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ERGODIC ROBUST MAXIMIZATION OF ASYMPTOTIC GROWTH

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We consider the problem of robustly maximizing the growth rate of investor wealth in the presence of model uncertainty. Possible models are all those under which the assets' region E and instantaneous covariation c are known, and where the assets are stable with an exogenously given limiting density p, in that their occupancy time measures converge to a law governed by p. This latter assumption is motivated by the observed stability of ranked relative market capitalizations for equity markets. We seek to identify the robust optimal growth rate, as well as a trading strategy which achieves this rate in all models. Under minimal assumptions upon (E, c, p), which in particular allow for an arbitrary number of assets, we identify the robust growth rate with the Donsker–Varadhan rate function from occupancy time large deviations theory. We also explicitly obtain the optimal trading strategy. We apply our results to the case of drift uncertainty for ranked relative market capitalizations. Here, assuming regularity under symmetrization for the covariance and limiting density of the ranked capitalizations, we explicitly identify the robust optimal trading strategy.

Introduction. In this work, we identify portfolios which maximize long-term growth rate of investor wealth in the presence of model uncertainty. Optimal portfolios are robust, as they achieve the largest possible uniform growth across all models. In our earlier work [20], beliefs ranged across models where assets have common state space and common instantaneous covariance; hence, model uncertainty was tantamount to lack of knowledge regarding the assets' drift. Presently, we obtain optimal portfolios when, in addition to the state space and covariance structure, assets are "stable" in that their occupancy time measures converge to a known, exogenously given, probability density.

Our work is motivated by the remarkable temporal stability of the ranked relative market capitalizations for equities in the United States. This stability was a primary factor behind the development of stochastic portfolio theory in [14, 15] and, as numerous subsequent articles have shown, it can be achieved by modelling market capitalizations via interacting diffusions, where interactions occur though the ranks. For example, [21] considers Brownian particle systems with rank-dependent drifts, and proves ergodicity with limiting exponential distribution, for the process of spacings between ranked particles. Extending the spacing analysis, [18] proves stability of the ranked relative capitalizations, as well as long horizon growth estimates for a large class of wealth processes. Additionally, [18] identifies the stable limiting density via its Laplace transform, with explicit results for a number of particular models, including the Atlas model of [14].

It is natural to ask if an investor may use stability to her advantage when seeking portfolios which maximize the growth of wealth. Furthermore, as it is notoriously difficult to estimate asset drifts, can one essentially only use stability and covariance in order to derive optimal policies? To this last point, it is worthwhile mentioning that, while accurate estimates of the drift (at least for Markovian diffusive models) are possible given the covariance and limiting

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density in *one* dimension, this fails spectacularly as the dimension increases. Indeed, even in two dimensions, there are uncountable families of Markovian diffusion models with common covariance and stable limiting distribution, and where by choosing particular models (drifts) within the family, arbitrarily large growth rates are possible: see Example 2.2 below. Therefore, should robust optimal policies be obtained using only stability and covariance, they enforce efficient investment in the face of severe informational loss, when compared to explicitly knowing the assets' drift.

Broadly speaking, there are two approaches to obtaining growth optimal policies in the face of model uncertainty. The first extends the notion of Cover's "universal" portfolio in [1], to construct portfolios which are growth optimal in a path-wise, model-free environment. The second seeks to construct growth optimal portfolios that are "functionally generated" in the sense of [14], producing optimal policies which are functions of the underlying price process, and thus easily implemented using observations of the current state. Universal portfolio constructions in relative capitalization models are given in the recent articles [5, 26] (in fact, each of these treat functionally generated portfolios as well), while the functionally generated approach, aside from being pioneered in [12–15], has been applied to long horizon problems in [4, 18, 20].

In this work, we follow the functionally generated approach. As in [20], we assume the "price" process X of the traded assets takes values in an arbitrary region $E \subseteq \mathbb{R}^d$, and has both instantaneous covariance rate $c(X)$ and limiting density p. More precisely, fix the region E, covariance function $c : E \to \mathbb{S}^d_{++}$ (the set of positive definite matrices) and probability density $p : E \to (0, \infty)$. On the canonical space of E-valued continuous functions, we consider the class Π of all probability measures $\mathbb{P}$ under which:

- X is a semi-martingale with covariation $\int_0^\cdot c(X_t)\,dt$.
- The laws of $\{X_t; t \geq 0\}$ are tight.
- $\lim_{T\to\infty}(1/T)\int_0^T h(X_u)\,du = \int_E h(y)p(y)\,dy$ almost surely for all h with $h^+ \in L^1(E, p)$.

Wealth processes $V^\vartheta = \mathcal{E}(\int_0^\cdot \vartheta_t'\,dX_t)$ are defined for predictable strategies ϑ in the class Θ ensuring X-integrability under every $\mathbb{P} \in \Pi$. For a given $\vartheta \in \Theta$ and $\mathbb{P} \in \Pi$ we denote by $G(V^\vartheta, \mathbb{P})$ the growth rate of V^ϑ in $\mathbb{P}$-probability

$$G(V^\vartheta, \mathbb{P}) := \sup\left\{\gamma \in \mathbb{R} : \lim_{T\uparrow\infty} \mathbb{P}\left[\frac{1}{T}\log V_T^\vartheta \geq \gamma\right] = 1\right\},$$

and we seek to identify both the optimal robust growth rate

$$\lambda := \sup_{\vartheta\in\Theta}\inf_{\mathbb{P}\in\Pi} G(V^\vartheta, \mathbb{P}),$$

and a strategy $\hat{\vartheta} \in \Theta$ which achieves λ robustly across all $\mathbb{P} \in \Pi$. Our main result, Theorem 1.8 proves that

$$\lambda = I(p), \tag{0.1}$$

where I is the Donsker–Varadhan rate function associated to the second-order linear operator $L^c = (1/2)\,\mathrm{Tr}(D^2 c)$ on E. Introduced in the series of papers [7–9] on ergodic Markov processes, the rate function I governs large deviations for the occupancy time measures. Presently, we do not assume L^c is ergodic (if L^c were ergodic, $\lambda = 0$ as shown in Example 2.4), but rather use the explicit form

$$I(p) = -\inf\left\{\int_E \frac{L^c u}{u}(y)p(y)\,dy \;\middle|\; u \in C^2(E), u > 0, \frac{(L^c u)^+}{u} \in L^1(p)\right\}. \tag{0.2}$$

A heuristic argument in Section 1.2 leads one to expect $\lambda = I(p)$, provided there exists a function $\hat{u}$ such that the E-valued diffusion

$$d\hat{X}_t = (c\nabla \log \hat{u})(\hat{X}_t)\,dt + \sigma(\hat{X}_t)\,d\hat{W}_t \tag{0.3}$$

is ergodic with limiting density p (here σ is a square root of c). However, as innocuous as this statement might seem, proving such a $\hat{u}$ exists for general (multi-dimensional) domains E, covariation functions c and densities p is a challenging task which takes up the bulk of the paper. Interestingly, essentially the only $\hat{u}$ (up to a multiplicative constant) which can possibly lead to ergodicity is the optimizer of the right hand side of (0.2). Furthermore, $\hat{u}$ cannot lead to ergodicity without a priori assuming that there exists at least one symmetric diffusion X^R whose law is in Π for any starting point $x \in E$: see Remark 1.6 below. This follows from the remarkable results in [23], Chapter 6, on necessary and sufficient conditions for multi-dimensional diffusions to be transient or recurrent.

Provided X^R is ergodic, Theorem 1.8 shows that, under mild integrability assumptions (see Assumption 1.5), not only does (0.1) hold, but also there exists an optimizer to the right hand side of (0.2) such that (0.3) is ergodic, and the functionally generated trading strategy $\hat{\vartheta}_\cdot = (\nabla \log \hat{u})(\hat{X}_\cdot)$ is robust growth optimal, achieving growth rate λ under all models in Π.

In Section 1.4 we reinforce the importance of the ergodicity of X^R, by proving that without it, the robust growth optimal problem is in effect ill-posed. More precisely, if X^R is not ergodic, then, at least in one dimension, either $\Pi = \varnothing$ (there exist no measures in our robust class) or $\lambda = \infty$ (infinite robust growth). Section 1.5 provides a general outline for establishing continuity (of the robust growth rate, optimal trading strategy and function $\hat{u}$) with respect to the model parameters. A special case concludes, motivated from stochastic portfolio theory, where the covariance matrix and region are known, and one wants to study the effects of a small deviation in the limiting density.

Section 2 contains important and clarifying examples. First, we show that, under a "gradient" condition (which always holds in the one-dimensional case), $\hat{u}$ is a simple function of c, p and the diffusion driven by $\hat{u}$ is simply X^R. In the second example, we construct a family of diffusions with common covariance and stable distribution, and show arbitrarily large growth rates are possible if one specifies to particular drifts within the family, in contrast to the (single) robust growth rate. The third example shows infinite robust growth is possible, and the fourth connects robust growth with explosion and ergodicity of the drift-less process $dX_t = \sigma(X_t)\,dW_t$. Indeed, strictly positive robust growth is possible for essentially all densities p should this diffusion explode, but if the diffusion is ergodic then robust growth is not possible.

Section 3 specifies to when X represents relative market capitalizations. Here there are two subtleties. First, trading in the assets is equivalent to trading in both the market portfolio and the relative capitalizations. Thus, by restricting trading to the relative capitalizations we are both assuming portfolios are fully invested in the market, and obtaining a wealth process which is not absolute, but rather relative to the market portfolio. As such, the robust maximization problem is to find an investment strategy in the ranked capitalizations which robustly maximizes the growth rate with respect to the market. Second, the observed phenomena is stability of the *ranked* relative capitalizations, not of the relative capitalizations themselves. However, trading does not happen in the ranked capitalizations, rather in the relative capitalizations and the market portfolio. Therefore, even though the natural inputs to the problem are the triple $(\Delta^{d-1}_{+,\leq}, \kappa, q)$, where $\Delta^{d-1}_{+,\leq}$ is the ordered unit simplex (the ranked relative capitalizations' state space), κ a covariation function, and q a density on $\Delta^{d-1}_{+,\leq}$; one must work on the unit simplex Δ^{d-1}_{+} (the relative capitalizations' state space), and use a covariation function c and density p defined on this region. To obtain c and p on Δ^{d-1}_{+}, we appropriately symmetrize (κ, q). In order to apply the abstract theory, we ask in Assumption 3.3 that such symmetrization preserves regularity in (c, p). However, as reinforced in Remark 3.4, we do not require that all models have limiting stable distribution p, where all rankings are equally likely; our only stability assumption is on the ranked process.

Under Assumption 3.3, Proposition 3.6 identifies the robust growth rate, as well as optimal strategy in the rank-based set-up. It also proves that optimal portfolios are functions solely of the ranked relative capitalizations, as one would expect. The section then closes with a useful result stating that one can start with an arbitrary pair (κ, q) on $\Delta^{d-1}_{+,\leq}$, which might not satisfy Assumption 3.3 (cf. [18, 21]), and then create a related pair which satisfies Assumption 3.3 by only modifying (κ, q) arbitrarily close near the boundary $\partial\Delta^{d-1}_{+,\leq}$. Thus, our results allow for general covariances and densities on an arbitrary open subsets of $\Delta^{d-1}_{+,\leq}$. The price of the modification is that optimal policies are combinations of the equally weighted and market portfolios near where relative capitalizations cross ranks. However, an advantage of this modification is that it rules out sudden portfolio changes on capitalization crossings, which in practice would be infeasible over a long horizon, due to transactions costs.

The paper is organized as follows: Section 1 outlines the model, heuristic arguments and main result in the abstract setting. Section 2 contain examples, while Section 3 specifies to the rank-based case. Appendix A contains the lengthy proof of the main abstract result, while Appendix B deals with proofs related to the rank-based model.

1. Problem set-up and main result.

1.1. *The problem.* There are three inputs to the problem: a region $E \subseteq \mathbb{R}^d$ where the underlying stochastic process X takes values; an instantaneous covariance function $c : E \mapsto \mathbb{S}^d_{++}$ for X; and a "limiting" probability density p for X. We make the following *standing assumptions* on (E, c, p).

ASSUMPTION 1.1. For some fixed constant $\gamma \in (0, 1]$:

(1) $E = \bigcup_{n=1}^{\infty} E_n$, where for each n, E_n is open, connected, bounded and has $C^{2,\gamma}$ boundary. Furthermore, $\bar{E}_n \subset E_{n+1}$ and E_1 is simply connected.
(2) $c \in C^{2,\gamma}(E, \mathbb{S}^d_{++})$.
(3) $p \in C^{2,\gamma}(E, (0, \infty))$ and $\int_E p = 1$.[1]

As in [20], we work on the canonical space of continuous functions $\Omega = C([0, \infty), E)$ equipped with its Borel sigma-algebra $\mathcal{F}$, where the topology is the one consistent with uniform convergence on compact subsets of $[0, \infty)$. The coordinate mapping process is denoted by X, and $\mathbb{F}$ is the right-continuous enlargement of the natural filtration generated by X. Allowable models are probability measures on $(\Omega, \mathcal{F})$ in the set Π as below.

