbb{P}}\left[1 \wedge |f - g|\right]. \tag{1.1}$$

Exercise 1.1. With the above notation, show the following:

(1) $d_{\mathbb{P}}$ is a metric on $\mathbb{L}^0$.
(2) If $(f_n)_{n \in \mathbb{N}}$ is a sequence in $\mathbb{L}^0$ and $f \in \mathbb{L}^0$, $\lim_{n \to \infty} d_{\mathbb{P}}(f_n, f) = 0$ holds if and only if $\lim_{n \to \infty} \mathbb{P}[|f_n - f| > \epsilon] = 0$ holds for every $\epsilon > 0$.
(3) If $\mathbb{Q} \in \Pi$ is another probability, $d_{\mathbb{Q}}$ (which is also a metric) induces the same topology on $\mathbb{L}^0$. (Hint: use Exercise 0.1.)

We endow $\mathbb{L}^0$ with the topology induced by $d_{\mathbb{P}}$. The previous exercise implies that this topology does not depend on the representative probability, and that convergence of sequences coincides with the well-known convergence in probability. In particular, a.s.-convergent sequences are converging in $\mathbb{L}^0$. Furthermore, any $\mathbb{L}^0$-convergent sequence has an a.s. converging subsequence.

[7] We shall only consider nonnegative measures.

NB: Unless otherwise explicitly mentioned, limits involving elements of $\mathbb{L}^0$ throughout the notes are (by convention) understood in the aforementioned topology.

Exercise 1.2. In what follows, $(f_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{L}^0$.

(1) If $\lim_{n\to\infty} \mathbb{E}_{\mathbb{P}}[|f_n - f|] = 0$ holds for some $f \in \mathbb{L}^0$ and $\mathbb{P} \in \Pi$, show that $\lim_{n\to\infty} f_n = f$.

(2*) Construct a sequence $(f_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} f_n = 0$, but where $\lim_{n\to\infty} \mathbb{E}_{\mathbb{P}}[|f_n|] = 0$ fails for *all* $\mathbb{P} \in \Pi$.

Proposition 1.3. $\mathbb{L}^0$ *is a complete metric space.*

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be Cauchy in $\mathbb{L}^0$. Fix $\mathbb{P} \in \Pi$ and pick a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ such that $d_{\mathbb{P}}(f_{n_{k+1}}, f_{n_k}) \le 2^{-k}$ holds for all $k \in \mathbb{N}$. Then,

$$\mathbb{E}_{\mathbb{P}}\left[\sum_{k\in\mathbb{N}} 1 \wedge |f_{n_{k+1}} - f_{n_k}|\right] = \sum_{k\in\mathbb{N}} \mathbb{E}_{\mathbb{P}}\left[1 \wedge |f_{n_{k+1}} - f_{n_k}|\right] = \sum_{k\in\mathbb{N}} d_{\mathbb{P}}(f_{n_{k+1}}, f_{n_k}) \le 1,$$

which implies that $\sum_{k\in\mathbb{N}} 1 \wedge |f_{n_{k+1}} - f_{n_k}| < \infty$ (a.s.) and, a fortiori, that $\sum_{k\in\mathbb{N}} |f_{n_{k+1}} - f_{n_k}| < \infty$. It follows that the limit $f := \lim_{k\to\infty} f_{n_k}$ exists in the a.s. sense; therefore, also in $\mathbb{L}^0$. Since $(f_n)_{n\in\mathbb{N}}$ is Cauchy and has a convergent subsequence, the whole sequence is convergent. □

1.2. **The "double subsequence" method.** The following trick in showing $\mathbb{L}^0$-convergence is often useful.

Proposition 1.4. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{L}^0$, and $f \in \mathbb{L}^0$. The following are equivalent:*

(1) $\lim_{n\to\infty} f_n = f$.

(2) *For* any *subsequence $(f_{n_k})_{k\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ one may extract a further subsequence $(f_{n_{k_m}})_{m\in\mathbb{N}}$ of $(f_{n_k})_{k\in\mathbb{N}}$ such that $\lim_{m\to\infty} f_{n_{k_m}} = f$.*

Proof. Implication (1) ⇒ (2) is obvious because $\mathbb{L}^0$-convergence comes from a metric topology. Assume (2) and let $d_{\mathbb{P}}$ be the metric of (1.1). If $\lim_{n\to\infty} f_n = f$ were not true, one would be able to find $\epsilon > 0$ and a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ such that $d_{\mathbb{P}}(f_{n_k}, f) > \epsilon$, for all $k \in \mathbb{N}$. But then, there would not exist any subsequence $(f_{n_{k_m}})_{m\in\mathbb{N}}$ of $(f_{n_k})_{k\in\mathbb{N}}$ such that $\lim_{m\to\infty} f_{n_{k_m}} = f$. □

Although checking condition (2) of Proposition 1.4 seems quite more involved than checking condition (1), we can nevertheless obtain certain properties of $\mathbb{L}^0$-convergence from the corresponding properties of a.s. convergence easily, using the fact that any $\mathbb{L}^0$-convergent sequence has an a.s. converging subsequence. For practice, use Proposition 1.4 in the following exercise.

Exercise 1.5. For $d \in \mathbb{N}$, and each $i \in \{1, \dots, d\}$, let $(f_n^i)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{L}^0$ converging to $f^i \in \mathbb{L}^0$. If $\Phi : \mathbb{R}^d \mapsto \mathbb{R}$ is continuous, show that $\lim_{n\to\infty} \Phi(f_n^1, \dots f_n^d) = \Phi(f^1, \dots f^d)$.

Proposition 1.3 and Exercise 1.5 combined imply that $\mathbb{L}^0$ *is a complete metrizable topological vector space*, i.e., a vector space equipped with a topology coming from a complete metric such that the operations $\mathbb{R} \times \mathbb{L}^0 \ni (a, f) \mapsto af \in \mathbb{L}^0$ and $\mathbb{L}^0 \times \mathbb{L}^0 \ni (f, g) \mapsto (f + g) \in \mathbb{L}^0$ are continuous. Thus, the algebraic structure of $\mathbb{L}^0$ ties well together with its topological structure.

1.3. **Special subsets of $\mathbb{L}^0$.** Open and closed sets are defined as usual in topological spaces; we shall use $\overline{K}$ to denote the closure of $K \subseteq \mathbb{L}^0$.

A set $K \subseteq \mathbb{L}^0$ is *convex* if for all $f \in K$, $g \in K$ and $\lambda \in [0,1]$, we have $((1-\lambda)f + \lambda g) \in K$. The space $\mathbb{L}^0$ is itself convex, as is its nonnegative orthant $\mathbb{L}^0_+$. The intersection of an arbitrary collection of convex sets is again convex. The closure of a convex set is convex. Since $\mathbb{L}^0$ is convex itself, for $K \subseteq \mathbb{L}^0$ its *convex hull* $\mathsf{conv}K$ is defined the smallest convex set containing K: $\mathsf{conv}K$ consists exactly of all elements of the form $\sum_{i=1}^n \alpha_i f_i$, where $n \in \mathbb{N}$, $(\alpha_i)_{i \in \{1,\ldots,n\}} \in \mathbb{R}^n_+$ is such that $\sum_{i=1}^n \alpha_i = 1$ and $f_i \in K$ for all $i \in \{1, \ldots, n\}$.

*Exercise** 1.6. Show by example that the convex hull of a closed set in $\mathbb{L}^0$ may fail to be closed.

We continue with another important notion. A set $B \subseteq \mathbb{L}^0$ will be called *bounded* if for some (and then, for all[8]) $\mathbb{P} \in \Pi$, $\lim_{\ell \to \infty} \sup_{f \in B} \mathbb{P}[|f| > \ell] = 0$.

*Exercise** 1.7. Show that a set $B \subseteq \mathbb{L}^0$ is bounded if and only if for every open $O \subseteq \mathbb{L}^0$ with $0 \in O$ it holds that $B \subset aO$ for some large enough $a > 0$ (that may depend on O).

The geometric picture revealed by the previous exercise is apparent: B is bounded if and only if any open set around $0 \in \mathbb{L}^0$ may be inflated enough so that it contains B.

Exercise 1.8. If $B \subseteq \mathbb{L}^0$ is bounded, show that $\overline{B}$ is also bounded.