DEFINITION 1.2. Π is the class of probability measures $\mathbb{P}$ on $(\Omega, \mathcal{F})$ such that:

(1) X is a $\mathbb{P}$-semimartingale with covariation process $[X, X] = \int_0^\cdot c(X_t)\,dt$, $\mathbb{P}$-a.s.
(2) For all Borel measurable functions h on E with $\int_E h^+ p < \infty$

$$\lim_{T\uparrow\infty} \frac{1}{T}\int_0^T h(X_t)\,dt = \int_E hp; \quad \mathbb{P}\text{-a.s.}$$

(3) The laws of $\{X_t; t \geq 0\}$ under $\mathbb{P}$ are tight.

[1] Throughout, all integrals over E or its subsets are with respect to Lebesgue measure.

Although condition (2) of Definition 1.2 above can be interpreted as p being a limiting density for X under $\mathbb{P} \in \Pi$, we stress that we do not ask for any Markovian or stationary structure from the probabilities in Π. In fact, while condition (1) of Definition 1.2 implies that the instantaneous covariation is a function of the current state of X, the drift of X under $\mathbb{P} \in \Pi$ can be quite general, as long as the tenets of Definition 1.2 are satisfied. For example, Π may contain laws such where X is a semi-martingale with finite variation component singular (in time) to Lebesgue measure.

We think of X as an underlying process related to tradeable entities. For example, X may be the stochastic logarithm of the discounted stock price in a local stochastic volatility model [10], where the drift is unknown and the local volatility a function of X. Here, it is natural to take $E = \mathbb{R}^d$, and identify a proportion of wealth trading strategy which, robustly, yields the largest discounted growth rate. Alternatively, and this is the primary example we have in mind, we can let X denote the relative market capitalizations for a group of stocks, in a model with unknown drift, but stable ranked capitalization curve. Here, $E = \Delta_+^{d-1}$ and Section 3 considers the robust growth optimization problem in detail.

In terms of trading, we use the following strategies.

DEFINITION 1.3. Θ is the class of predictable process that are X-integrable with respect to every $\mathbb{P} \in \Pi$, where Π is from Definition 1.2.

For a process $\vartheta \in \Theta$ and measure $\mathbb{P} \in \Pi$, we set

$$V^{\vartheta} := \mathcal{E}\left(\int_0^{\cdot} \vartheta_t' \, dX_t\right), \tag{1.1}$$

where, for any continuous semi-martingale M, $\mathcal{E}(M)_\cdot = \exp(M_\cdot - (1/2)[M, M]_\cdot)$ is the stochastic exponential (cf. [24], Chapter II.8). Note that the version of V^{ϑ} may also depend on $\mathbb{P} \in \Pi$, but we do not explicitly mention this dependency above, as it will be clear in each case which probability in Π is considered. As discussed in detail in Section 3, the interpretation is that V^{ϑ} is the wealth process generated starting from unit initial capital, and investing a proportion $\vartheta_t^i X_t^i$ of current wealth in X^i at time $t \geq 0$, for all $i = 1, \dots, d$.

For $\vartheta \in \Theta$ and $\mathbb{P} \in \Pi$, define

$$G(V^{\vartheta}, \mathbb{P}) := \sup\left\{\gamma \in \mathbb{R} : \lim_{T \uparrow \infty} \mathbb{P}\left[\frac{1}{T} \log V_T^{\vartheta} \geq \gamma\right] = 1\right\}. \tag{1.2}$$

As such, $G(V^{\vartheta}, \mathbb{P})$ is the *long-run growth rate* (in probability) of the wealth generated by following the strategy ϑ, when security prices evolve according to the probability measure $\mathbb{P}$.

REMARK 1.4. In [20] we defined $G(V^{\vartheta}, \mathbb{P})$ by the formulas

$$\mathbb{P}\text{-}\liminf_{T \to \infty} \zeta_T := \operatorname*{ess\,sup}_{\mathbb{P}}\left\{\chi \text{ is } \mathcal{F} \text{ measurable} : \lim_{T \uparrow \infty} \mathbb{P}[\zeta_T \geq \chi] = 1\right\},$$

$$G(V^{\vartheta}, \mathbb{P}) := \sup\left\{\gamma \in \mathbb{R} : \mathbb{P}\text{-}\liminf_{T \to \infty}\left(\frac{1}{T} \log V_T^{\vartheta}\right) \geq \gamma, \mathbb{P}\text{-a.s.}\right\}.$$

However, as [20], Lemma 1.3, showed, the above definition coincides with the simpler (1.2), and this is what we use here. It should be noted that we are looking for the supremum of those γ for which the limit in (1.2) exists (and is equal to one), just as in the alternate definition where we take the essential supremum over those χ where the limit exists.

Our goal is to compute

$$\lambda := \sup_{\vartheta \in \Theta} \inf_{\mathbb{P} \in \Pi} G(V^{\vartheta}, \mathbb{P}), \tag{1.3}$$

and obtain a robust maximizing strategy $\hat{\vartheta} \in \Theta$.

1.2. *Heuristics.* We first provide a heuristic argument for how the optimal strategy and robust growth rate are obtained. To this end, set

$$L^c := \frac{1}{2}\sum_{i,j} c^{ij}\partial_{ij} = \frac{1}{2}\operatorname{Tr}(cD^2). \tag{1.4}$$

Note that L^c is the second order operator associated to the driftless diffusion with covariance function c. Furthermore, while a solution to the *generalized* martingale problem (cf. [23], Chapter 1) for L^c on E exists by Assumption 1.1, the solution may be exploding. Next, consider the class of functions

$$\mathcal{D} := \left\{ u \in C^2(E) \,\middle|\, u > 0, \int_E \left(\frac{L^c u}{u}\right)^+ p < \infty \right\}.$$

With this notation, and since p is fixed throughout, we define

$$I := -\inf_{u\in\mathcal{D}} \int_E \frac{L^c u}{u} p,$$

which is essentially the Donsker–Varadhan rate function from occupancy time large deviations (LDP) theory evaluated at p.[2] Now, let $u \in \mathcal{D}$, and set $\vartheta^u_\cdot = (\nabla \log u)(X_\cdot)$, which is a process belonging in Θ due to its path-continuity. Itô's formula implies for all $\mathbb{P} \in \Pi$ that

$$\frac{1}{T}\log(V_T^{\vartheta^u}) = \frac{1}{T}\log\left(\frac{u(X_T)}{u(X_0)}\right) - \frac{1}{T}\int_0^T \frac{L^c u}{u}(X_t)\,dt. \tag{1.5}$$

Note that the function u "generates" the portfolio ϑ^u and wealth process V^{ϑ^u}, and hence is closely related to the functionally generated portfolios from stochastic portfolio theory, described in [14]. Thus, under Assumption 1.1, we conclude $G(V^{\vartheta^u}, \mathbb{P}) = -\int_E (L^c u/u)p$. As this holds for all $u \in \mathcal{D}$ and $\mathbb{P} \in \Pi$, by (1.3) we obtain

$$\lambda \geq \sup_{u\in\mathcal{D}} \inf_{\mathbb{P}\in\Pi} G(V^{\vartheta^u}, \mathbb{P}) = \sup_{u\in\mathcal{D}} \left\{ -\int_E \frac{L^c u}{u} p \right\} = I. \tag{1.6}$$

For the upper bound, let σ denote the unique positive definite symmetric square root of c, and assume that for some $\hat{u} \in \mathcal{D}$ and $\hat{X}_0 \in E$, the diffusion with dynamics

$$d\hat{X}_t = (c\nabla \log \hat{u})(\hat{X}_t)\,dt + \sigma(\hat{X}_t)\,d\hat{W}_t, \tag{1.7}$$

is ergodic with invariant density p. This implies the probability measure $\hat{\mathbb{P}}$ induced by the law of $\hat{X}$ is in Π. The wealth process $V^{\hat{\vartheta}}$ obtained by $\hat{\vartheta}_\cdot = (\nabla \log \hat{u})(X_\cdot)$ is in Θ and is growth-optimal for the model $\hat{\mathbb{P}}$. Therefore, (1.3) gives

$$\lambda \leq \sup_{\vartheta\in\Theta} G(V^{\vartheta}, \hat{\mathbb{P}}) = G(V^{\hat{\vartheta}}, \hat{\mathbb{P}}) = -\int_E \frac{L^c \hat{u}}{\hat{u}} p \leq \sup_{u\in\mathcal{D}} \left\{ -\int_E \frac{L^c u}{u} p \right\} = I.$$

Note that, if the discussion of this paragraph is valid, then *a posteriori* $\hat{u}$ has to be a minimizer of $\mathcal{D} \ni u \mapsto \int_E (L^c u/u)p$. We also regard $\hat{\mathbb{P}}$ as a "worst-case" model, in the sense that the maximal growth achievable under $\hat{\mathbb{P}} \in \Pi$ is λ.

From the above discussion, we conjecture that $\lambda = I$. As $I \leq \lambda$ follows from (1.6), the difficulty is in establishing existence of a minimizer $\hat{u} \in \mathcal{D}$ of the mapping $\mathcal{D} \ni u \mapsto \int_E (L^c u/u)p$, and showing that the corresponding diffusion in (1.7) is ergodic with invariant measure p.

[2] We say "essentially" because the domain $\mathcal{D}$ slightly differs from that used to prove occupancy time LDP in, for example, [6], Chapter 4.

1.3. *The main result.* In order to carry out the plan outlined in Section 1.2, we must make additional assumptions on how (E, c, p) interact. To simplify the presentation, set

$$\ell := \nabla \log p + c^{-1}\,\mathsf{div}(c) \quad \text{where } \mathsf{div}(c)^i = \sum_j \partial_j c^{ij}, i = 1, \dots, d. \tag{1.8}$$

ASSUMPTION 1.5. The following hold:

(i) $\int_E \ell' c \ell p < \infty$.
(ii) $\int_E (\nabla \cdot (pc\ell))^+ < \infty$.
(iii) For the symmetric second order linear operator

$$L^R := \frac{1}{2}\nabla \cdot (c\nabla) + \frac{1}{2}(\nabla \log p)' c \nabla = \frac{1}{2}\mathsf{Tr}(cD^2) + \frac{1}{2}\ell' c \nabla,$$

a (nonexplosive) solution to the Martingale problem for L^R on E exists.

REMARK 1.6. Recall that σ denotes the unique positive definite symmetric square root of c. The diffusion X^R associated to L^R has dynamics

$$dX_t^R = \frac{1}{2}(c\ell)(X_t^R)\,dt + \sigma(X_t^R)\,dW_t. \tag{1.9}$$

For any Brownian motion W (on some probability space), Assumptions 1.1, 1.5(iii) imply that there exists a unique strong solution for any initial condition $X_0^R \in E$. Furthermore, as formally p is a candidate invariant density for X^R, Assumption 1.5(iii) also implies the seemingly stronger result that X^R is ergodic with invariant density p: see [23], Corollary 4.9.4. As for verification of Assumption 1.5(iii), the most common way to identify if a diffusion does not explode is to use Lyapunov functions: see [25], Chapter 10, and [23], Chapter 6.7, for sufficient conditions in the multi-dimensional case, and [23], Theorem 5.1.5, for necessary and sufficient conditions in one dimension.

Therefore, given (1.7), under Assumption 1.5(iii) we see that (1.9) is a worst case model if and only if $c^{-1}\,\mathsf{div}(c)$ is a gradient, in which case $X^R = \hat{X}$. Absent this, X^R is not $\hat{X}$. However, there is still a very good reason why we enforce Assumption 1.5(iii). As shown in [23], Theorem 6.6.2(ii), if Assumption 1.5(iii) fails, then there are *no* time-homogeneous diffusions whose laws are in Π. Therefore, *a fortiori*, the candidate for the "worst case" model of (1.7) will not belong to Π, making it impossible to prove Theorem 1.8 that follows. In fact, if Assumption 1.5(iii) fails to hold, it is not clear whether the class Π contains any elements whatsoever, and even if it did, it is also not clear if the robust problem is well-posed. To reinforce these points, Proposition 1.9 below will have more to reveal for the one-dimensional case.

REMARK 1.7. Neither of conditions (i) and (iii) in Assumption 1.5 implies the other. That (i) does not imply (iii) follows by letting $E = (0, 1)$, and $p(x) = c(x) = 1$. To show that (iii) does not imply (i), let $E = (0, \infty)$, and $p(x) = Be^{-Bx}$, $c(x) = \xi^2 x$ for $B, \xi > 0$. Then,

$$\int_E \ell' c \ell p = \int_0^\infty B\xi^2 x e^{-Bx}\left(\frac{1}{x} - B\right)^2 = \infty,$$

but X^R has dynamics $dX_t^R = (1/2)(\xi^2 - BX_t^R)\,dt + \xi\sqrt{X_t^R}\,dW_t$ and hence is nonexplosive from the well-known properties of the CIR process.

What follows is our main result, the proof of which is in Appendix A.

THEOREM 1.8. *Let Assumptions 1.1 and 1.5 hold. Then, there exists a unique (up to a multiplicative constant) $\hat{u} \in \mathcal{D}$ such that*

$$\hat{u} = \operatorname*{argmin}_{u \in \mathcal{D}} \int_E \frac{L^c u}{u} p. \tag{1.10}$$

Furthermore, it holds that

$$\lambda = I = \frac{1}{2} \int_E (\nabla \log \hat{u})' c (\nabla \log \hat{u}) p, \tag{1.11}$$

and the trading strategy

$$\hat{\vartheta}. = (\nabla \log \hat{u})(X.) \in \Theta \tag{1.12}$$

is such that $G(V^{\hat{\vartheta}}, \mathbb{P}) = \lambda$, for all $\mathbb{P} \in \Pi$.