Exercise 1.9. Find a bounded $B \subseteq \mathbb{L}^0_+$, with the property that $\mathsf{conv}B$ fails to be bounded.

Exercise 1.10. Suppose that B_i, $i \in \{1, \ldots, d\}$ are bounded subsets of $\mathbb{L}^0$. Then, show that:

- $\bigcup_{i \in \{1,\ldots,n\}} B_i$ is bounded.
- $B_1 + \ldots + B_n := \{f_1 + \ldots + f_n \mid f_i \in B_i \text{ for all } i \in \{1, \ldots, n\}\}$ is bounded.

Exercise 1.11. If $(f_n)_{n \in \mathbb{N}}$ is a convergent sequence, $\{f_n \mid n \in \mathbb{N}\}$ is bounded.

*Exercise** 1.12. If $\mathbb{L}^0$ is infinite-dimensional (as a vector space), show that *no* open and bounded subsets of $\mathbb{L}^0$ exist.

1.4. **Forward convex combinations and Komlos lemma.** Let $(f_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{L}^0$. A sequence $(g_n)_{n \in \mathbb{N}}$ with the property that $g_n \in \mathsf{conv}\{f_k \mid k \geq n\}$ for all $n \in \mathbb{N}$ will be called *a sequence of forward convex combinations of* $(f_n)_{n \in \mathbb{N}}$. A subsequence of $(f_n)_{n \in \mathbb{N}}$ is in particular a sequence of forward convex combinations of $(f_n)_{n \in \mathbb{N}}$, where the forward convex weights are of "Dirac" type. Furthermore, a sequence of forward convex combinations of a sequence of forward convex combinations of $(f_n)_{n \in \mathbb{N}}$ is itself a sequence of forward convex combinations of $(f_n)_{n \in \mathbb{N}}$.

The following result, whose beautiful proof we borrow from [DS94, Lemma A1.1] will be extremely important in the sequel.

[8]Prove this "and then, for all" claim using Exercise 0.1.

Lemma 1.13 (Baby Komlos). *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{L}^0_+$ such that $\mathsf{conv}\{f_n \mid n \in \mathbb{N}\}$ is bounded. Then, there exists an a.s. convergent sequence of forward convex combinations of $(f_n)_{n\in\mathbb{N}}$.*

Proof. Fix $\mathbb{P} \in \Pi$. Also, let $\mathbb{R}_+ \ni x \mapsto \phi(x) := 1 - \exp(-x) \in [0, 1)$, as well as $\mathbb{L}^0_+ \ni f \mapsto u(f) := \mathbb{E}_{\mathbb{P}}[\phi(f)] \in [0, 1)$. For each $n \in \mathbb{N}$, define $K_n := \mathsf{conv}(\{f_k \mid k \geq n\})$ and $v_n = \sup_{g\in K_n} u(g) \in [0, 1)$. Clearly, $(v_n)_{n\in\mathbb{N}}$ is a nonincreasing sequence; define $v := \lim_{n\to\infty} v_n \in [0, 1)$.

For each $n \in \mathbb{N}$, pick $g_n \in K_n$ with $u(g_n) \geq v_n - 1/n$. We shall show that $(g_n)_{n\in\mathbb{N}}$ is Cauchy, which will imply that it is convergent in view of Proposition 1.3. For every $\epsilon > 0$, pick $\ell_\epsilon > 0$ so that $\sup_{n\in\mathbb{N}} \mathbb{P}[g_n > \ell_\epsilon] < \epsilon/2$, which is possible because K_1 is bounded. Let

$$D_\epsilon = \left\{(x, y) \in \mathbb{R}^2_+ \mid |x - y| > \epsilon,\ x \leq \ell_\epsilon,\ y \leq \ell_\epsilon\right\}.$$

Strict concavity of ϕ implies that there exists $\gamma_\epsilon > 0$ such that

$$\phi\left(\frac{x+y}{2}\right) \geq \frac{\phi(x) + \phi(y)}{2} + \gamma_\epsilon \mathbb{I}_{D_\epsilon}(x, y). \tag{1.2}$$

We then obtain, for all $m \in \mathbb{N}$ and $n \in \mathbb{N}$,

$$\gamma_\epsilon \mathbb{P}[(g_n, g_m) \in D_\epsilon] \leq u\left(\frac{g_n + g_m}{2}\right) - \frac{u(g_n) + u(g_m)}{2}.$$

When $n \leq m$, we have $u((g_n + g_m)/2) \leq v_n$, $u(g_n) \geq v_n - 1/n \geq v - 1/n$ and $u(g_m) \geq v_m - 1/m \geq v - 1/n$; therefore, for fixed $n \in \mathbb{N}$,

$$\sup_{m\geq n} \mathbb{P}[(g_n, g_m) \in D_\epsilon] \leq \frac{1}{\gamma_\epsilon}\left(v_n - v + \frac{1}{n}\right). \tag{1.3}$$

It follows that $\lim_{n\to\infty} \sup_{m\geq n} \mathbb{P}[(g_n, g_m) \in D_\epsilon] = 0$. Recalling that $\sup_{n\in\mathbb{N}} \mathbb{P}[g_n > \ell_\epsilon] < \epsilon/2$ and the definition of D_ϵ, we obtain $\mathbb{P}[|g_n - g_m| > \epsilon] \leq \mathbb{P}[(g_n, g_m) \in D_\epsilon] + \epsilon$ holds for all $m \in \mathbb{N}$ and $n \in \mathbb{N}$. It follows that $\limsup_{n\to\infty} \sup_{m\geq n} \mathbb{P}[|g_n - g_m| > \epsilon] \leq \epsilon$, for all $\epsilon > 0$. Since $d_{\mathbb{P}}(g_n, g_m) = \mathbb{E}_{\mathbb{P}}[1 \wedge |g_n - g_m|] \leq \epsilon/2 + \mathbb{P}[|g_n - g_m| > \epsilon/2]$ holds for all $\epsilon \in (0, 1)$, it follows that $\limsup_{n\to\infty} \sup_{m\geq n} d_{\mathbb{P}}(g_n, g_m) \leq \epsilon$ for all $\epsilon > 0$, i.e., $\limsup_{n\to\infty} \sup_{m\geq n} d_{\mathbb{P}}(g_n, g_m) = 0$.

We have constructed a sequence of forward convex combinations of $(f_n)_{n\in\mathbb{N}}$ that is $\mathbb{L}^0$-convergent; by passing to a subsequence is necessary, we obtain an a.s. convergent sequence of forward convex combinations of $(f_n)_{n\in\mathbb{N}}$, concluding the proof. □

Remark 1.14. Let I be an index set of at most countable cardinality. Let $K \subseteq \mathbb{L}^0_+$ be a convex and bounded set, and for each $i \in I$ consider a K-valued sequence $(f^i_n)_{n\in\mathbb{N}}$. By Lemma 1.13, for each individual $i \in I$ there exists an a.s. convergent sequence $(g^i_n)_{n\in\mathbb{N}}$ of forward convex combinations of $(f^i_n)_{n\in\mathbb{N}}$. In general, the forward convex weights used to pass from $(f^i_n)_{n\in\mathbb{N}}$ to $(g^i_n)_{n\in\mathbb{N}}$ will depend on $i \in I$. We claim that, in fact, one can choose these forward convex weights to be the same for all $i \in I$. Probably the easiest way to see this is the following: Let $(I, \mathcal{I}, \iota)$ be the measure space where $\mathcal{I}$ is the powerset of I and ι is the counting measure, and consider the product space $(S \times I, \mathcal{S} \otimes \mathcal{I}, \sigma \otimes \iota)$. The collection of sequences $(f^i_n)_{n\in\mathbb{N}}$ on $\mathbb{L}^0_+(S, \Sigma, \sigma)$ for all $i \in I$ may be seen as a single sequence on $\mathbb{L}^0_+(S \times I, \mathcal{S} \otimes \mathcal{I}, \sigma \otimes \iota)$, living on the bounded convex set $K \times I$. The latter set is actually bounded on $\mathbb{L}^0_+(S \times I, \mathcal{S} \otimes \mathcal{I}, \sigma \otimes \iota)$—one may prove this[9] either directly or by use of Proposition 1.16. An application of Lemma 1.13 immediately implies our claim.

As an application of Lemma 1.13, we have the following result on maximization for quasi-concave and upper-semicontinuous functionals on $\mathbb{L}^0_+$.