1.4. *On Assumption 1.5*(iii). We elaborate here on the importance of Assumption 1.5(iii), already hinted in Remark 1.6, by investigating deeper the one-dimensional case.

PROPOSITION 1.9. *Assume that $d = 1$ and $E = (\alpha, \beta)$ for $-\infty \le \alpha < \beta \le \infty$. Let (c, p) satisfy Assumption 1.1 and Assumptions 1.5*(i) *and* (ii). *Then, if Assumption 1.5*(iii) *fails, it either holds that $\Pi = \varnothing$ or $\lambda = \infty$.*

In words, the conclusion of Proposition 1.9 is that, absent Assumption 1.5(iii), either there are no models in the robust class, or infinite robust growth is possible.

PROOF. Assume that $\Pi \neq \varnothing$. Let $x_0 \in (\alpha, \beta)$. From [23], Theorem 5.1.1, Assumption 1.5(iii) failing is equivalent to either $\int_\alpha^{x_0} 1/(pc) < \infty$ or $\int_{x_0}^\beta 1/(pc) < \infty$. We shall only consider the case where $\int_\alpha^\beta 1/(pc) < \infty$ as the other cases are similar. To this end, from (1.5) with $u = \sqrt{pc}$, it follows that

$$\frac{L^c u}{u} p = \frac{1}{2}(\ddot{pc}) - \frac{1}{4} \frac{(\dot{pc})^2}{4pc},$$

and hence Assumption 1.5(ii) implies $(L^c u/u)^+ \in L^1(E, p)$ so that $u \in \mathcal{D}$. Next, from (1.6), it is clear that if $(L^c u/u)^- \notin L^1(E, p)$ then $\lambda = \infty$. If $(L^c u/u)^- \in L^1(E, p)$, which, along with Assumption 1.5(ii), implies that $(L^c u/u) \in L^1(E, p)$, for $\varepsilon > 0$ consider the function

$$v_\epsilon(x) := \sqrt{\varepsilon + \int_\alpha^x \frac{1}{pc}}, \quad x \in (\alpha, \beta).$$

A straightforward calculation shows that

$$\frac{L^c(uv_\epsilon)}{uv_\epsilon} = \frac{L^c u}{u} + \frac{L^R v_\epsilon}{v_\epsilon} = \frac{L^c u}{u} - \frac{1}{8} \frac{c}{(pc)^2(\varepsilon + \int_\alpha^x (pc)^{-1})^2}.$$

It thus follows that $(L^c(uv_\epsilon)/(uv_\epsilon))^+ \in L^1(E, p)$ so that $uv_\epsilon \in \mathcal{D}$. Furthermore,

$$\begin{aligned}
-\int_\alpha^\beta \frac{L^c(uv_\epsilon)}{uv_\epsilon} p &= -\int_\alpha^\beta \frac{L^c u}{u} p + \frac{1}{8} \int_\alpha^\beta \frac{1}{pc(\varepsilon + \int_\alpha^x (pc)^{-1})^2} \\
&= -\int_\alpha^\beta \frac{L^c u}{u} p + \frac{1}{8} \left(-\frac{1}{\varepsilon + \int_\alpha^x (pc)^{-1}} \Bigg|_{x=\alpha}^{x=\beta} \right) \\
&= -\int_\alpha^\beta \frac{L^c u}{u} p - \frac{1}{8} \frac{1}{\varepsilon + \int_\alpha^\beta (pc)^{-1}} + \frac{1}{8\varepsilon}.
\end{aligned}$$

Thus, we see from (1.6) and $(L^c u/u) \in L^1(E, p)$ that

$$\lambda \geq \lim_{\varepsilon \downarrow 0}\left(- \int_\alpha^\beta \frac{L^c u}{u} p - \frac{1}{8}\frac{1}{\varepsilon + \int_\alpha^\beta (pc)^{-1}} + \frac{1}{8\varepsilon}\right) = \infty,$$

concluding the proof. □

1.5. *Sensitivity with respect to inputs*. The proof of Theorem 1.8 shows $\hat{u} = \exp(\hat{\phi}/2)$, where $\hat{\phi} : E \mapsto \mathbb{R}$ is the unique (up to an additive constant) function satisfying

$$\hat{\phi} = \operatorname*{argmin}_{C^2(E)} \int_E (\nabla\phi - \ell)' c (\nabla\phi - \ell) p. \tag{1.13}$$

The above quadratic variational problem is more convenient (in terms of theoretical properties such as existence/uniqueness/regularity) to solve[3] than identifying $\hat{u}$ via the right hand side of (1.10). It also allows one to establish continuity with respect to the problem inputs, as we now explain.

First (cf. Section 2.1 below), if $c^{-1}\,\mathsf{div}(c)$ is a gradient of some function $H \in C^{1,\gamma}(E, \mathbb{R})$ then (1.8) trivially implies $\hat{\phi} = \log(p) + H$ and continuity (with respect to (p, H)) of $\hat{\phi}$, $\nabla\hat{\phi}$ and $\lambda = (1/8)\int_E \nabla\hat{\phi}' c \nabla\hat{\phi} p$ can easily be checked.

Absent the gradient case, one may establish continuity using the following simple argument. Fix E and for $i = 1, 2$ let (E, c_i, p_i) satisfy Assumptions 1.1, 1.5 and denote by $\hat{\phi}_i$, the corresponding optimizer of (1.13). Assume furthermore that $\int_E \nabla\hat{\phi}_j' c_i \nabla\hat{\phi}_j p_i < \infty$.[4] This, in conjunction with the respective optimality of $\hat{\phi}_i$ implies (cf. Remark A.6 below)

$$0 = \int_E (\nabla\hat{\phi}_i - \ell_i)' c_i (\nabla\hat{\phi}_j - \nabla\hat{\phi}_i) p_i.$$

Using this, straightforward calculations yield the upper bound

$$\begin{aligned} &\int_E (\nabla\hat{\phi}_1 - \nabla\hat{\phi}_2)' c_1 (\nabla\hat{\phi}_1 - \nabla\hat{\phi}_2) p_1 \\ &\quad \leq \int_E \left(\ell_1 - \ell_2 + \left(\frac{p_2}{p_1} c_1^{-1} c_2 - 1\right)(\nabla\hat{\phi}_2 - \ell_2)\right)' \\ &\qquad \times c_1 \left(\ell_1 - \ell_2 + \left(\frac{p_2}{p_1} c_1^{-1} c_2 - 1\right)(\nabla\hat{\phi}_2 - \ell_2)\right) p_1. \end{aligned} \tag{1.14}$$

To see how (1.14) may be used to establish continuity, consider the following special case, motivated from stochastic portfolio theory, where the covariance matrix is fixed, and one wishes to account for deviations in the limiting density. Fix a triple (E, c, p) which satisfy Assumptions 1.1, 1.5, and take a sequence of functions $\{\upsilon_n\}$ such that

$$\lim_{n\to\infty} \sup_{x\in E} |\upsilon_n(x)| = 0; \qquad \lim_{n\to\infty} \int_E \nabla\upsilon_n' c \nabla\upsilon_n p = 0,$$

and such that, for each n, with $p_n = pe^{\upsilon_n}$ the triple (E, c, p_n) also satisfies Assumptions 1.1 and 1.5. Here, (1.14) and the optimality of $\hat{\phi}_n$ imply

$$\begin{aligned} &\int_E (\nabla\hat{\phi} - \nabla\hat{\phi}_n)' c (\nabla\hat{\phi} - \nabla\hat{\phi}_n) p \\ &\quad \leq 2\int_E \nabla\upsilon_n' c \nabla\upsilon_n p + 4\sup_E\left((e^{\upsilon_n} - 1)^2 e^{2|\upsilon_n|}\right)\left(\int_E \ell' c \ell p + \int_E \nabla\upsilon_n' c \nabla\upsilon_n p\right). \end{aligned}$$

[3]In fact, one solves the variational problem over $W^{1,2}_{\mathrm{Loc}}(E)$ the set of weakly differentiable functions ϕ with ϕ, $\nabla\phi$ locally square integrable, and then proves regularity of the minimizer.

[4]Throughout, $\{i, j\} = \{1, 2\}$ or $\{2, 1\}$, and $\{i\} = \{1\}$ or $\{2\}$.

Therefore, $\lim_{n\to\infty}\int_E(\nabla\hat{\phi}-\nabla\hat{\phi}_n)'c(\nabla\hat{\phi}-\nabla\hat{\phi}_n)p=0$, which along with $\lim_{n\to\infty}\int_E \ell_n' c \times \ell_n p_n = \int_E \ell' c\ell p$ implies continuity in the worst case growth rate. Last, for more general models, from (1.14) one sees the relevant quantities to control are $\int_E(\ell-\ell_n)'c(\ell-\ell_n)p$ and $\sup_E((pc)^{-1}p_n c_n - 1)$.

2. Examples.

2.1. *One-dimensional and "gradient" cases.* As mentioned above, assume c satisfies the special condition $c^{-1}\operatorname{div}(c)=\nabla H$, for some function $H\in C^{1,\gamma}(E,\mathbb{R})$. Note this always holds in the one-dimensional case with $H=\log(c)$. Here, $\hat{\phi}=\log(p)+H$ and hence $\hat{u}=\sqrt{pe^H}$. Furthermore, ergodicity under the candidate worst-case model holds directly by Assumption 1.5(iii), since in this case the reversing diffusion of Assumption 1.5(iii) is in fact the worst-case model. Expanding on the one dimensional case where $E=(\alpha,\beta)$ for $-\infty\le\alpha<\beta\le\infty$, we have $\hat{u}=\sqrt{pc}$, and using [23], Theorem 5.1.5, it follows that Assumptions 1.1 and 1.5 hold provided

- $\int_\alpha^\beta(\dot{pc})^2/pc<\infty$;
- for some $x_0\in E$ we have $\lim_{x\downarrow\alpha}\int_x^{x_0}(pc)^{-1}=\infty=\lim_{x\uparrow\beta}\int_{x_0}^x(pc)^{-1}$;
- $\int_\alpha^\beta(\ddot{pc})^+<\infty$.

Therefore, the great difficultly in establishing Theorem 1.8 lies in treating the multi-dimensional setting, absent the highly particular case when $c^{-1}\operatorname{div}(c)$ is a gradient.

2.2. *A multi-dimensional example*: *Langevin diffusion.* Consider the case where $E=\mathbb{R}^d$, $c\in\mathbb{S}^d_{++}$ is constant and $p=e^{-V}$ for a smooth function V such that $\int_E e^{-V}=1$. We assume that $|\nabla V|$ is of linear growth, parts (i), (ii) of Assumption 1.5 hold (part (iii) holds in view of the assumed linear growth) and

$$\lim_{n\uparrow\infty}\int_{\partial B_n}|V|e^{-V}|\nabla V'\nu|\,dS=0, \tag{2.1}$$

where B_n is the open ball of radius n, ν is an outward normal unit vector and dS denotes surface measure. Here, X^R is the Langevin diffusion $dX_t^R=-(1/2)c\nabla V(X_t^R)\,dt+\sigma\,dW_t$. As this setting falls into the gradient case of Example 2.1, Theorem 1.8 implies $\hat{u}=e^{-V/2}$, the robust strategy is $\hat{\vartheta}_\cdot=-\nabla V(X_\cdot)/2$ and

$$\lambda=\frac{1}{8}\int_{\mathbb{R}^d}\nabla V'c\nabla Ve^{-V}.$$

Now, let B be *any* antisymmetric matrix ($B+B'=0$) and consider the diffusion with dynamics $dX_t=(-(1/2)c+B)\nabla V(X_t)\,dt+\sigma\,dW_t$. Due to the linear growth assumption, X is nonexplosive and simple calculation shows that, should X be ergodic, p is the invariant density. Thus, as mentioned in Remark 1.6, X is in fact ergodic with invariant density p for all antisymmetric B. This demonstrates that, for $d\ge 2$, the class of time homogenous diffusions whose law is in Π is uncountably large. Next, for the nonrobust model with B fixed, the growth optimal portfolio is $\hat{\vartheta}_\cdot^B=(-1/2+c^{-1}B)\nabla V(X_\cdot)$. Straightforward calculations using (2.1) show the long run growth rate for this portfolio is

$$\lambda^B=\lambda+\frac{1}{2}\int_{\mathbb{R}^d}\nabla V'B'c^{-1}B\nabla Ve^{-V}.$$

As V cannot be identically constant, this growth rate can be made arbitrarily large varying over B. However, the robust growth rate of λ is achievable in every model, via the portfolio $\hat{\vartheta}$.

2.3. *Cox–Ingersoll–Ross model under uncertainty*: *Infinite robust growth*. Let $E = (0, \infty)$ and $c(x) = \xi^2 x$, $x \in (0, \infty)$, where $\xi > 0$. For $A > 1$ and $B > 0$, set

$$p(x) = \frac{B^A}{\Gamma(A)} x^{A-1} e^{-Bx}; \qquad \Gamma(A) = \int_0^\infty y^{A-1} e^{-y}\, dy.$$

Assumptions 1.1 and 1.5 hold, and a straightforward calculation using $\hat{u} = \sqrt{pc}$ (cf. Section 2.1) shows that $\lambda = \lambda(A, B) = \xi^2 AB/(8(A-1))$. In [20] we considered a wider class of models, where E and c are known, but no assumption is made regarding a limiting density p (not even whether such a density exists). In this example, as $\lim_{B\downarrow 0} \lambda(A, B) = 0$, there is no possibility to achieve strictly positive growth in the setting of [20]. However, once a density p (i.e., a choice of A, B) is specified, strictly positive growth is possible.