Proposition 1.15. *Let $K \subseteq \mathbb{L}^0_+$ be nonempty, convex, closed and bounded, and $u : K \mapsto [-\infty, \infty]$ be quasi-concave[10] and upper-semicontinuous. Then, there exists $g \in K$ with $u(g) = \sup_{f \in K} u(f)$.*

Proof. Let $v := \sup_{f \in K} u(f)$. If $v = -\infty$ there is nothing to show, so assume $v > -\infty$. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence in K such that $\lim_{n \to \infty} u(f_n) = v$. Since K is convex and bounded, $\mathsf{conv}\, \{f_n \mid n \in \mathbb{N}\}$ is bounded. Lemma 1.13 provides a sequence $(g_n)_{n \in \mathbb{N}}$ of forward convex combinations of $(f_n)_{n \in \mathbb{N}}$ such that $g := \lim_{n \to \infty} g_n$ exists. Since K is convex, $g_n \in K$ for all $n \in \mathbb{N}$; then, since K is closed, $g \in K$ follows. Let $i_n := \inf_{k \geq n} u(f_k)$ for all $n \in \mathbb{N}$, so that $f_k \in \{u \geq i_n\}$ for all $n \leq k$. Since u is quasi-concave and $g_n \in \mathsf{conv}\, \{f_k \mid k \geq n\}$, $g_n \in \{u \geq i_n\}$ for all $n \in \mathbb{N}$ follows; in particular, $\limsup_{n \to \infty} u(g_n) \geq \limsup_{n \to \infty} i_n = v$. Finally, $u(g) \geq \limsup_{n \to \infty} u(g_n) = v$ follows from the upper semi-continuity of u. □

For convex and closed sets of $\mathbb{L}^0_+$, boundedness appears as the proper relaxation of compactness that enables a satisfactory theory to be developed. In fact, [Žit10] calls convex, closed and bounded sets of $\mathbb{L}^0_+$ "convexly compact."

The following result is folklore, albeit usually (for example, in [HWY92]) proved using functional-analytic arguments.

Proposition 1.16. *Let $K \subseteq \mathbb{L}^0_+$. Then, $\mathsf{conv} K$ is bounded if and only if there exists $\mathbb{P} \in \Pi$ such that $\sup_{f \in K} \mathbb{E}_\mathbb{P}[f] < \infty$.*

Proof. Assuming that $\sup_{f \in K} \mathbb{E}_\mathbb{P}[f] < \infty$, it follows that $\sup_{f \in \mathsf{conv} K} \mathbb{E}_\mathbb{P}[f] < \infty$; therefore,

$$\lim_{\ell \to \infty} \sup_{f \in \mathsf{conv} K} \mathbb{P}[f > \ell] \leq \lim_{\ell \to \infty} \frac{\sup_{f \in \mathsf{conv} K} \mathbb{E}_\mathbb{P}[f]}{\ell} = 0.$$

Conversely, assume that $\mathsf{conv} K$ is bounded. Then, the closure $\overline{\mathsf{conv}} K$ of $\mathsf{conv} K$ is still convex and bounded. Fix $\mathbb{Q} \in \Pi$, and let $\mathbb{R}_+ \ni x \mapsto \phi(x) := 1 - \exp(-x) \in [0, 1)$, as well as $\mathbb{L}^0_+ \ni f \mapsto u(f) := \mathbb{E}_\mathbb{Q}[\phi(f)] \in [0, 1)$ as in the proof of Lemma 1.13. Note that u is concave and continuous; therefore, from Proposition 1.15, there exists $g \in \overline{\mathsf{conv}} K$ such that $u(f) \leq u(g)$ for all $f \in \mathsf{conv} K$. Using first-order conditions for this concave maximization problem, and noting that ϕ has bounded derivative on $[0, \infty)$, it follows that $\mathbb{E}_\mathbb{Q}[\exp(-g)(f - g)] \leq 0$ holds for all $f \in \mathsf{conv} K$; in particular, for all $f \in K$. Defining $\mathbb{P} \in \Pi$ via $\mathrm{d}\mathbb{P}/\mathrm{d}\mathbb{Q} = \exp(-g)/\mathbb{E}_\mathbb{Q}[\exp(-g)]$, it follows that

$$\sup_{f \in K} \mathbb{E}_\mathbb{P}[f] = \sup_{f \in K} \frac{\mathbb{E}_\mathbb{Q}[\exp(-g) f]}{\mathbb{E}_\mathbb{Q}[\exp(-g)]} \leq \frac{\mathbb{E}_\mathbb{Q}[\exp(-g) g]}{\mathbb{E}_\mathbb{Q}[\exp(-g)]} < \infty,$$

completing the argument. □

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[9] Please try it!

[10] Meaning that $\{u \geq a\}$ is a convex subset of K for all $a \in \mathbb{R}$.

1.5. **The "inner product" of $\mathbb{L}^0_+$.** The convex cone $\mathbb{L}^0_+$ will act as "dual" to itself. Define

$$\langle g, f\rangle := \int_S fg \, \mathrm{d}\sigma \in [0,\infty], \quad f \in \mathbb{L}^0_+, \quad g \in \mathbb{L}^0_+.$$

It is clear that $\langle \cdot, \cdot \rangle : \mathbb{L}^0_+ \times \mathbb{L}^0_+ \mapsto [0,\infty]$ is[11] *bilinear*.

There is a natural identification of $\mathbb{L}^0_+$ with the space $\mathbb{M}^0_+$ of sigma-finite measures that are absolutely continuous with respect to σ. To wit, for any $g \in \mathbb{L}^0_+$ we define $\mu_g \in \mathbb{M}^0_+$ via the recipe $\mathrm{d}\mu_g = g\mathrm{d}\sigma$; conversely, for any $\mu \in \mathbb{M}^0_+$ we define $g_\mu \in \mathbb{L}^0_+$ via $g_\mu := \mathrm{d}\mu/\mathrm{d}\sigma$. The previous operations are inverse to each other, and preserve linearity[12], making the convex cones $\mathbb{M}^0_+$ and $\mathbb{L}^0_+$ copies of each other.[13] With this understanding, the "formal dual" of $\mathbb{L}^0_+$ becomes $\mathbb{M}^0_+$, and we shall indulge ourselves in mostly using notation "μ" (or "ν") in the first argument of $\langle \cdot, \cdot \rangle$, although μ and ν are still considered elements of $\mathbb{L}^0_+$, using its identification with $\mathbb{L}^0_+$ via the Radon-Nikodym theorem, as explained previously.

Lemma 1.17 (Fatou)**.** *The mapping $\langle \cdot, \cdot \rangle : \mathbb{L}^0_+ \times \mathbb{L}^0_+ \mapsto [0,\infty]$ is (jointly) lower semicontinuous.*

Proof. Let $(\mu_n, f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{L}^0_+ \times \mathbb{L}^0_+$ that converges to (μ, f). Extract a subsequence $(\mu_{n_k}, f_{n_k})_{k\in\mathbb{N}}$ of $(\mu_n, f_n)_{n\in\mathbb{N}}$ such that $\liminf_{n\to\infty} \langle \mu_n, f_n\rangle = \lim_{k\to\infty} \langle \mu_{n_k}, f_{n_k}\rangle$, and then extract a further subsequence of $(\mu_{n_k}, f_{n_k})_{k\in\mathbb{N}}$ that converges to (μ, f) in the a.s. sense. By Fatou's lemma, $\langle \mu, f\rangle \leq \lim_{k\to\infty} \langle \mu_{n_k}, f_{n_k}\rangle = \liminf_{n\to\infty} \langle \mu_n, f_n\rangle$, completing the argument. □

For $\mu \in \mathbb{L}^0_+$, define $B(\mu) := \left\{f \in \mathbb{L}^0_+ \mid \langle \mu, f\rangle \leq 1\right\}$ to be the budget set associated with linear prices μ and unit capital. The next result, combined with Proposition 1.15, implies that utility maximization problems over budget sets have solutions.