In fact, we now show infinite robust growth is possible when $A = 1$, in which case $p(x) = Be^{-Bx}$, $x > 0$. From Remark 1.7, we know Assumption 1.5(i) does not hold. However, Assumption 1.5(iii) does hold, and this implies that $\Pi \neq \varnothing$ (cf. [22] for verification of item (3) with unbounded functions). The candidate optimal trading strategy is $\vartheta^u = (\nabla u/u)(X.)$ for $u = \sqrt{pc}$ which specifies here to $u(x) = \xi\sqrt{Bx}\exp(-Bx/2)$, $x > 0$. A calculation shows that $L^c u/u = -(1/8)\xi^2(1/x + 2B - B^2 x)$, implying $(L^c u/u)^+ \in L^1(E, p)$ and hence $u \in \mathcal{D}$. But, it is clear $(L^c u/u)^- \notin L^1(E, p)$ and hence (1.5) implies that $G(V^{\vartheta^u}, \mathbb{P}) = \infty$ for all $\mathbb{P} \in \Pi$, and hence $\lambda = \infty$.

2.4. *Explosion*, *ergodicity and robust growth*. If the diffusion with dynamics $dX_t = \sigma(X_t)\, dW_t$, associated to L^c from (1.4), explodes in finite time, then robust growth is achievable for all densities p such that Assumptions 1.1, 1.5 hold. Indeed, it follows from [19], Lemma 3.5, and [17], Lemma 33, that for all such p

$$0 < -\inf_{u\in\tilde{\mathcal{D}}} \int_E \frac{L^c u}{u} p \leq -\inf_{u\in\mathcal{D}} \int_E \frac{L^c u}{u} p = I = \lambda.$$

Here, $\tilde{D} \subset \mathcal{D}$ contains those $u \in C^2(E)$, $u > 0$ such that $L^c u/u$ is bounded from above.

Note also that, if the diffusion with dynamics $dX_t = \sigma(X_t)\, dW_t$ is positive recurrent with invariant density p then positive robust growth is not possible. Indeed, we may take $\hat{u} \equiv 1$ in (1.7), and as such, the trading strategy $\hat{\vartheta} = \nabla\hat{u}/\hat{u} \equiv 0$ achieves maximal growth under $\hat{\mathbb{P}} \in \Pi$. No trading trivially leads to $G(V^{\hat{\vartheta}}, \hat{\mathbb{P}}) = 0$; hence, $\lambda = 0$. For example, this situation occurs when $E = \mathbb{R}$, c is any positive smooth function such that $\int_{\mathbb{R}}(1/c) = 1$ and $p = 1/c$ (cf. [23], Chapter 5).

3. An application in ranked-based models.

3.1. *Relative market capitalizations*. To motivate the results of this section, start with a collection $S = (S^i; i = 1, \ldots, d)$ of processes representing market capitalizations of d stocks, and set $M = \sum_{i=1}^d S^i$ as the total capitalization. With $X^i := S^i/M$, $i = 1, \ldots, d$, the process $X = (X^i; i = 1, \ldots, d)$ denotes *relative* market capitalizations. We assume that no stock capitalization vanishes, so that X takes values in the open simplex

$$\Delta_+^{d-1} := \left\{ x \in \mathbb{R}^d \,\middle|\, \min_{i=1,\ldots,d} x^i > 0, \sum_{i=1}^d x^i = 1 \right\}. \tag{3.1}$$

As already noted in the Introduction, wealth from investment (as well as growth rates) will *not* be absolute, but rather relative to market capitalization. In fact, investment is defined with respect to the relative capitalizations X, and not with respect to the original prices S, through the usual *change-of-numéraire* technique. As direct calculations show, for any d-dimensional predictable process π with $\sum_{i=1}^{d} \pi^i = 1$,

$$\frac{dU_t}{U_t} = \sum_{i=1}^{d} \pi_t^i \frac{dS_t^i}{S_t^i} \quad \Longleftrightarrow \quad \frac{d(U/M)_t}{(U/M)_t} = \sum_{i=1}^{d} \pi_t^i \frac{dX_t^i}{X_t^i}.$$

To wit, if a strategy of portfolio weights $\pi = (\pi^i; i = 1, \dots, d)$ is fully invested in stocks, the same strategy applied to the relative capitalizations results in wealth relative to the total market capitalization. Note also that, as the vector-valued process X is degenerate (the sum of its components equals one), there is no loss of generality in assuming the strategies ϑ resulting in (1.1) are such that $\sum_{i=1}^{d} X^i \vartheta^i = 1$, where to connect with the above, we have set $\vartheta^i = \pi^i / X^i$. Indeed, for any predictable strategy ϑ, if one defines a strategy η via $\eta^i = \vartheta^i + (1 - \sum_{j=1}^{d} X^j \vartheta^j)$ for $i = 1, \dots, d$, then we have $\sum_{i=1}^{d} X^i \eta^i = 1$, and

$$\sum_{i=1}^{d} \eta_t^i \, dX_t^i = \sum_{i=1}^{d} \vartheta_t^i \, dX_t^i + \left(1 - \sum_{j=1}^{d} X^j \vartheta_t^j\right) \sum_{i=1}^{d} dX_t^i = \sum_{i=1}^{d} \vartheta_t^i \, dX_t^i,$$

implying that $V^{\vartheta} = V^{\eta}$.

3.2. *Ranked capitalizations*. As has been observed in [14], Chapter 5, empirical time-series data suggest that the capital distribution curve (i.e., the log-log plot of ranked relative capitalizations versus rank, in decreasing order) is stable for U.S. equities. This leads to the introduction of so-called *ranked based* models for financial markets. Here, we shall not go into the details of ranked based models; for a thorough treatment, see [14], Chapters 4, 5. Rather, we introduce assumptions on the ranked capitalizations, as opposed to the actual capitalizations, and consider questions of robust growth.

Define the ordered simplex

$$\Delta_{+,\leq}^{d-1} := \{x \in \Delta_+^{d-1} \mid x^1 \leq x^2 \leq \cdots \leq x^d\}. \tag{3.2}$$

For $x \in \Delta_+^{d-1}$, we write $x^{()} = (x^{(1)}, \dots, x^{(d)})$ for the corresponding ordered point in $\Delta_{+,\leq}^{d-1}$; also, for $i = 1, \dots, d$, let $r(x^i)$ denote the rank of x^i among $x^1, \dots, x^d$, with ties resolved in lexicographic order.

As aforementioned, it is natural to assume the vector $X^{()} = (X^{(i)}; i = 1, \dots, d)$ of ranked relative capitalizations, which takes values in $\Delta_{+,\leq}^{d-1}$, is stable in the long run. Thus, we shall take as inputs a pair (κ, q) where $\kappa : \Delta_{+,\leq}^{d-1} \mapsto \mathbb{S}_{++}^d$ and $q : \Delta_{+,\leq}^{d-1} \mapsto (0, \infty)$ with $\int_{\Delta_{+,\leq}^{d-1}} q = 1$. Similar to Definition 1.2, we consider the class of measures $\Pi_{\leq}$ on $C([0, \infty), \Delta_+^{d-1})$, equipped with the Borel σ-algebra $\mathcal{F}$, such that for $\mathbb{P} \in \Pi_{\leq}$:

(1) X is a $\mathbb{P}$-semimartingale and, for $i = 1, \dots, d$ and $j = 1, \dots, d$:

$$[X^i, X^j] = \int_0^{\cdot} \kappa^{r(X_t^i) r(X_t^j)}(X_t^{()}) \, dt; \quad \mathbb{P}\text{-a.s.} \tag{3.3}$$

(2) For all Borel measurable functions h on $\Delta_{+,\leq}^{d-1}$ with $\int_{\Delta_{+,\leq}^{d-1}} h^+ q < \infty$, it holds that

$$\lim_{T \uparrow \infty} \frac{1}{T} \int_0^T h(X_t^{()}) \, dt = \int_{\Delta_{+,\leq}^{d-1}} hq; \quad \mathbb{P}\text{-a.s.}$$

(3) The laws of $\{X_t; t \geq 0\}$ under $\mathbb{P}$ are tight, where compact sets are those compactly contained within Δ_+^{d-1}.

REMARK 3.1. We pause here to discuss two issues arising from the degeneracy of Δ_+^{d-1}. First, note that we can identify Δ_+^{d-1} with a region $E \subseteq \mathbb{R}^{d-1}$ which satisfies Assumption 1.1(1), by replacing x^d with $1 - x^1 - \cdots - x^{d-1}$. However, since Δ_+^{d-1} is a flat $(d-1)$-dimensional manifold, and for ease of notation, we will not work with E, preferring to work directly with Δ_+^{d-1}.

Second, in view of the $(d-1)$-dimensionality of Δ_+^{d-1}, one has to appropriately understand our assumption that $\kappa(x^{()}) \in \mathbb{S}_{++}^d$ for $x^{()} \in \Delta_{+,\leq}^{d-1}$. In fact, $z'\kappa(x^{()})z \geq \ell(x^{()})z'z$ for some Borel function $\ell : \Delta_{+,\leq}^{d-1} \mapsto (0, \infty)$ need not hold on the whole of $\mathbb{R}^d$, but rather on the $(d-1)$-dimensional subspace

$$\left\{ z \in \mathbb{R}^d \,\Big|\, \sum_{i=1}^{d} z_i = 0 \right\},$$

which (up to an affine translation) is tangent to every point $x \in \Delta_+^{d-1}$. Despite the clear abuse of notation, we prefer to write $\kappa(x) \in \mathbb{S}_{++}^d$, understanding that it need only hold on the tangent space to Δ_+^{d-1}.

REMARK 3.2. Regarding (3.3) above, it may seem more natural to require

$$[X^{()}, X^{()}]_\cdot = \int_0^\cdot \kappa(X_t^{()})\, dt. \tag{3.4}$$

Indeed, as can be deduced from [3], Theorem 2.3, (3.4) implies X has instantaneous covariations

$$\frac{d[X^i, X^j]_t}{dt} = \kappa^{r(X_t^i) r(X_t^j)}(X_t^{()}); \quad i = 1, \ldots, d, j = 1, \ldots, d, t \geq 0,$$

when $X^{()}$ is in the interior of $\Delta_{+,\leq}^{d-1}$. However, without additional assumptions (e.g., almost surely zero Lebesgue measure for ranks coinciding), one cannot assume that κ is nondegenerate, or even recover $(d[X, X]_t/dt; t \geq 0)$ from κ on the boundary of $\Delta_{+,\leq}^{d-1}$. For this reason, we define $\Pi_\leq$ using (3.3) rather than (3.4). Morally, we regard the two definitions as equivalent.

3.3. *Growth in rank-based models*. In accordance to Definition 1.3, let $\Theta_\leq$ be the class of predictable processes ϑ that are X-integrable with respect to every $\mathbb{P} \in \Pi_\leq$. The growth rate $G(V^\vartheta, \mathbb{P})$ for $\vartheta \in \Theta_\leq$ is defined as in (1.2), and we set

$$\lambda_\leq := \sup_{\vartheta \in \Theta_\leq} \inf_{\mathbb{P} \in \Pi_\leq} G(V^\vartheta, \mathbb{P}). \tag{3.5}$$

We wish to use the results of Section 1 in the current setting. Of course, one really invests in the relative capitalizations X, and not in its ranked counterpart $X^{()}$; therefore, a transformation of the inputs (κ, q) from $\Delta_{+,\leq}^{d-1}$ to (c, p) in Δ_+^{d-1} is in order. Additionally, we must ensure that (c, p) satisfy the regularity and integrability requirements of Assumptions 1.1 and 1.5.

In view of (3.3), we first naturally extend κ from $\Delta_{+,\leq}^{d-1}$ to Δ_+^{d-1} by defining

$$c(x) = \{c^{ij}(x)\}_{i,j=1}^d; \qquad c^{ij}(x) := \kappa^{r(x^i) r(x^j)}(x^{()}); \quad x \in \Delta_+^{d-1}. \tag{3.6}$$

The extension of q from $\Delta_{+,\leq}^{d-1}$ to Δ_+^{d-1} is performed in a "symmetric" way, by defining

$$p(x) := \frac{1}{d!} q(x^{()}); \quad x \in \Delta_+^{d-1}. \tag{3.7}$$

We make the following assumption (see Remark 3.1 regarding part (2) of Assumption 1.1).

ASSUMPTION 3.3. The pair (κ, q) is such that, with $E = \Delta_+^{d-1}$, c as in (3.6) and p as in (3.7), Assumptions 1.1 and 1.5 are satisfied.

REMARK 3.4. We stress that the passage from (κ, q) to (c, p) is made only in order to use the results from Section 1 under the validity of Assumption 3.3. In particular, we do *not* require that models have limiting stable distribution p for X, where all rankings are equally likely, each with probability $1/d!$. Note, however, that the worst-case model will have this structure.

REMARK 3.5. Consider the model of [21] and [18], Section 3, where the market capitalizations $S = (S^1, \dots, S^d)$ are given by $S^i = e^{Y^i}$ where $dY_t^i = \sum_{j=1}^d 1_{Y_t^i = Y_t^{(j)}}(\delta_j\, dt + \sigma\, dW_t^i)$. In other words, the ith log capitalization is driven by its own Brownian motion, but the drift depends upon its relative rank amongst all the log capitalizations. With $X^i = S^i/(S^1 + \cdots + S^d)$ as the relative capitalization, Ito's formula implies $d[X^i, X^j]_t = c(X_t)^{ij}\, dt$ where

$$c^{ij}(x) = \sigma^2 x^i x^j \big(\delta_{ij} - x^i - x^j + x'x\big).$$

Calculation verifies (3.6) with

$$\kappa^{pq}\big(x^{()}\big) := \sigma^2 x^{(p)} x^{(q)} \big(\delta_{pq} - x^{(p)} - x^{(q)} + \big(x^{()}\big)' x^{()}\big).$$

Now, c is locally elliptic (it can only degenerate when $\sum_{i=1}^d x_i = 1$—see Remark 3.1), and smoothness is evident. Therefore, κ satisfies Assumption 3.3. However, the invariant measure q of [18] does not satisfy Assumption 3.3 as the associated p is not smooth on the rank-switching boundaries. Indeed, in the two-dimensional case, one may identify $\Delta_{+,\le}^{d-1}$ with $(0, 1/2)$, and in this region q is the distribution of $1/(1 + e^Y)$ where Y is exponentially distributed with some parameter $\lambda > 0$. This implies $q(x) = \lambda x^{\lambda-1}(1 - x)^{-\lambda-1}$, and since $\dot{q}(1/2) = 16\lambda^2 \neq 0$ it follows that p is not differentiable at $x = 1/2$. However, as $1/2$ is the only problematic point, we may smooth p at $1/2$ (cf. Proposition 3.8 below) to apply our results.

Under the force of Assumption 3.3 on (κ, q), build (c, p) as in (3.6) and (3.7), respectively, as well as Π, Θ according to Definitions 1.2 and 1.3. It is immediate that

$$\Pi \subseteq \Pi_\le; \qquad \Theta_\le \subseteq \Theta.$$

Thus, we always have

$$\lambda_\le = \sup_{\vartheta \in \Theta_\le} \inf_{\mathbb{P} \in \Pi_\le} G\big(V^\vartheta, \mathbb{P}\big) \le \sup_{\vartheta \in \Theta} \inf_{\mathbb{P} \in \Pi} G\big(V^\vartheta, \mathbb{P}\big) = \lambda. \tag{3.8}$$

From Theorem 1.8, we know that for $\hat{u}$ solving (1.10) and $\hat{\vartheta}$ defined in (1.12), we have (1.11) holding for all $\mathbb{P} \in \Pi$. The following result, the proof of which can be found in Appendix B, implies that the portfolio generating function $\hat{u}$ is permutation invariant, giving also that $\lambda_\le = \lambda$.

PROPOSITION 3.6. *For the pair (κ, q), let Assumption 3.3 hold. For the associated (c, p) constructed above, let $\hat{u}$ be as in (1.10), from Theorem 1.8. Then, $\hat{u}(x) = \hat{u}(x^{()})$ for all $x \in \Delta_+^{d-1}$. Furthermore,*

$$\lambda_\le = \frac{1}{2} \int_{\Delta_{+,\le}^{d-1}} q(\nabla \log \hat{u})' \kappa (\nabla \log \hat{u}) = \frac{1}{2} \int_{\Delta_+^{d-1}} p(\nabla \log \hat{u})' c (\nabla \log \hat{u}) = \lambda, \tag{3.9}$$

and the trading strategy $\hat{\vartheta}_\cdot = (\nabla \log \hat{u})(X_\cdot) \in \Theta_\le$ is such that

$$G\big(V^{\hat{\vartheta}}, \mathbb{P}\big) = \lambda_\le \quad \forall \mathbb{P} \in \Pi_\le. \tag{3.10}$$

REMARK 3.7. The importance of $\hat{u}$ being a function of the ranked weights is that it implies the optimal strategy is rank-generated, in the sense of [14], Section 4.2. Because $\hat{u}$ is permutation-invariant and twice continuously differentiable, the local time terms in [14], Theorem 4.2.1, vanish. In effect, we have $\{X^i = X^j\} \subseteq \{\hat{\vartheta}^i = \hat{\vartheta}^j\}$ for all $i = 1, \dots, d$ and $j = 1, \dots, d$ in our result, which is a desirable feature. Indeed, if optimal positions were different when the ranks of two stocks are the same, then, upon collisions of ranked market capitalizations, one would need to change large positions with very high frequency. Not only is this practically infeasible, it also would lead to unsustainable transaction costs (which, admittedly, we do not model here).

We close this section with an important observation. It is not hard to see that Assumption 3.3 only concerns (κ, q) near $\partial\Delta^{d-1}_{+,\le}$. However, and especially in view of X^R being nonexplosive in Assumption 1.5(iii), it is natural to wonder if there is *ever* a way to modify an arbitrary (κ, q) near $\partial\Delta^{d-1}_{+,\le}$ so that Assumption 3.3 holds. To this end we have the following result.

PROPOSITION 3.8. *Let* $\kappa : \Delta^{d-1}_{+,\le} \mapsto \mathbb{S}^d_{++}$ *and* $q : \Delta^{d-1}_{+,\le} \mapsto (0, \infty)$ *be such that for all open subsets* $V \subset \Delta^{d-1}_{+,\le}$ *we have* $\kappa \in C^{2,\gamma}(\bar{V}, \mathbb{S}^d_{++})$ *and* $q \in C^{1,\gamma}(\bar{V}, (0, \infty))$. *Then, for any open subset* $V \subset \Delta^{d-1}_{+,\le}$ *there are* (κ_V, q_V) *such that*:

(i) q_V *is strictly positive in* $\Delta^{d-1}_{+,\le}$ *with* $\int_{\Delta^{d-1}_{+,\le}} q_V(x)\,dx = 1$;
(ii) $\kappa = \kappa_V$, $q = q_V$ *on* V;
(iii) (κ_V, q_V) *satisfy Assumption* 3.3.

In fact, (κ_V, q_V) in Proposition 3.8 admit explicit formulas in (B.13). Upon inspection of Lemma B.1, the modified pair (κ_V, q_V) is such that, for some constant $K > 0$, the optimizer $\hat{u}(x) = (\prod_{i=1}^d x^i)^K$ for $x^{()}$ lying near $\partial\Delta^{d-1}_{+,\le}$. This leads to an optimal strategy $\hat{\vartheta}$ such that $\hat{\vartheta}^i = K/X^i + (1 - Kd)$, $i = 1, \dots, d$, when $X^{()}$ is near $\partial\Delta^{d-1}_{+,\le}$. Qualitatively, the investor holds a combination of the market and equally weighted portfolios. This, of course, is entirely consistent with the set inclusion $\{X^i = X^j\} \subseteq \{\hat{\vartheta}^i = \hat{\vartheta}^j\}$ for $i, j = 1, \dots, d$ of Remark 3.7.

APPENDIX A: PROOF OF THEOREM 1.8

We first provide a brief road-map on how Theorem 1.8 is proved, starting with the variational problem (1.10). Consider when $u = e^{(1/2)\phi}$ for $\phi \in C^\infty_c(E)$. Clearly, $u \in \mathcal{D}$, and from (1.5) we deduce that for all $\mathbb{P} \in \Pi$

$$G(V^{\vartheta^u}, \mathbb{P}) = -\frac{1}{8}\int_E \left(\nabla\phi' c \nabla\phi + 2\,\mathrm{Tr}(cD^2\phi)\right)p.$$

As $\phi \in C^\infty_c(E)$, integration-by-parts yields

$$\begin{aligned} G(V^{\vartheta^u}, \mathbb{P}) &= -\frac{1}{8}\int_E \left(\nabla\phi' c\nabla\phi - 2\nabla\phi' c\left(\frac{\nabla p}{p} + c^{-1}\,\mathrm{div}(c)\right)\right)p \\ &= \frac{1}{8}\int_E \ell' c \ell p - \frac{1}{8}\int_E (\nabla\phi - \ell)' c (\nabla\phi - \ell) p, \end{aligned}$$

where ℓ is from (1.8). Thus, we conjecture that $\hat{u}$ in (1.10) is found by solving

$$\inf_\phi \int_E (\nabla\phi - \ell)' c (\nabla\phi - \ell) p, \tag{A.1}$$

and setting $\hat{u} = e^{(1/2)\hat{\phi}}$ if a minimizer exists. Of course, since we actually need the minimizer, we cannot take the infimum over $C^\infty_c(E)$. Instead we use $W^{1,2}_{\mathrm{Loc}}(E)$, the space of weakly differentiable functions ϕ so that ϕ^2, $|\nabla\phi|^2$ are locally integrable. The first result we shall provide, Lemma A.1 in Section A.1, identifies a unique (up to an additive constant) minimizer $\hat{\phi} \in W^{1,2}_{\mathrm{Loc}}(E)$, which is in fact twice continuously differentiable with Hölder second-order derivative.

Given a minimizer $\hat{\phi}$, the first order condition for optimality in the minimization problem of (A.1) suggests that

$$\nabla \cdot \big(pc(\nabla\hat{\phi} - \ell)\big) = 0. \tag{A.2}$$

This is indeed shown to hold in Lemma A.1. Therefore, [23], Corollary 4.9.4, implies that, *if* $\hat{X}$ from (1.7) does not explode, it cannot be transient. Thus, [23], Theorem 2.8.1, implies $\hat{X}$ is recurrent, hence ergodic with invariant measure p in light of (A.2) and $\int_E p < \infty$. Therefore, the second result, Lemma A.2 in Section A.2, will establish the fact that $\hat{X}$ from (1.7) does *not* explode.

Given the previous two auxiliary results, Section A.3 will conclude the proof of Theorem 1.8.

A.1. The variational problem. We first consider the minimization problem in (A.1) and obtain the following result.

LEMMA A.1. *Let Assumption* 1.1 *and Assumption* 1.5(i) *hold, and recall* ℓ *from* (1.8). *Then, there exists a unique* (*up to an additive constant*) $\hat{\phi} \in W^{1,2}_{\mathrm{Loc}}(E)$ *which solves*

$$\inf_{\phi \in W^{1,2}_{\mathrm{Loc}}(E)} \int_E (\nabla\phi - \ell)' c (\nabla\phi - \ell) p. \tag{A.3}$$

Furthermore, $\hat{\phi} \in C^{2,\gamma'}(E)$ *for some* $0 < \gamma' \le \gamma$ *and satisfies the second order linear elliptic equation*

$$\nabla \cdot \big(pc(\nabla\hat{\phi} - \ell)\big) = 0; \quad x \in E. \tag{A.4}$$

PROOF. To make the notation cleaner set

$$J(\phi) := \int_E (\nabla\phi - \ell)' c (\nabla\phi - \ell) p, \tag{A.5}$$

so that (A.3) becomes $\hat{J} := \inf_{\phi \in W^{1,2}_{\mathrm{Loc}}(E)} J(\phi)$. Note that Assumption 1.5(i) gives $\hat{J} < \infty$. In what follows K will be a constant which changes from line to line. Also, where appropriate, K_n will be a constant which depends only upon E_n and the model coefficients on E_n.