Proposition 1.18. *For $\mu \in \mathbb{L}^0_+$, $B(\mu)$ is convex and closed. If $\mu \in \mathbb{L}^0_{++}$, $B(\mu)$ is further bounded.*

Proof. $B(\mu)$ is clearly convex, and closedness follows from Lemma 1.17. Assume that $\mu > 0$, let $0 < \nu \leq \mu$ be such that[14] $\langle \nu, 1\rangle \equiv \int_S \nu \mathrm{d}\sigma < \infty$, and define a probability $\mathbb{P} \in \Pi$ via $\mathrm{d}\mathbb{P} = \langle \nu, 1\rangle^{-1} \nu \mathrm{d}\sigma$. For any $f \in B(\mu)$, it follows that $\mathbb{E}_\mathbb{P}[f] \leq \langle \nu, f\rangle / \langle \nu, 1\rangle \leq \langle \mu, f\rangle / \langle \nu, 1\rangle \leq 1/\langle \nu, 1\rangle$, which implies $\sup_{f\in B(\mu)} \mathbb{E}_\mathbb{P}[f] < \infty$, and thus that $B(\mu)$ is bounded by Proposition 1.16. □

2. Separation of Monotone Convex Subsets of $\mathbb{L}^0_+$

2.1. **Strong separation.** A set $C \subseteq \mathbb{L}^0_+$ is called *solid* if $g \in C$ and $0 \leq f \leq g$ implies $f \in C$.

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[11] Since we are working in the positive cone $\mathbb{L}^0_+$, "linearity" has to be defined by multiplication with nonnegative real numbers only, but it is clear what is meant. Note also that the convention $0 \times \infty = 0$ is used.

[12] They even preserve the lattice structure, i.e., $\mu_{f\wedge g} = \mu_f \wedge \mu_g$ and $\mu_{f\vee g} = \mu_f \vee \mu_g$, where for $\mu \in \mathbb{M}^0_+$ and $\mu \in \mathbb{M}^0_+$ one defines $(\mu\wedge\nu)[A] = \inf_{B\in\mathcal{S}, B\subseteq A} (\mu[B] + \nu[A \setminus B])$ and $(\mu\vee\nu)[A] = \sup_{B\in\mathcal{S}, B\subseteq A} (\mu[B] + \nu[A \setminus B])$, where $A \in \mathcal{S}$.

[13] It should be noted that the identification itself depends (in an obvious way) on the choice of the baseline measure σ. The situation is similar to the identification of the dual of a Hilbert space with the Hilbert space itself via the Riesz representation theorem, which depends on the inner product.

[14] Make sure you understand why we can pick such ν.

Theorem 2.1. *Let C and K be nonempty convex and closed subsets of $\mathbb{L}^0_+$. Additionally, suppose that one of the following two conditions is satisfied:*

(A) *C is solid and K is bounded.*
(B) *C is bounded and $K = K + \mathbb{L}^0_+$.*

Then, the following statements are equivalent:

(1) *$C \cap K = \emptyset$.*
(2) *There exists $\mu \in \mathbb{L}^0_+$ such that*

$$\sup_{f \in C} \langle \mu, f \rangle < \inf_{g \in K} \langle \mu, g \rangle < \infty.$$

Furthermore, under condition (B), one may choose $\mu \in \mathbb{L}^0_{++}$.

Proof. Fix $f_0 \in C$ and $g_0 \in K$. By Proposition 1.16, there exists a probability $\mathbb{Q} \in \Pi$ such that:

- Under condition (A), $\mathbb{E}_{\mathbb{Q}}[f_0] < \infty$ and $\sup_{g \in K} \mathbb{E}_{\mathbb{Q}}[g] < \infty$.
- Under condition (B), $\mathbb{E}_{\mathbb{Q}}[g_0] < \infty$ and $\sup_{f \in C} \mathbb{E}_{\mathbb{Q}}[f] < \infty$.

Let $\phi : \mathbb{R} \mapsto \mathbb{R}_+$ be defined via $\phi(y) = (y + \exp(-y) - 1)_+$ for all $y \in \mathbb{R}$. Note that ϕ is nondecreasing and convex and continuously differentiable function with $\phi'(y) = (1 - \exp(-y))_+ \in [0, 1)$ for $y \in \mathbb{R}$, with $\phi(y) = 0$ for $y \leq 0$. Define $r : C \times K \mapsto [0, \infty]$ via $r(f, g) := \mathbb{E}_{\mathbb{Q}}[\phi(g - f)]$ for $f \in C$ and $g \in K$, and set $\rho := \inf_{(f,g) \in C \times K} e(f, g)$. Since $r(f_0, g_0) < \infty$, it follows that $\rho < \infty$. We shall show below that the infimum is actually attained on $C \times K$.

Let $(f_n)_{n \in \mathbb{N}}$ and $(g_n)_{n \in \mathbb{N}}$ be C-valued and K-valued (respectively) sequences such that $r(f_n, g_n) \leq \rho + n^{-1}$ for all $n \in \mathbb{N}$. It may, and will, be assumed that $f_n \leq g_n$. Indeed, under condition (A), $g_n \in K$ implies $(f_n \vee g_n) \in K$, and it is immediate that $r(f_n, f_n \vee g_n) \leq r(f_n, g_n)$; similarly, under condition (B), $f_n \in C$ implies $(f_n \wedge g_n) \in C$, and $r(f_n \wedge g_n, g_n) \leq r(f_n, g_n)$.

Continuing, $\rho < \infty$, implies $\sup_n \mathbb{E}_{\mathbb{Q}}[g_n - f_n] < \infty$, which implies that the convex hull of the sequence $(g_n - f_n)_{n \in \mathbb{N}}$ is bounded. Under condition (A), the convex hull of $(g_n)_{n \in \mathbb{N}}$ is bounded; since $f_n \leq g_n$ holds for all $n \in \mathbb{N}$, the convex hull of $(f_n)_{n \in \mathbb{N}}$ is bounded. Under condition (B), the convex hull of $(f_n)_{n \in \mathbb{N}}$ is bounded, which also implies that the convex hull of $(g_n)_{n \in \mathbb{N}}$ is bounded. It follows that, under the force of either condition (A) or condition (B), the convex hull of both $(f_n)_{n \in \mathbb{N}}$ and $(g_n)_{n \in \mathbb{N}}$ is bounded. Therefore, Komlos' Lemma 1.13 allows to extract a sequence $(f'_n)_{n \in \mathbb{N}}$ of forward convex combinations of $(f_n)_{n \in \mathbb{N}}$, as well as $f_\infty \in \mathbb{L}^0_+$, such that $\lim_{n \to \infty} f'_n = f_\infty$ holds in the a.s. sense. Let $(g'_n)_{n \in \mathbb{N}}$ denote the sequence of forward convex combinations of $(g_n)_{n \in \mathbb{N}}$ that is constructed by using the same convex weights that were used for $(f'_n)_{n \in \mathbb{N}}$; then, note that $f'_n \leq g'_n$ holds for all $n \in \mathbb{N}$. Using Komlos' Lemma again, one may extract a sequence $(g''_n)_{n \in \mathbb{N}}$ of forward convex combinations of $(g'_n)_{n \in \mathbb{N}}$, as well as $g_\infty \in \mathbb{L}^0_+$, such that $\lim_{n \to \infty} g''_n = g_\infty$ holds in the a.s. sense. Letting $(f''_n)_{n \in \mathbb{N}}$ denote the sequence of forward convex combinations of $(f'_n)_{n \in \mathbb{N}}$ that is constructed by using the same convex weights that were used for $(g''_n)_{n \in \mathbb{N}}$, it follows that $f''_n \leq g''_n$ for all $n \in \mathbb{N}$, and $\lim_{n \to \infty} f''_n = f_\infty$ in the a.s. sense. Since C and K are convex and closed, we obtain that $f_\infty \in C$ and $g_\infty \in K$. Fatou's lemma gives $\mathbb{E}_{\mathbb{Q}}[g_\infty - f_\infty] < \infty$. Furthermore, convexity of r implies that the sequence $(f''_n, g''_n)_{n\in\mathbb{N}}$ is also minimising. Since $\lim_{n\to\infty} r(f''_n, g''_n) = \rho$, Fatou's lemma implies that $r(f_\infty, g_\infty) = \rho$ holds for some $(f_\infty, g_\infty) \in C \times K$ with $f_\infty \le g_\infty$. Define $\zeta := (g_\infty - f_\infty) \in \mathbb{L}^0_+$. Since $C \cap K = \emptyset$, note that $\zeta \neq 0$.