Let $\{\phi_m\}_{m\in\mathbb{N}} \subset W^{1,2}_{\mathrm{Loc}}(E)$ be such that $\lim_{m\uparrow\infty} J(\phi_m) = \hat{J}$. Assumption 1.5(i) and the Cauchy–Schwarz inequality then imply

$$\sup_m \int_E \nabla\phi_m' c \nabla\phi_m p \le K,$$

and hence for all n

$$\sup_m \int_{E_n} \nabla\phi_m' c \nabla\phi_m p \le K.$$

Next, since $p \ge c_n > 0$ and $c \ge \lambda_n > 0$ on E_n we have that

$$\sup_m \int_{E_n} \nabla\phi_m' \nabla\phi_m \le K_n. \tag{A.6}$$

Denote by

$$\psi_m^n := \phi_m - \fint_{E_n} \phi_m,$$

as ϕ_m less its average over E_n. From the classical Poincaré inequality [11], Chapter 5.8, and (A.6) it follows that

$$\sup_m \int_{E_n} \left((\psi_m^n)^2 + (\nabla\psi_m^n)'\nabla\psi_m^n\right) \le K_n.$$

The Rellich–Kondrachov theorem [11], Chapter 5.7, and the fact that $\nabla\psi_m^n$ is norm bounded in $L^2(E_n, \mathbb{R}^d)$ imply the existence of $\eta^n \in W^{1,2}(E_n)$ such that for some subsequence $m(n)$:

$$\psi_{m(n)}^n \to \eta^n \quad \text{strongly in } L^2(E_n),$$
$$\nabla\psi_{m(n)}^n \to \nabla\eta^n \quad \text{weakly in } L^2(E_n, \mathbb{R}^d).$$

Thus, by [2], Theorem 13.1.1, which shows that

$$L^2(E_n, \mathbb{R}^d) \ni v \mapsto \int_{E_n} (v-\ell)'c(v-\ell)p, \tag{A.7}$$

is weakly lower-semicontinuous it follows that

$$\int_{E_n} (\nabla\eta^n - \ell)'c(\nabla\eta^n - \ell)p \le \liminf_{m(n)\to\infty} \int_{E_n} (\nabla\phi_m - \ell)'c(\nabla\phi_m - \ell)p. \tag{A.8}$$

Now, fix $n < n'$. There exists a common subsequence $m(n, n')$ such that

$$\phi_{m(n,n')} - \oint_{E_n} \phi_{m(n,n')} \to \eta^n; \quad \text{s-}L^2(E_n), \text{w-}W^{1,2}(E_n),$$
$$\phi_{m(n,n')} - \oint_{E_{n'}} \phi_{m(n,n')} \to \eta^{n'}; \quad \text{s-}L^2(E_{n'}), \text{w-}W^{1,2}(E_{n'})$$

(we have used "s" and "w" to denote strong and weak convergence). We now claim that $\nabla\eta^n = \nabla\eta^{n'}$ a.e. in E_n. Indeed, we have for all $v \in L^2(E_n; \mathbb{R}^d)$ that

$$\int_{E_n} (\nabla\eta^n - \nabla\eta^{n'})'v = \lim_{m(n,n')\to\infty} \int_{E_n} (\nabla\phi_{m(n,n')} - \nabla\phi_{m(n,n')})'v = 0,$$

upon which the result follows by taking $v = \nabla\eta^n - \nabla\eta^{n'}$. Thus, since E_n is connected we know [11], Chapter 5, that for some constant $C(n, n')$

$$\begin{aligned} \eta^{n'} &= \eta^n + C(n, n'); \quad \text{a.e. in } E_n, \\ \nabla\eta^{n'} &= \nabla\eta^n; \quad \text{a.e. in } E_n. \end{aligned} \tag{A.9}$$

Now, using the double-subsequence trick we can find a single subsequence (which we will label m) such that the above convergences holds for all $n \in \mathbb{N}$. For this subsequence (and the resultant η^n) define (for a.e. $x \in E$) v by

$$v(x) := \nabla\eta^n(x); \quad x \in E_n; n = 1, 2, \ldots. \tag{A.10}$$

Note that v is well defined: indeed we have

$$v(x) = \nabla\eta^1(x) = \nabla\eta^2(x) = \cdots; \quad x \in E_1,$$
$$v(x) = \nabla\eta^2(x) = \nabla\eta^3(x) = \cdots; \quad x \in E_2,$$
$$\vdots$$

Next, define (for a.e. $x \in E$) η by

$$\eta(x) := \eta^n(x) - \sum_{k=1}^{n-1} C(k, k+1); \quad x \in E_n; n = 1, 2, \ldots. \tag{A.11}$$

Again, η is well defined. This follows because for any $n = 1, 2, \dots$ and $q = 0, 1, 2, \dots$ we have on E_n that

$$\begin{aligned}\eta^{n+q}(x) - \sum_{k=1}^{n+q-1} C(k, k+1) &= \eta^{n+q-1}(x) + C(n+q-1, n+q) - \sum_{k=1}^{n+q-1} C(k, k+1),\\ &= \eta^{n+1-q}(x) - \sum_{k=1}^{n+q-2} C(k, k+1),\\ &\vdots\\ &= \eta^n(x) - \sum_{k=1}^{n-1} C(k, k+1).\end{aligned}$$

We now claim that $\eta \in W^{1,2}_{\mathrm{Loc}}(E)$. First $\nabla\eta = v$. To see this, let $\theta \in C_c^\infty(E)$ and choose n so that $\theta \in C_c^\infty(E_n)$. For $i \in 1, \dots, d$ write D^i as the derivative with respect to x_i. We have

$$\begin{aligned}\int_E \eta D^i\theta = \int_{E_n} \eta D^i\theta &= \int_{E_n} \left(\eta^n - \sum_{k=1}^{n-1} C(k, k+1)\right) D^i\theta\\ &= -\int_{E_n} D^i\eta^n\theta = -\int_{E_n} v^i\theta.\end{aligned}$$

Given that $\nabla\eta = v$ the fact that $\eta \in W^{1,2}_{\mathrm{Loc}}(E)$ is immediate. From (A.8) we thus have for each n that

$$\begin{aligned}\int_{E_n} (\nabla\eta - \ell)'c(\nabla\eta - \ell)p &\le \liminf_{m\to\infty} \int_{E_n} (\nabla\phi_m - \ell)'c(\nabla\phi_m - \ell)p,\\ &\le \liminf_{m\to\infty} \int_E (\nabla\phi_m - \ell)'c(\nabla\phi_m - \ell)p = \hat{J}.\end{aligned}$$

Taking $n \uparrow \infty$ and using the monotone convergence theorem we see that

$$\int_E (\nabla\eta - \ell)'c(\nabla\eta - \ell)p \le \hat{J},$$

and hence $\hat{\phi} := \eta$ minimizes J over $W^{1,2}_{\mathrm{Loc}}(E)$. The uniqueness up to an additive constant follows by the strict convexity of $(\nabla\phi - \ell)'c(\nabla\phi - \ell)p$ in $\nabla\phi$.

We now turn to the regularity for $\hat{\phi}$ which essentially is a standard argument and hence just a broad overview is given. Let $\theta \in C_c^1(E_n) \subset W^{1,2}_{\mathrm{Loc}}(E)$. By varying J at $\hat{\phi} \pm \varepsilon\theta$ and taking $\varepsilon \downarrow 0$ we see that

$$0 = \int_{E_n} \nabla\theta' c(\nabla\hat{\phi} - \ell)p. \tag{A.12}$$

It thus follows [16], Chapter 8, page 178, that $u = \hat{\phi}$ is a weak solution of the PDE

$$\begin{aligned}\nabla \cdot (pc\nabla u - pc\ell) = 0; &\quad x \in E_n,\\ u = \hat{\phi}; &\quad x \in \partial E_n.\end{aligned} \tag{A.13}$$

Here, the boundary condition is interpreted to mean that $u - \hat{\phi} \in W_0^{1,2}(E_n)$. Under the given regularity and ellipticity assumptions in E_n it follows by [16], Theorem 8.22, that $u = \hat{\phi}$ is locally Holder continuous in E_n for some exponent $0 < \gamma' \le \gamma$. Next, consider the problem of finding classical solutions to the same PDE but in E_{n-1}: that is,

$$\nabla \cdot (pc\nabla u - pc\ell) = 0; \quad x \in E_{n-1};$$
$$u = \hat{\phi}; \quad x \in \partial E_{n-1}.$$

Since $\hat{\phi}$ is Holder continuous in $\bar{E}_{n-1}$ it follows from [16], Theorem 6.13, that there is a unique solution $u \in C^{2,\gamma'}(E_{n-1}) \cap C(\bar{E}_{n-1})$ to the above PDE. But, this means that u is a weak solution to the above PDE as well. By the uniqueness of weak solutions (which also holds in the current set—see [16], Theorem 8.3, and the fact that $\hat{\phi}$ is also a weak solution in E_{n-1}) it follows that $\hat{\phi} = u$ a.e. in E_{n-1}. Since we already know $\hat{\phi}$ is Holder continuous, this in fact proves that $\hat{\phi} \in C^{2,\gamma}(E_{n-1})$ and solves the differential expression in (A.4) in E_{n-1}. Since this works for any n the result follows. □

A.2. An ergodic diffusion. Having established existence of minimizer $\hat{\phi}$ we now consider the diffusion as in (1.7) with $\hat{u} = e^{(1/2)\hat{\phi}}$: that is,

$$d\hat{X}_t = c\frac{\nabla \hat{u}}{\hat{u}}(\hat{X}_t)\,dt + \sigma(\hat{X}_t)\,d\hat{W}_t = \frac{1}{2}c\nabla\hat{\phi}(\hat{X}_t)\,dt + \sigma(\hat{X}_t)\,dW_t. \tag{A.14}$$

Our goal is to show $\hat{X}$ is ergodic with invariant measure p. More precisely, we let $\hat{\mathbb{P}} = (\hat{\mathbb{P}}^x)_{x\in E}$ denote the solution to the generalized martingale problem for the second order linear operator $\hat{L}$ associated to $\hat{X}$ on E: that is,

$$\hat{L} := \frac{1}{2}\mathsf{Tr}\big(cD^2\big) + \frac{1}{2}\nabla\hat{\phi}'c\nabla. \tag{A.15}$$

We then have the following which proves $\hat{\mathbb{P}}^x \in \Pi$ for all $x \in E$.

LEMMA A.2. *Let Assumption 1.1 and Assumptions 1.5*(i) *and* (iii) *hold. Let $\hat{\phi}$ be as in Lemma A.1. Set $\hat{\mathbb{P}}$ as the solution to the generalized martingale problem for the operator $\hat{L}$ in* (A.15). *Then $\hat{\mathbb{P}}$ solves the martingale problem for $\hat{L}$ and in fact, $\hat{\mathbb{P}}^x \in \Pi$ for all $x \in E$.*

The rest of this subsection is devoted to the proof of Lemma A.2. We retain the notation of (1.8), (A.5). Since $\hat{\phi} \in C^{2,\gamma'}(E)$ and solves (A.4) it follows that

$$\nabla \cdot \left(p\left(\frac{1}{2}c\nabla\hat{\phi}\right)\right) = \frac{1}{2}\nabla \cdot \big(c\nabla p + p\,\mathsf{div}(c)\big),$$

and hence p is an invariant density for $\hat{X}$. Since $\int_E p = 1$ it will follow that $\mathbb{P}^x \in \Pi$ for all $x \in E$ if it can be shown that X is recurrent in E; see [23], Theorem 4.9.5. To this end, we use the results of [23], Section 6.6, which provide necessary and sufficient conditions for $\hat{X}$ to be recurrent in the current setup.

We first state a consequence of Assumption 1.5(iii). Denote by $E_n - E_1 := E_n \setminus \bar{E}_1^c$ and L^R the second order linear elliptic operator associated to the diffusion X^R from Assumption 1.5(iii): that is, in divergence form

$$L^R = \frac{1}{2}\nabla \cdot (c\nabla) + \frac{1}{2}\frac{\nabla p'}{p}c\nabla. \tag{A.16}$$

Since X^R is assumed ergodic with invariant density p and reversing by construction, it follows from [23], Theorem 6.4.1, that

$$\lim_{n\uparrow\infty}\frac{1}{2}\int_{E_n - E_1}(\nabla u_n)'c\nabla u_n p = 0, \tag{A.17}$$

where $u_n \in C^{2,\gamma}(E_n - E_1)$ is the unique (strictly positive in $E_n - E_1$) solution to

$$L^R u = 0, \quad x \in E_n - E_1; \qquad u = 1, \quad x \in \partial E_1; \qquad u = 0, \quad x \in \partial E_n. \tag{A.18}$$

In fact, one has

$$u_n(x) = \mathbb{P}_x^R[\tau_{E_1} < \tau_{E_n}], \tag{A.19}$$

where $\{\mathbb{P}_x^R\}_{x\in E}$ is the solution to the Martingale problem for L^R on E and τ_{E_i} is the first hitting time to ∂E_i. Note that this implies $0 \le u^R \le 1$. To show that X is recurrent we use the following result, as can be found in [23], Theorem 6.6.1.

THEOREM A.3. *Let Assumptions 1.1—1.5(i) hold and let $\hat{\phi}$ be as in Lemma A.1. Let $\hat{L}$ be operator associated to $\hat{X}$ in (A.14). For each n define the convex sets*

$$\begin{aligned} A_n &:= \{g \in W^{1,2}(E_n - E_1) : g = \sqrt{p} \text{ on } \partial E_1, g = 0 \text{ on } \partial E_n, \\ &\qquad \operatorname{dist}(x, \partial E_n)^{-1} g(x) \in L^\infty(E_n - E_1)\}; \\ B_n &:= \left\{h \in W^{1,2}(E_n - E_1, g^2) : h = \frac{1}{2}\log(p) \text{ on } \partial E_1\right\}. \end{aligned}$$

Now, define

$$\begin{aligned} \mu_n &:= \inf_{g\in A_n} \sup_{h\in B_n} \frac{1}{2}\int_{E_n-E_1} g^2\left(\frac{\nabla g}{g} - \frac{1}{2}(\nabla\hat{\phi} - c^{-1}\operatorname{div}(c))\right)' c\left(\frac{\nabla g}{g} - \frac{1}{2}(\nabla\hat{\phi} - c^{-1}\operatorname{div}(c))\right) \\ &\quad - \frac{1}{2}\int_{E_n-E_1} g^2\left(\nabla h - \frac{1}{2}(\nabla\hat{\phi} - c^{-1}\operatorname{div}(c))\right)' c\left(\nabla h - \frac{1}{2}(\nabla\hat{\phi} - c^{-1}\operatorname{div}(c))\right). \end{aligned}$$

Then, $\hat{L}$ is recurrent if and only if $\lim_{n\uparrow\infty}\mu_n = 0$.

REMARK A.4. Above, $W^{1,2}(E_n - E_1, g^2)$ is the space of weakly differentiable functions h satisfying

$$\int_{E_n-E_1} g^2(h^2 + \nabla h'\nabla h) < \infty.$$

Also, the boundary conditions are interpreted to hold in the trace sense. Lastly, as shown right above [23], Theorem 6.6.1, μ_n takes the simpler form

$$\mu_n = \frac{1}{2}\int_{E_n-E_1} \frac{\tilde{v}_n}{v_n}\nabla v_n' c \nabla v_n, \tag{A.20}$$

where v_n solves $\hat{L}v = 0$ in $E_n - \bar{E}_1$ with $v = 0$ on ∂E_n and $v = 1$ on ∂E_1; and, with $\tilde{L}$ denoting the formal adjoint to $\hat{L}$, where $\tilde{v}_n$ solves $\tilde{L}\tilde{v} = 0$ on $E_n - \bar{E}_1$ with $\tilde{v} = 0$ on ∂E_n and 1 on ∂E_1. Thus, if $\{\hat{\mathbb{P}}^x\}_{x\in E}$ denotes the solution to the generalized Martingale problem for $\hat{L}$ on E then $v_n(x) = \hat{\mathbb{P}}^x[\tau_{E_1} < \tau_{E_n}]$ where τ_{E_i} is the first hitting time of ∂E_i. As solutions to the generalized martingale problem remain in a cemetery state upon explosion, $v_n(x) \uparrow v_\infty(x) = \hat{\mathbb{P}}^x[\tau_{E_1} < \infty]$. Furthermore, since one can show $\tilde{v}_n$ is locally uniformly bounded from below, the convergence of v_n to v_∞ is such that if $\mu_n \to 0$ then $v_\infty \equiv 1$, and [23], Theorems 2.8.1, 2.8.2, imply $\hat{L}$ is recurrent, hence positive recurrent because p is an invariant probability density. If $\mu_n \nrightarrow 0$ then we cannot conclude $v_\infty \equiv 1$, and in fact (this is the difficult part in proving Theorem A.3) $\hat{L}$ is transient.

PROOF OF LEMMA A.2. (A.20) implies $\mu_n \geq 0$ and hence $\liminf_{n\uparrow\infty} \mu_n \geq 0$. Assume by way of contradiction that $\limsup_{n\uparrow\infty} \mu_n > 0$. Thus, for some sub-sequence (still labelled n) and for some $\delta > 0$ we have $\mu_n \geq \delta$ for all n. Taking $g := u_n\sqrt{p}$ (which by the global Schauder estimates for u_n is in A_n: see [23], Theorem 3.2.8) one obtains

$$\begin{aligned}
\delta &\leq \frac{1}{8}\int_{E_n-E_1} u_n^2\Big(\nabla\hat{\phi} - \Big(\frac{\nabla p}{p} + c^{-1}\operatorname{div}(c)\Big) - 2\frac{\nabla u_n}{u_n}\Big)' \\
&\quad \times c\Big(\nabla\hat{\phi} - \Big(\frac{\nabla p}{p} + c^{-1}\operatorname{div}(c)\Big) - 2\frac{\nabla u_n}{u_n}\Big)p \\
&\quad - \frac{1}{2}\inf_{h\in B_n}\int_{E_n-E_1} u_n^2\Big(\nabla h - \frac{1}{2}(\nabla\hat{\phi} - c^{-1}\operatorname{div}(c))\Big)' c\Big(\nabla h - \frac{1}{2}(\nabla\hat{\phi} - c^{-1}\operatorname{div}(c))\Big)p \\
&= \frac{1}{8}\int_{E_n-E_1} u_n^2\Big(\nabla\hat{\phi} - \ell - 2\frac{\nabla u_n}{u_n}\Big)' c\Big(\nabla\hat{\phi} - \ell - 2\frac{\nabla u_n}{u_n}\Big)p \\
&\quad - \frac{1}{2}\inf_{h\in B_n}\int_{E_n-E_1} u_n^2\Big(\nabla h - \frac{1}{2}(\nabla\hat{\phi} - c^{-1}\operatorname{div}(c))\Big)' c\Big(\nabla h - \frac{1}{2}(\nabla\hat{\phi} - c^{-1}\operatorname{div}(c))\Big)p,
\end{aligned}$$

where we have used the definition of ℓ in (A.5). Next, for $h \in B_n$ define $\phi := \log(p) + \hat{\phi} - 2h$. Under the given regularity assumptions on $p, \hat{\phi}$ we have by the linearity of the trace operator that

$$h \in B_n \quad \Leftrightarrow \quad \phi \in B_n' := \{\phi \in W^{1,2}(E_n - E_1, pu_n^2) : \phi = \hat{\phi} \text{ on } \partial E_1\}.$$

The change of variables $h = (1/2)(\log(p) + \hat{\phi} - \phi)$ and simple algebra in the previous inequality gives

$$\begin{aligned}
&\inf_{\phi\in B_n'}\int_{E_n-E_1} u_n^2(\nabla\phi - \ell)'c(\nabla\phi - \ell)p \\
&\quad \leq \int_{E_n-E_1} u_n^2\Big(\nabla\hat{\phi} - \ell - 2\frac{\nabla u_n}{u_n}\Big)' c\Big(\nabla\hat{\phi} - \ell - 2\frac{\nabla u_n}{u_n}\Big)p - 8\delta.
\end{aligned}$$

Since $\hat{\phi} \in W^{1,2}_{\mathrm{Loc}}(E)$ and, according to Lemma A.1 satisfies $\int_E p\nabla\hat{\phi}'c\nabla\hat{\phi} < \infty$, by (A.17) we know that

$$\lim_{n\uparrow\infty}\int_{E_n-E_1} u_n^2\Big(\nabla\hat{\phi} - \ell - 2\frac{\nabla u_n}{u_n}\Big)' c\Big(\nabla\hat{\phi} - \ell - 2\frac{\nabla u_n}{u_n}\Big)p = \int_{E-E_1}(\nabla\hat{\phi} - \ell)'c(\nabla\hat{\phi} - \ell)p.$$

Thus, for n large enough we have

$$\inf_{\phi\in B_n'}\int_{E_n-E_1} u_n^2(\nabla\phi - \ell)'c(\nabla\phi - \ell)p \leq \int_{E-E_1}(\nabla\hat{\phi} - \ell)'c(\nabla\hat{\phi} - \ell)p - 4\delta.$$

Now, by Assumption 1.5(i) it follows that $\int_{E_n-E_1}\ell' c\ell p < \infty$. Thus, as shown in [23], Theorem 6.5.1, page 264, there exists an a.e. unique (up to an additive constant) solution $\phi_n \in W^{1,2}(E_n - E_1, pu_n^2)$ to the minimization problem above. Indeed, to connect with the proof therein take $g = u_n\sqrt{p}$, $f = \hat{\phi}$, $\phi = 1$ on ∂E_1 and $\phi = 0$ on ∂E_n and lastly $a = c$, $b = \ell$. Therefore, we have

$$\int_{E_n-E_1} u_n^2(\nabla\phi_n - \ell)'c(\nabla\phi_n - \ell)p \leq \int_{E-E_1}(\nabla\hat{\phi} - \ell)'c(\nabla\hat{\phi} - \ell)p - 4\delta.$$

Next, extend u_n to all of E_n by setting $u_n = 1$ on E_1. It is well known (see [2], Proposition 5.1.1) that since $u_n = 1$ on ∂E_1 that this extension is in $W^{1,2}(E_n)$ (in fact, it is continuous, though not continuously differentiable because of the Hopf maximum principle).

Similarly, for $\phi \in B_n'$ we have $\phi = \hat{\phi}$ on ∂E_1 and hence we may extend ϕ to E_n by setting $\phi = \hat{\phi}$ in E_1 and it still holds that $\ell \in W^{1,2}(E_n, pu_n^2)$. This gives for n large enough, say $n \geq N_0(\delta)$ that

$$\int_{E_n} u_n^2(\nabla\phi_n - \ell)'c(\nabla\phi_n - \ell)p \leq \int_E (\nabla\hat{\phi} - \ell)'c(\nabla\hat{\phi} - \ell)p - 4\delta = J(\hat{\phi}) - 4\delta, \tag{A.21}$$

where we recall the definition of J in (A.5). We now use (A.21) to derive a contradiction to Lemma A.1. To do this, fix an integer m. Since $Lu_n = 0$, Harnack's inequality, $u_n \leq 1$ (which follows by the probabilistic representation for u_n in (A.19)) and $u_n(x) = 1$ on E_1 yields the existence of a constant $c_m > 0$ so that $u_n(x)^2 \geq c_m$ on E_m for all $n \geq m + 1$.[5] We thus have for $n \geq N_0(\delta) \vee (m + 1)$ that

$$\begin{aligned} J(\hat{\phi}) - 4\delta &\geq \int_{E_n} u_n^2(\nabla\phi_n - \ell)'c(\nabla\phi_n - \ell)p \\ &\geq \int_{E_m} u_n^2(\nabla\phi_n - \ell)'c(\nabla\phi_n - \ell)p \\ &\geq c_m \int_{E_m} (\nabla\phi_n - \ell)'c(\nabla\phi_n - \ell)p \\ &\geq c_m \int_{E_m} (\nabla\phi_n - \ell)'(\nabla\phi_n - \ell), \end{aligned} \tag{A.22}$$

where c_m has changed to the last line, taking into account that $p, c \geq c_m > 0$ on E_m. Thus, from the Cauchy–Schwarz inequality we have

$$\sup_{n \geq N_0(\delta)\vee(m+1)} \int_{E_m} \nabla\phi_n' \nabla\phi_n \leq K_m.$$

Copying the argument below (A.6) in Lemma A.1 (note the roles of m and n have reversed) there exists a function $\eta^m \in W^{1,2}(E_m)$ so that, for some subsequence $n(m)$

$$\begin{aligned} \phi_n - \fint_{E_m} \phi_n \to \eta^m; &\quad \text{s-}L^2(E_m); \\ \nabla\phi_n \to \nabla\eta^m; &\quad \text{w-}L^2\big(E_m; \mathbb{R}^d\big). \end{aligned}$$

Furthermore, if $m < m'$ then by taking a common subsequence $\eta^{m'} = \eta^m + C(m, m')$ and $\nabla\eta^{m'} = \nabla\eta^m$ almost everywhere in E_m. In fact, there exists a single subsequence labelled n such that the convergence holds for all m on this subsequence and we can construct a function $\eta \in W^{1,2}_{\mathrm{Loc}}(E)$, exactly as in Lemma A.1, so that $\nabla\eta = \nabla\eta^m$ on E_m for each m.

Now, come back to (A.21). For the common subsequence $\{\phi_n\}_{n\in\mathbb{N}}$ where all the convergences take place, for each m we have for $n \geq m$ that (similar to (A.22))

$$J(\hat{\phi}) - 4\delta \geq \int_{E_m} u_n^2(\nabla\phi_n - \ell)'c(\nabla\phi_n - \ell)p \geq \inf_{E_m} u_n^2 \int_{E_m} (\nabla\phi_n - \ell)'c(\nabla\phi_n - \ell)p. \tag{A.23}$$

We now claim that for each m

$$\lim_{n\uparrow\infty} \inf_{E_m} u_n^2 = 1. \tag{A.24}$$

First of all, for $x \in \bar{E}_1$ we have $u_n(x) = 1$ by construction. Second, in $E_m - E_1$ we have, since $L^R u_n = 0$, $u_n = 1$ on ∂E_1, and $u_n \leq 1$ on $E_n - \bar{E}_1$, from the global Schauder estimates [23],

[5] Technically, Harnack's inequality holds in $E_m - E_2$ where u_n is smooth. The extension to all of E_m follows since by the extension, $u_n = 1$ on $\bar{E}_1$ and since u_n is larger in $E_2 - E_1$ than in $E_m - E_2$, as the probabilistic representation shows.

Theorem 3.2.8, there is a constant K_m so that $\sup_{n\geq m}\|u_n\|_{2,\gamma,E_m}\leq K_m$, where $\|\cdot\|_{2,\gamma,E_m}$ is the $C^{2,\gamma}$ Hölder norm on E_n. Now, assume there is some subsequence (still labelled n) so that $\lim_{n\uparrow\infty}\inf_{E_m} u_n^2 = 1-\varepsilon$ for some $\varepsilon>0$. By the Schauder estimates, $\{u_n\}_{n\in\mathbb{N}}$ is pre-compact in the $\|\cdot\|_{2,\gamma,E_m}$ norm and there is a further subsequence (still labelled n) and a function $u_\infty\in C^{2,\gamma}(E_m)$ so that $\|u_n-u_\infty\|_{2,\gamma,E_m}\to 0$. But, from Assumption 1.5(iii) and [23], Theorem 6.4.1, we a priori know that $u_n(x)\to 1$ so that $u_\infty(x)=1$. But this gives

$$0=\lim_{n\uparrow\infty}\sup_{E_m}\left|u_n(x)-1\right|=0,$$

which contradicts the fact that $\lim_{n\uparrow\infty}\inf_{E_m}u_n(x)=1-\varepsilon$. Thus, (A.24) holds.