Let $(f, g) \in C \times K$ be such that $\mathbb{E}_{\mathbb{Q}}[f] < \infty$ and $\mathbb{E}_{\mathbb{Q}}[g] < \infty$. (Such pairs certainly exist; recall the beginning of the proof.) Note that

$$\frac{\mathbb{E}_{\mathbb{Q}}[\phi((1-\epsilon)\zeta + \epsilon(g-f))] - \mathbb{E}_{\mathbb{Q}}[\phi(\zeta)]}{\epsilon} \ge 0, \quad \forall \epsilon \in (0,1),$$

and $|\phi((1-\epsilon)\zeta + \epsilon(g-f)) - \phi(\zeta)| \le \epsilon|g-f|$ holds in view of the fact that ϕ' is $[0,1)$-valued. Therefore, using the dominated convergence theorem as $\epsilon \downarrow 0$, we obtain that $\mathbb{E}_{\mathbb{Q}}[\phi'(\zeta)(g-f-\zeta)] \ge 0$. In other words, defining $\nu := \phi'(\zeta)(\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P})$ and $\gamma := \langle \nu, \zeta\rangle > 0$, and noting that $\langle \nu, f\rangle < \infty$ and $\langle \nu, g\rangle < \infty$, it follows that $\langle \nu, f\rangle + \gamma \le \langle \nu, g\rangle$ holds for all $(f, g) \in C \times K$ such that $\mathbb{E}_{\mathbb{Q}}[f] < \infty$ and $\mathbb{E}_{\mathbb{Q}}[g] < \infty$.

Assume condition (A). For arbitrary $(f, g) \in C \times K$, it holds that $(f \wedge n) \in C$ for all $n \in \mathbb{N}$ and $\mathbb{E}_{\mathbb{Q}}[g] < \infty$. Since $\mathbb{E}_{\mathbb{Q}}[f \wedge n] < \infty$ holds for all $n \in \mathbb{N}$, the monotone convergence theorem gives

$$\langle \nu, f\rangle + \gamma = \lim_{n\to\infty} (\langle \nu, f \wedge n\rangle + \gamma) \le \langle \nu, g\rangle,$$

which is exactly what was required with $\mu = \nu$.

Assume now condition (B). Recall that $\sup_{f\in C} \mathbb{E}_{\mathbb{Q}}[f] < \infty$, and define $\mu \in \mathbb{L}^0_{++}$ via

$$\mu = \nu + \frac{\gamma}{2(1 + \sup_{f\in C} \mathbb{E}_{\mathbb{Q}}[f])} \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}.$$

Then, for any $(f, g) \in C \times K$ such that $\mathbb{E}_{\mathbb{Q}}[g] < \infty$,

$$\langle \mu, f\rangle + \gamma/2 \le \langle \nu, f\rangle + \gamma \le \langle \nu, g\rangle \le \langle \mu, g\rangle.$$

If $g \in K$ is such that $\mathbb{E}_{\mathbb{Q}}[g] = \infty$, then $\langle \mu, g\rangle = \infty$ and the above inequality trivially holds. This concludes the proof. □

For later applications, we shall need a strengthening of Theorem 2.1 for subsets of $\mathbb{L}^0_+ \times \mathbb{R}$, which is stated below. The proof is similar to (but more notationally involved than) Theorem 2.1, so it will be skipped. Boundedness on $\mathbb{L}^0_+ \times \mathbb{R}$ is defined in the obvious way (in the product topology), and $C \subseteq \mathbb{L}^0_+ \times \mathbb{R}$ is solid if whenever $(g, b) \in C$ and $0 \le f \le g$, $a \le b$, then $(f, a) \in C$.

Theorem 2.2. *Let C and K be nonempty convex and closed subsets of $\mathbb{L}^0_+ \times \mathbb{R}$. Additionally, suppose that one of the following two conditions is satisfied:*

(A) *C is solid and K is bounded.*

(B) *C is bounded and $K = K + \mathbb{L}^0_+ \times \mathbb{R}_+$.*

Then, the following statements are equivalent:

(1) *$C \cap K = \emptyset$.*

(2) *there exists $(\mu, \beta) \in \mathbb{M}^0_+ \times \{0, 1\}$ such that*

$$\sup_{(f,a)\in C} (\langle \mu, f\rangle + \beta a) < \inf_{(g,b)\in K} (\langle \mu, g\rangle + \beta b) < \infty.$$

Furthermore, under condition (B), one may choose $\mu \in \mathbb{L}^0_{++}$.

2.2. **Polars; the Bipolar Theorem.** For $K \subseteq \mathbb{L}^0_+$, we define its *polar* to be

$$K^\circ := \left\{ \mu \in \mathbb{L}^0_+ \mid \langle \mu, f \rangle \le 1, \text{ for all } f \in K \right\}.$$

The following exercise provides practice with polars.

Exercise 2.3. Show the following:

(1) $(\mathbb{L}^0_+)^\circ = \{0\}$ and $\{0\}^\circ = \mathbb{L}^0_+$.
(2) If $A \subseteq B \subseteq \mathbb{L}^0_+$, then $B^\circ \subseteq A^\circ$.
(3) For every $K \subseteq \mathbb{L}^0_+$, K° is convex, solid and closed.
(4) $K \subseteq \mathbb{L}^0_+$ is bounded if and only if $K^\circ \cap \mathbb{L}^0_{++} \neq \emptyset$.

For $K \subseteq \mathbb{L}^0_+$, consider now the *bipolar* $K^{\circ\circ} \equiv (K^\circ)^\circ$ of K:

$$K^{\circ\circ} := \left\{ f \in \mathbb{L}^0_+ \mid \langle \mu, f \rangle \le 1, \text{ for all } \mu \in K^\circ \right\}.$$

Exercise 2.4. For $K \subseteq \mathbb{L}^0$, $K^{\circ\circ}$ is convex, solid and closed. Furthermore, $K \subseteq K^{\circ\circ}$.

It is clear from Exercise 2.4 that, if $K = K^{\circ\circ}$ is to hold, it is necessary that K is convex, solid and closed. In fact, the following result shows the sufficiency of this structural conditions. Its statement first appeared in [BS99], via a functional-analytic proof.

Theorem 2.5 (Bipolar). *A set $K \subseteq \mathbb{L}^0_+$ is convex, solid and closed if and only if $K = K^{\circ\circ}$.*

Proof. Only one direction has to be shown. Let $K \subseteq \mathbb{L}^0_+$ be convex, solid and closed. Suppose there exists $g \in K^{\circ\circ} \setminus K$. From Theorem 2.1, there exists $\nu \in \mathbb{L}^0_+$ such that $\sup_{f \in K} \langle \mu, f \rangle < \langle \mu, g \rangle$. By multiplying μ with a strictly positive constant if necessary, we may assume that $\sup_{f \in K} \langle \mu, f \rangle \le 1 < \langle \mu, g \rangle$. The last inequality implies that $\mu \in K^\circ$; but then $\langle \mu, g \rangle > 1$ would imply that $g \notin K^{\circ\circ}$, which is a contradiction. □

2.3. **Strict separation.** The next result complements Theorem 2.1, where now one of the non-intersecting convex sets is open and monotone above.

Theorem 2.6. *Let $f \in \mathbb{L}^0_{++}$, and $O \subseteq \mathbb{L}^0_{++}$ be nonempty, convex, open (in the relative topology of $\mathbb{L}^0_{++}$), and such that $O = O + \mathbb{L}^0_+$. Then, the following statements are equivalent.*

(1) *$f \notin O$.*
(2) *There exists $\mu \in \mathbb{L}^0_+$ such that $\langle \mu, f \rangle < \langle \mu, g \rangle$ holds for all $g \in O$.*

Proof. Implication (2) $\Rightarrow$ (1) is trivial. We focus below on proving the reverse implication (1) $\Rightarrow$ (2).

Without loss of generality, we may assume in the sequel that $f = 1$; indeed, note that $(1/f)O := \{g/f \mid g \in O\}$ satisfies all the corresponding properties that O has. Suppose one can find $\nu \in \mathbb{L}^0_+$ such that $\langle \nu, 1 \rangle < \langle \nu, g \rangle$ holds for all $g \in (1/f)O$. Then, with $\mu = (1/f)\nu$, it holds that $\langle \mu, f \rangle < \langle \mu, g \rangle$ for all $g \in O$. To recapitulate, we shall additionally assume that $f = 1 \in C$ from now on.

Let $x := \sup \{y \in (0, \infty) \mid y \notin O\}$. Clearly $x \geq 1$. Furthermore, we claim that $x < \infty$; to wit, pick $g \in O$, and note that $g \wedge y_0 \in O$ for large enough $y_0 > 0$ follows from the fact that O is open, which since $g \wedge y_0 \leq y_0$ implies $y_0 \in O$. The fact that O is open also implies that $x \notin O$. Of course, $(x + 1/n) \in O$ holds for all $n \in \mathbb{N}$.

Define $O^\infty := O \cap \mathbb{L}^\infty$, and note that $x \notin O^\infty$. Furthermore, if $g \in O$, then $g \wedge n \in O^\infty$ for large enough $n \in \mathbb{N}$, since O is open. In particular, O^∞ is a nonempty convex subset of $\mathbb{L}^\infty$. Furthermore, O^∞, when seen as a subset of $\mathbb{L}^\infty$, has an *internal point*: indeed, fix any $g \in O^\infty$, and note that $(g + 1) \in O^\infty$ is such that for any $h \in \mathbb{L}^\infty$ with $\|h - (g + 1)\|_{\mathbb{L}^\infty} \leq 1$ we have $h \in O^\infty$. According to the "plain vanilla" algebraic separation theorem [AB06, Theorem 5.61], there exists a nonzero linear functional $\pi : \mathbb{L}^\infty \mapsto \mathbb{R}$, such that $\pi(x) \leq \pi(g)$ holds for all $g \in O^\infty$. Fixing $g \in O^\infty$, for any $h \in \mathbb{L}^\infty_+$ and $n \in \mathbb{N}$, $(g + nh) \in O^\infty$; therefore, $\pi(x) - \pi(g) \leq n\pi(h)$ holds for all $n \in \mathbb{N}$, which gives $\pi(h) \geq 0$. We therefore obtain that π is a positive linear functional.

Note that $\pi(1) > 0$ has to hold; otherwise, if $\pi(1) = 0$, $\pi(h) = 0$ would holds for all $h \in \mathbb{L}^\infty_+$, which would give $\pi = 0$. Upon normalisation, we may assume that π is a finitely additive probability. Now, let $(h_n)_{n\in\mathbb{N}}$ be any nonincreasing $[0, 1]$-valued sequence in $\mathbb{L}^0_+$ that converges to zero in $\mathbb{L}^0$. Note that $(x + 1/k - h_n) \in \mathbb{L}^\infty_{++}$ for all $k \in \mathbb{N}$ and $n \in \mathbb{N}$, and $(x + 1/k) \in O^\infty$ for all $k \in \mathbb{N}$; therefore, for every $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ such that $(x + 1/k - h_{n_k}) \in O^\infty$. It follows that $\pi(x) \leq \pi(x + 1/k - h_{n_k})$, which gives $\pi(h_{n_k}) \leq 1/k$. We deduce that $\lim_{n\to\infty} \pi(h_n) = 0$. Since the sequence $(h_n)_{n\in\mathbb{N}}$ with the aforementioned properties is arbitrary, we obtain that π is actually countably additive; therefore, it can be extended to a countably additive probability measure; in other words, there exists $\mu \in \mathbb{L}^0_+ \setminus \{0\}$ such that $\langle \mu, f\rangle = \pi(f)$ holds for all $f \in \mathbb{L}^\infty_+$. Therefore, we have $\langle \mu, 1\rangle \leq \langle \mu, x\rangle \leq \langle \mu, g\rangle$ holding for all $g \in O^\infty$. For arbitrary $g \in O$, and since $(g \wedge n) \in O^\infty$ holds for all large enough $n \in \mathbb{N}$, the monotone convergence theorem gives $\langle \mu, 1\rangle \leq \langle \mu, g\rangle$. Finally, for arbitrary $g \in O$, pick large enough $n \in \mathbb{N}$ so that $(1 - 1/n)g \in O$, and note that $\langle \mu, 1\rangle \leq \langle \mu, (1 - 1/n)g\rangle < \langle \mu, g\rangle$, with the last strict inequality following due to the fact that $\mu \in \mathbb{L}^0_+ \setminus \{0\}$ and $g \in \mathbb{L}^0_{++}$. This concludes the proof. □

Remark 2.7. Let O satisfy the requirements of Theorem 2.6. Let $f \in \mathbb{L}^0_{++}$ be such that $f \notin O$, but $(f + h) \in O$ holds for all $h \in \mathbb{L}^0_+ \setminus \{0\}$. By Theorem 2.6, there exists $\mu \in \mathbb{L}^0_+$ such that $\langle \mu, f\rangle < \langle \mu, g\rangle$ holds for all $g \in O$. Then, $\langle \mu, h\rangle > 0$ for all $h \in \mathbb{L}^0_+ \setminus \{0\}$, which gives $\mu \in \mathbb{L}^0_{++}$.

As was the case for strong separation, there is a more general version than Theorem 2.6 that involves subsets of $\mathbb{L}^0_{++} \times \mathbb{R}$ which is going to be useful later on. Its proof is very similar to the one of Theorem 2.8; therefore, it is skipped.

Theorem 2.8. *Let $(f, a) \in \mathbb{L}^0_{++} \times \mathbb{R}$, and $O \subseteq \mathbb{L}^0_{++} \times \mathbb{R}$ be nonempty, convex, open, and such that $O = O + \mathbb{L}^0_+ \times \mathbb{R}_+$. Furthermore, assume that $(f, b) \in O$ for some $b \in \mathbb{R}$. Then, the following statements are equivalent.*

(1) *$(f, a) \notin O$.*
(2) *There exists $\mu \in \mathbb{L}^0_+$ such that $\langle \mu, f\rangle + a < \langle \mu, g\rangle + b$ holds for all $(g, b) \in O$.*

3. Monotone Convex Functionals

3.1. **First definitions.** Define $\mathcal{C}$ as the class of all functions $c : \mathbb{L}^0_+ \mapsto [0, \infty]$ that are nondecreasing, convex, and satisfy $c(0) = 0$. Elements in $\mathcal{C}$ should be though of as *cost functional*, with $c(f)$ the cost associated to the consumption bundle $f \in \mathbb{L}^0_+$. A primary example is the linear cost functional $\mathbb{L}^0_+ \ni f \mapsto \langle \mu, f \rangle \in [0, \infty]$ for $\mu \in \mathbb{L}^0_+$. Clearly, $\langle \mu, \cdot \rangle \in \mathcal{C}$ has the extra property that $\langle \mu, af \rangle = a \langle \mu, f \rangle$ holds for $f \in \mathbb{L}^0_+$ and $a \in \mathbb{R}_+$; furthermore, $\langle \mu, \cdot \rangle$ is continuous from below, in view of the monotone convergence theorem. We isolate the last properties in definitions. Define $\mathcal{C}^\uparrow$ as the class of all functions $c \in \mathcal{C}$ that are furthermore continuous from below: for all nondecreasing $(f_n)_{n \in \mathbb{N}}$ such that $f := \lim_{n \to \infty} f_n \in \mathbb{L}^0_+$, it holds that $\lim_{n \to \infty} c(f_n) = c(f)$. A functional $c \in \mathcal{C}$ will be called *positively homogeneous* if $c(af) = ac(f)$ holds for all $a \in \mathbb{R}_+$ and $f \in \mathbb{L}^0_+$.

Exercise 3.1. Every $c \in \mathcal{C}^\uparrow$ is lower semicontinuous.

Both set $\mathcal{C}$ and $\mathcal{C}^\uparrow$ are convex cones, and are closed under arbitrary suprema and sums.

Exercise 3.2. Let $c \in \mathcal{C}$ (but not necessarily $c \in \mathcal{C}^\uparrow$.) Define $\overline{c} : \mathbb{L}^0_+ \mapsto [0, \infty]$ via

$$\overline{c}(f) = \inf \left\{ \lim_{n \to \infty} c(f_n) \mid (f_n)_{n \in \mathbb{N}} \text{ is nondecreasing, and } \lim_{n \to \infty} f_n = f \right\}, \quad f \in \mathbb{L}^0_+.$$

Show that $\overline{c} \in \mathcal{C}^\uparrow$. In fact, show that $\overline{c}$ is the largest function in $\mathcal{C}^\uparrow$ that is dominated by c. If c is positively homogeneous, then so is $\overline{c}$.

3.2. **Conjugacy.** Define the *convex conjugate* of $c \in \mathcal{C}$ via

$$c^*(\mu) := \sup_{f \in \{c < \infty\}} \left(\langle \mu, f \rangle - c(f) \right), \quad \mu \in \mathbb{L}^0_+.$$

Since $\langle \cdot, f \rangle - c(f)$ is a monotone, convex and continuous from below functional for all $f \in \{c < \infty\}$, and $c^*(0) = -\inf_{f \in \{c < \infty\}} c(f) = -c(0) = 0$, it follows that $c^* \in \mathcal{C}^\uparrow$.

Example 3.3. Let $K \subseteq \mathbb{L}^0_+$ be convex and solid, and let δ_K be the *convex indicator* of K, defined via $\delta_K(f) = 0$ if $f \in K$ and $\delta_K(f) = \infty$ if $f \notin K$. It is straightforward to check that $\delta_K \in \mathcal{C}$; furthermore, $\delta_K \in \mathcal{C}^\uparrow$ if and only if K is further closed. For $\mu \in \mathbb{L}^0_+$, $\delta^*_K(\mu) = \sup_{f \in K} \langle \mu, f \rangle$, which is a positively homogeneous functional in $\mathcal{C}^\uparrow$.

Set $c^{**} = (c^*)^*$. Since $\langle \mu, f \rangle - c^*(\mu) \leq c(f)$ holds for all $f \in \mathbb{L}^0_+$ and $\mu \in \{c^* < \infty\}$, it follows that $c^{**} \leq c$. The next result shows that c^{**} and c are, in fact, equal, when $c \in \mathcal{C}^\uparrow$.

Theorem 3.4. *Let $c \in \mathcal{C}^\uparrow$. Then, $c^{**} = c$.*

Proof. We already know that $c^{**} \leq c$. By way of contradiction, fix $g \in \mathbb{L}^0_+$ and assume that $c^{**}(g) < c(g)$ holds; in this case, pick $x \in \mathbb{R}$ such that $c^{**}(g) < x < c(g)$.

Define $C := \left\{ (f, a) \in \mathbb{L}^0_+ \times \mathbb{R} \mid c(f) \leq -a \right\}$, which is a convex, closed and solid subset of $\mathbb{L}^0_+ \times \mathbb{R}$. Furthermore, define $K := \{(g, -x)\}$, which is a closed, convex and bounded subset of $\mathbb{L}^0_+ \times \mathbb{R}$. Since $C \cap K = \emptyset$, Theorem 2.2, with condition (A) in its statement valid, implies the existence of $\mu \in \mathbb{L}^0_+$, $\beta \in \{0, 1\}$ and $\gamma > 0$, such that $\langle \mu, f \rangle - \beta c(f) + \gamma \leq \langle \mu, g \rangle - \beta x < \infty$ holds for all $f \in \{c < \infty\}$.

Assume first that $\beta = 0$. In this case, $\langle \mu, f \rangle + \gamma \leq \langle \mu, g \rangle$ would hold for all $f \in \{c < \infty\}$. In particular, $c^*(n\mu) \leq \langle n\mu, g \rangle - n\gamma - c(0) < \infty$ would hold for all $n \in \mathbb{N}$, which would imply that $c^{**}(g) \geq \langle n\mu, g \rangle - c^*(n\mu) \geq c(0) + n\gamma$, for all $n \in \mathbb{N}$, in turn implying that $c^{**}(g) = \infty$. However, the last equality would contradict the fact that $x < c^{**}(g)$.

Now, assume that $\beta = 1$. Then, $\langle \mu, f \rangle - c(f) \leq \langle \mu, g \rangle - x$ holds for all $f \in \{c < \infty\}$; in other words, $c^*(\mu) \leq \langle \mu, g \rangle - x$, which gives $c^{**}(g) \geq \langle \mu, g \rangle - c^*(\mu) \geq x$, which is again a contradiction to $c^{**}(g) < x$. We conclude that $c^{**} = c$ holds. □

Exercise 3.5. Let $c \in \mathcal{C}$. Recalling the definition of $\overline{c} \in \mathcal{C}$ from Exercise 3.2, show that $\overline{c} = c^{**}$.

Use the previous exercise to establish the following.

Exercise 3.6. Let $K \subseteq \mathbb{L}^0_+$ be convex and solid. Recalling the convex indication $\delta_K \in \mathcal{C}$ of Example 3.3, show that $\delta_K^{**} = \delta_{\overline{K}}$.

Example 3.7 that follows builds further on Example 3.3.

Example 3.7. Suppose that $c \in \mathcal{C}^\uparrow$ is positively homogeneous. In this case, it is straightforward to check that c^* is $\{0, \infty\}$-valued; in fact, $c^*(\mu) = 0$ if $\langle \mu, \cdot \rangle \leq c$ (meaning $\langle \mu, f \rangle \leq c(f)$ for all $f \in \mathbb{L}^0_+$), and $c^*(\mu) = \infty$ otherwise. Because both $\langle \mu, \cdot \rangle$ and c are positively homogeneous, $\langle \mu, \cdot \rangle \leq c$ is equivalent to the statement that $\{c \leq 1\} \subseteq \{\langle \mu, \cdot \rangle \leq 1\}$; in other words, with $K := \{c \leq 1\} \in \mathbb{L}^0_+$ being convex, closed and solid, $\langle \mu, f \rangle \leq 1$ holds for all $f \in K$, which is equivalent to that $\mu \in K^\circ$. In the notation of Example 3.3, $c^* = \delta_{K^\circ}$. A combination of Example 3.3 and Theorem 3.4 gives

$$c(f) = c^{**}(f) = \delta^*_{K^\circ}(f) = \sup_{\mu \in K^\circ} \langle \mu, f \rangle, \quad f \in \mathbb{L}^0_+,$$

which implies that the positively homogeneous functionals in $\mathcal{C}^\uparrow$ are exactly the ones of the form $\sup_{\mu \in D} \langle \mu, \cdot \rangle$, where $D \subseteq \mathbb{L}^0_+$ is a convex, closed, and solid set.

Here is another interesting example.

Example 3.8. Let I be a finite set, and consider a collection $(c_i)_{i \in I}$ with $c_i \in \mathcal{C}$ for all $i \in I$. Define

$$A_I(f) := \left\{ (f^i)_{i \in I} \in \left(\mathbb{L}^0_+\right)^I \;\middle|\; \sum_{i \in I} f^i = f \right\}, \quad f \in \mathbb{L}^0_+, \tag{3.1}$$

as the class of all *allocations of f in I*. The *inf-convolution of* $(c_i)_{i \in I}$ is the functional $\square_{i \in I} c_i : \mathbb{L}^0_+ \mapsto [0, \infty]$ defined via

$$\left(\square_{i \in I} c_i\right)(f) = \inf \left\{ \sum_{i \in I} c_i(f^i) \;\middle|\; (f_i)_{i \in I} \in A_I(f) \right\}, \quad f \in \mathbb{L}^0_+.$$

It is straightforward to check that $\square_{i \in I} c_i \in \mathcal{C}$, as well as that $\left(\square_{i \in I} c_i\right)^* = \sum_{i \in I} c_i^*$.

Assume now that $c_i \in \mathcal{C}^\uparrow$ holds for all $i \in I$. We claim that $\square_{i \in I} c_i$ is further continuous from below, which will imply that $(\square_{i \in I} c_i) \in \mathcal{C}^\uparrow$. To wit, let $f \in \mathbb{L}^0_+$ and a nondecreasing sequence $(f_n)_{n \in \mathbb{N}}$ such that $\lim_{n \to \infty} f_n = f$. Let $\ell := \lim_{n \to \infty} \square_{i \in I} c_i(f_n)$; we shall show that $\square_{i \in I} c_i(f) \leq \ell$;

coupled with the reverse inequality $\ell \leq \square_{i \in I} c_i(f)$ coming from monotonicity, the claim will be proved. For each $n \in \mathbb{N}$, let $(f_n^i)_{i \in I} \in A_I(f)$ be such that $\sum_{i \in I} c_i(f_n^i) \leq \square_{i \in I} c_i(f_n) + 1/n \leq \ell + 1/n$. Noting that the sequences $(f_n^i)_{n \in \mathbb{N}}$ are $[0, f]$-valued for all $i \in I$, Remark 1.14 implies the existence of convex weights $(\alpha_{n,k})_{k \geq n}$ for each fixed $n \in \mathbb{N}$, such that the sequences $(g_n^i)_{n \in \mathbb{N}}$ of forward convex combinations of $(f_n^i)_{n \in \mathbb{N}}$ defined via $g_n^i := \sum_{k \geq n} \alpha_{n,k} f_k^i$ are a.s. convergent to, say, $g^i \in \mathbb{L}^0_+$ for all $i \in I$. Since $(g_n^i)_{i \in I} \in A_I(f)$ for all $n \in \mathbb{N}$, $(g^i)_{i \in I} \in A_I(f)$ follows. Now, note that convexity of c_i for all $i \in I$ and the fact that $\sum_{i \in I} c_i(f_n^i) \leq \ell + 1/n$ imply that $\sum_{i \in I} c_i(g_n^i) \leq \ell + 1/n$. Lower semi-continuity of c_i for all $i \in I$ will them imply that $\sum_{i \in I} c_i(g^i) \leq \ell$; by definition of $\square_{i \in I} c_i$, $\square_{i \in I} c_i(f) \leq \ell$.

Suppose that $c_i = c$ for all $i \in I$. If $\# I = n$, we set $\square_n c \equiv \square_{i \in I} c_i$; by convexity and symmetry, it follows that $\square_n c(f) = n c(f/n)$ holds for $f \in \mathbb{L}^0_+$. In a limiting sense, one may define $\square_\infty c(f) = \lim_{n \to \infty} n c(f/n)$, which is an element of $\mathcal{C}$, but may fail to be continuous from below (even if c itself is). Since $c(0) = 0$, $\square_\infty c(f)$ actually coincides with the directional derivative at zero of c in the direction $f \in \mathbb{L}^0_+$.

3.3. **No arbitrage.** The following result can be seen as providing an equivalent formulation of a "no arbitrage" requirement on a cost functional.

Theorem 3.9. *For $c \in \mathcal{C}^\uparrow$, the following statements are equivalent:*

(1) *For all $f \in \mathbb{L}^0_+ \setminus \{0\}$, there exists $a \equiv a_f > 0$ such that $c(af) > 0$.*
(2) *There exist $x \in \mathbb{R}_+$ and $\nu \in \mathbb{L}^0_{++}$ such that $c(f) \geq -x + \langle \nu, f \rangle$ holds for all $f \in \mathbb{L}^0_+$.*

Remark 3.10. Statement (2) of Theorem 3.9 equivalently reads $\{c^* < \infty\} \cap \mathbb{L}^0_{++} \neq \emptyset$.

Proof. Assume statement (2), and fix $f \in \mathbb{L}^0_+ \setminus \{0\}$. Since $\nu \in \mathbb{L}^0_{++}$, it holds that $\langle \nu, f \rangle > 0$; therefore, with $a = (x+1)/\langle \nu, f \rangle$, it follows that $c(af) \geq -x + a \langle \nu, f \rangle = 1 > 0$.

In the sequel of the proof, assume statement (1). As a first step towards showing statement (2), we treat the special case where $c \in \mathcal{C}^\uparrow$ is further assumed to be positively homogeneous. In this case, Example 3.7 implies that there exists a convex and closed $D \subseteq \mathbb{L}^0_+$ such that $c(f) = \sup_{\mu \in D} \langle \mu, f \rangle$ holds for all $f \in \mathbb{L}^0_+$. Let $p := \sup \{\mathbb{P}[\mu > 0] \mid \mu \in D\} \in [0,1]$. Since D is closed and convex, it is straightforward to show[15] that there exists $\nu \in \mathcal{D}$ such that $\mathbb{P}[\nu > 0] = p$. Note that $\mu = 0$ holds on $\{\nu = 0\}$ for all $\mu \in D$; therefore, if $p < 1$, then $f := \mathbb{I}_{\{\nu = 0\}}$ is such that $f \neq 0$ and $c(f) = 0$, which contradicts statement (1). We conclude that $p = 1$, which means that there exists $\nu \in \mathbb{L}^0_{++}$ such that $c(f) \geq \langle \nu, f \rangle$ holds for all $f \in \mathbb{L}^0_+$, which concludes the proof of implication (1) $\Rightarrow$ (2), in fact with $x = 0$.

We proceed in treating the general case $c \in \mathcal{C}^\uparrow$. For each $n \in \mathbb{N}$, define $c_n \in \mathcal{C}^\uparrow$ via $c_n(f) = c(nf)/n$ for all $f \in \mathbb{L}^0_+$. Note that $(c_n)_{n \in \mathbb{N}}$ is a nondecreasing sequence, and define the positively homogeneous $c_\infty \in \mathcal{C}^\uparrow$ via $c_\infty = \lim_{n \to \infty} c_n$. Statement (1) amounts to $c_\infty(f) > 0$ whenever $f \in \mathbb{L}^0_+ \setminus \{0\}$, which, according to the first step of the proof above, is equivalent to the existence

[15]Have this as an exercise!

of $\nu \in \mathbb{L}^0_{++}$ such that $c^*_\infty(\nu) = 0$. Noting that $c^*_n = (1/n)c^*$ holds for all $n \in \mathbb{N}$, and recalling the notation of convex indicators for Example 3.3, it follows that $\lim_{n\to\infty} c^*_n = \delta_{\{c^*<\infty\}}$. Since $\delta_{\{c^*<\infty\}} \leq c^*_n$ implies $c_n = c^{**}_n \leq \delta^*_{\{c^*<\infty\}}$ from Theorem 3.4, it follows that $c_\infty \leq \delta^*_{\{c^*<\infty\}}$, which in view of Exercise 3.6 gives $\delta_{\overline{\{c^*<\infty\}}} = \delta^{**}_{\{c^*<\infty\}} \leq c^*_\infty$. It follows that $\nu \in \overline{\{c^* < \infty\}}$, which shows that $\overline{\{c^* < \infty\}} \cap \mathbb{L}^0_{++} \neq \emptyset$.

We finally show that $\overline{\{c^* < \infty\}} \cap \mathbb{L}^0_{++} \neq \emptyset$ implies $\{c^* < \infty\} \cap \mathbb{L}^0_{++} \neq \emptyset$. For each $n \in \mathbb{N}$, pick $\mu_n \in \mathbb{L}^0_+$ with $\mathbb{P}[\mu_n = 0] < 2^{-n}$ and $c^*(\mu_n) < \infty$. Upon substituting μ_n with $\mu_n \wedge 1$ if necessary, we may assume that $\mu_n \leq 1$. Define $\alpha_n = 2^{-n}(1 + c^*(\mu_n))^{-1}$ for each $n \in \mathbb{N}$, as well as $\mu := \sum_{n\in\mathbb{N}} \alpha_n \mu_n$, which is a well defined element of $\mathbb{L}^0_+$, since $\mu_n \leq 1$ for all $n \in \mathbb{N}$ and $\sum_{n\in\mathbb{N}} \alpha_n \leq 1$. Note that $\mu \in \mathbb{L}^0_{++}$, since $\mathbb{P}[\mu = 0] \leq \mathbb{P}[\mu_n = 0] \leq 2^{-n}$ holds for all $n \in \mathbb{N}$. Furthermore, for each $m \in \mathbb{N}$, and since $c^*(0) = 0$, it holds that $c^*(\sum_{n=1}^m \alpha_n \mu_n) \leq \sum_{n=1}^m \alpha_n c^*(\mu_n) \leq 1$; since $c^* \in \mathcal{C}^\uparrow$, it follows that $c^*(\mu) \leq 1 < \infty$, which shows that $\{c^* < \infty\} \cap \mathbb{L}^0_{++} \neq \emptyset$, completing the argument. □

The next exercise shows that the requirement $c \in \mathcal{C}^\uparrow$ in Theorem 3.9 may be weakened to $c \in \mathcal{C}$.

Exercise 3.11. For $c \in \mathcal{C}$, suppose that condition (1) in the statement of Theorem 3.9 holds. Show then that the same condition holds for $\overline{c} \in \mathcal{C}^\uparrow$ (with potentially different constants a_f for $f \in \mathbb{L}^0_+$.) Conclude that the equivalences of Theorem 3.9 are true under the weaker assumption $c \in \mathcal{C}$.

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