Now, come back to (A.23). In view of (A.24) and the lower-semicontinuity of the operator in (A.7) (with n there-in equal to m here) it follows that

$$\begin{aligned}J(\hat\phi)-4\delta &\geq \liminf_{n\uparrow\infty}\inf_{E_m}u_n^2\int_{E_m}(\nabla\phi_n-\ell)'c(\nabla\phi_n-\ell)p\\ &\geq \int_{E_m}(\nabla\eta^m-\ell)'c(\nabla\eta^m-\ell)p\\ &=\int_{E_m}(\nabla\eta-\ell)'c(\nabla\eta-\ell)p.\end{aligned}$$

Taking $m\uparrow\infty$ yields

$$J(\hat\phi)-4\delta\geq\int_E(\nabla\eta-\ell)'c(\nabla\eta-\ell)p,$$

contradicting Lemma A.1. Thus, it cannot be that $\limsup_{n\uparrow\infty}\mu_n>0$ and hence $\lim_{n\uparrow\infty}\mu_n=0$, proving the recurrence of $\hat X$. □

A.3. Proof of Theorem 1.8. Before proving Theorem 1.8 we state one equality and prove one technical fact. For the equality, let $\hat\phi$ be from Lemma A.1. In light of (A.4) we obtain

$$\begin{aligned}\frac{L^c\hat u}{\hat u}&=\frac14\operatorname{Tr}(cD^2\hat\phi)+\frac18\nabla\hat\phi'c\nabla\hat\phi\\ &=\frac{1}{4p}\nabla\cdot(pc\ell)-\frac14\nabla\hat\phi'c\ell+\frac18\nabla\hat\phi'c\nabla\hat\phi\\ &=\frac{1}{4p}\nabla\cdot(pc\ell)+\frac18(\nabla\hat\phi-\ell)'c(\nabla\hat\phi-\ell)-\frac18\ell'c\ell.\end{aligned}\tag{A.25}$$

As for the technical fact, we have

LEMMA A.5. *Let Assumptions* 1.1–1.5(iii) *hold. Then* $\int_E\nabla\cdot(pc\ell)=0$.

PROOF OF LEMMA A.5. If $\nabla\cdot(pc\ell)=0$ for all $x\in E$ then clearly the result holds. Else, let $\hat\phi$ be from Lemmas A.1, A.2 and note that (A.2) and $\int_E\nabla\cdot(pc\ell)\neq 0$ imply $\hat\phi$ is not identically constant, and hence $\int_E p\nabla\hat\phi'c\nabla\hat\phi>0$. Recalling X^R from Assumption 1.5(iii) and using (A.25):

$$\begin{aligned}\frac1T\hat\phi(X_T^R)&=\frac1T\hat\phi(X_0^R)+\frac{1}{2T}\int_0^T\nabla\hat\phi'c\ell(X_t^R)\,dt\\ &\quad+\frac1T\int_0^T\nabla\hat\phi'\sigma(X_t^R)\,dW_t+\frac{1}{2T}\int_0^T\operatorname{Tr}(cD^2\hat\phi)(X_t^R)\,dt\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{T}\hat{\phi}(X_0^R) + \frac{1}{2T}\int_0^T \frac{1}{p}\nabla\cdot(pc\ell)(X_t^R)\,dt + \frac{1}{T}\int_0^T \nabla\hat{\phi}'\sigma(X_t^R)\,dW_t \\
&= \frac{1}{T}\hat{\phi}(X_0^R) + \frac{1}{2T}\int_0^T \frac{1}{p}\nabla\cdot(pc\ell)(X_t^R)\,dt \\
&\quad + \frac{1}{T}\int_0^T \nabla\hat{\phi}'c\nabla\hat{\phi}(X_t^R)\,dt\left(\frac{\int_0^T \nabla\hat{\phi}'\sigma(X_t^R)\,dW_t}{\int_0^T \nabla\hat{\phi}'c\nabla\hat{\phi}(X_t^R)\,dt}\right).
\end{aligned}
$$

Since $(\nabla\cdot(pc\ell))^+ \in L^1(E,\text{leb})$, $\int_E p\nabla\hat{\phi}'c\nabla\hat{\phi} > 0$, the Dambis–Dubins–Schwarz theorem and strong law for Brownian motion imply almost surely:

$$
\lim_{T\uparrow\infty}\frac{1}{T}\hat{\phi}(X_T^R) = \frac{1}{2}\int_E \nabla\cdot(pc\ell).
$$

If the right hand side above were not zero, it would contradict the positive recurrence of X^R. □

PROOF OF THEOREM 1.8. From (1.6) we see that

$$
\lambda \ge I = -\inf_{u\in\mathcal{D}}\int_E \frac{L^c u}{u}p \ge -\int_E \frac{L^c\hat{u}}{\hat{u}}p. \tag{A.26}
$$

For now, assume $\hat{u} = e^{(1/2)\hat{\phi}} \in \mathcal{D}$. By Lemma A.2, the diffusion $\hat{X}$ from (1.7) is ergodic, and hence the associated $\hat{\mathbb{P}} \in \Pi$. As $V^{\hat{\vartheta}}$ enjoys the numéraire property under $\hat{\mathbb{P}}$, we know

$$
\lambda \le -\int_E \frac{L^c\hat{u}}{\hat{u}}p \le -\inf_{u\in D}\int_E \frac{L^c u}{u}p = I,
$$

which in conjunction with (A.26) establishes the first equality in (1.11), provided that $\hat{u}\in\mathcal{D}$. To show this latter fact, recall (A.25). Since $(\nabla\cdot(pc\ell))^+ \in L^1(E,\text{leb})$, Lemma A.1 implies $(L^c\hat{u}/\hat{u})^+ \in L^1(E,p)$, and hence $\hat{u}\in\mathcal{D}$.

It remains to prove the second equality in (1.11) as well as that $G(V^{\hat{\vartheta}},\mathbb{P}) = \lambda$ for all $\mathbb{P}\in\Pi$. From (A.25) and Lemma A.5 we see that

$$
\int_E \frac{L^c\hat{u}}{\hat{u}}p = \frac{1}{8}\int_E(\nabla\hat{\phi}-\ell)'c(\nabla\hat{\phi}-\ell)p - \frac{1}{8}\int_E \ell'c\ell p. \tag{A.27}
$$

Next, if $\nabla\cdot(pc\ell) = 0$ for $x\in E$ then $\hat{\phi}$ is constant and clearly, (A.27) implies the second equality in (1.11). Else, $\nabla\hat{\phi}$ is not identically 0, and $\int_E \nabla\hat{\phi}'c\nabla\hat{\phi}p > 0$. Continuing, note that for $\hat{X}$ as in (A.14) we have, using (A.25)

$$
\begin{aligned}
\hat{\phi}(\hat{X}_T) &= \hat{\phi}(\hat{X}_0) + \frac{1}{2}\int_0^T \nabla\hat{\phi}'c\nabla\hat{\phi}(\hat{X}_t)\,dt + \int_0^T \nabla\hat{\phi}'\sigma(\hat{X}_u)\,dW_u \\
&\quad + \frac{1}{2}\int_0^T \operatorname{Tr}(cD^2\hat{\phi})(\hat{X}_t)\,dt \\
&= \hat{\phi}(\hat{X}_0) + \frac{1}{2}\int_0^T \nabla\hat{\phi}'c\nabla\hat{\phi}(\hat{X}_t)\,dt + \int_0^T \nabla\hat{\phi}'\sigma(\hat{X}_u)\,dW_u \\
&\quad + \frac{1}{2}\int_0^T\left(\frac{1}{p}\nabla\cdot(pc\ell) - \nabla\hat{\phi}'c\ell\right)(\hat{X}_t)\,dt \\
&= \hat{\phi}(\hat{X}_0)
\end{aligned}
$$

$$+\int_0^T \left(\frac{1}{4}\nabla\hat{\phi}'c\nabla\hat{\phi} - \frac{1}{4}\ell'c\ell + \frac{1}{4}(\nabla\hat{\phi}-\ell)'c(\nabla\hat{\phi}-\ell) + \frac{1}{2p}\nabla\cdot(pc\ell)\right)(\hat{X}_u)\,du$$
$$+\int_0^T \nabla\hat{\phi}'\sigma(\hat{X}_t)\,dW_t.$$

So, we see by the Strong law for Brownian motion, the Dambis–Dubins–Schwarz theorem and the given assumptions, we have almost surely

$$\begin{aligned}\lim_{T\uparrow\infty}\frac{1}{T}\hat{\phi}(\hat{X}_T) &= \frac{1}{4}\int_E\big(\nabla\hat{\phi}'c\nabla\hat{\phi} - \ell'c\ell + (\nabla\hat{\phi}-\ell)'c(\nabla\hat{\phi}-\ell)\big)p + \frac{1}{2}\int_E \nabla\cdot(pc\ell)\\ &= \frac{1}{4}\int_E\big(\nabla\hat{\phi}'c\nabla\hat{\phi} - \ell'c\ell + (\nabla\hat{\phi}-\ell)'c(\nabla\hat{\phi}-\ell)\big)p,\end{aligned}$$

where the last inequality follows by Lemma A.5. Now, if the right hand side above was not zero it would violate the positive recurrence of $\hat{X}$. This gives

$$\int_E \nabla\hat{\phi}'c\nabla\hat{\phi}p = \int_E \ell'c\ell p - \int_E(\nabla\hat{\phi}-\ell)'c(\nabla\hat{\phi}-\ell)p \tag{A.28}$$

which, in view of (A.27), establishes the second equality in (1.11).

REMARK A.6. Note that (A.28) implies $\int_E \nabla\hat{\phi}'c\nabla\hat{\phi}p = \int_E \nabla\hat{\phi}'c\ell p$.

The last thing to show is $G(V^{\hat{\vartheta}}, \mathbb{P}) = \lambda$ for all $\mathbb{P}\in\Pi$. To this end, by Itô's formula and (A.25) we know

$$\begin{aligned}\frac{1}{T}\log V_T^{\hat{\phi}} &= \frac{1}{T}\log V_0 + \frac{1}{2T}\hat{\phi}(X_T) - \frac{1}{2T}\hat{\phi}(X_0)\\ &\quad - \frac{1}{8T}\int_0^T\big(2\operatorname{Tr}(CD^2\hat{\phi}) + \nabla\hat{\phi}'c\nabla\hat{\phi}\big)(X_t)\,dt\\ &= \frac{1}{T}\log V_0 + \frac{1}{2T}\hat{\phi}(X_T) - \frac{1}{2T}\hat{\phi}(X_0) - \frac{1}{4T}\int_0^T\frac{1}{p}\nabla\cdot(pc\ell)(X_t)\,dt\\ &\quad + \frac{1}{8T}\int_0^T \ell'c\ell(X_t)\,dt - \frac{1}{8T}\int_0^T(\nabla\hat{\phi}-\ell)'c(\nabla\hat{\phi}-\ell)(X_t)\,dt.\end{aligned} \tag{A.29}$$

Taking $T\uparrow\infty$ gives

$$\begin{aligned}G(V^{\hat{\phi}}, \mathbb{P}) &= -\frac{1}{4}\int_E\nabla\cdot(pc\ell) + \frac{1}{8}\int_E\ell'c\ell p - \frac{1}{8}\int_E(\nabla\hat{\phi}-\ell)'c(\nabla\hat{\phi}-\ell)p\\ &= \frac{1}{8}\int_E\ell'c\ell p - \frac{1}{8}\int_E(\nabla\hat{\phi}-\ell)'c(\nabla\hat{\phi}-\ell)p\\ &= \lambda,\end{aligned}$$

where the second equality came from Lemma A.5, and the third from (1.11), (A.28). This finishes the proof. □

APPENDIX B: PROOFS FROM SECTION 3

We keep all notation from Section 3. Additionally, we denote $\mathcal{T}$ as the set of all permutations τ of $\{1,\dots,d\}$. For $\tau\in\mathcal{T}$, and $x\in\Delta_+^{d-1}$, we define $x_\tau\in\Delta_+^{d-1}$ by $x_\tau^i = x(\tau^i)$, $i=1,\dots,d$.

B.1. Proof of Proposition 3.6. In the course of the proof, we shall use the sets

$$R_\tau := \{x \in \Delta_+^{d-1} \mid x_\tau \in \Delta_{+,\leq}^{d-1}\}, \quad \tau \in \mathcal{T}.$$

Note that the $\{R_\tau \mid \tau \in \mathcal{T}\}$ may not be disjoint, but their topological interiors are.

We first show that $\hat{u}$ from (1.10) is permutation invariant. To this end, recall that $\hat{u} = \exp(\hat{\phi}/2)$, where $\hat{\phi}$ solves the variational problem in Lemma A.1, and recall the functional $J(\phi)$ from (A.5). For a given $\tau \in \mathcal{T}$ and function ϕ, write $\phi_\tau(x) := \phi(x_\tau)$, $x \in \Delta_+^{d-1}$. We claim that

$$J(\phi) = J(\phi_\tau) \quad \forall \tau \in \mathcal{T}. \tag{B.1}$$

Admitting (B.1), that $\hat{\phi}(x) = \hat{\phi}(x_\tau)$ (and hence $\hat{u}(x) = \hat{u}(x_\tau)$) for all $\tau \in \mathcal{T}$ is easy to show. Indeed, as the functional $J(\phi)$ is evidently convex, we see that

$$J\left(\frac{1}{d!}\sum_\tau \phi_\tau\right) \leq \frac{1}{d!}\sum_\tau J(\phi_\tau) = J(\phi),$$

where the last equality follows by (B.1). Thus, if $\hat{\phi}$ is a minimizer then so is $(1/d!)\sum_\tau \hat{\phi}_\tau$ and by Lemma A.1 we can write

$$\hat{\phi} = \frac{1}{d!}\sum_\tau \hat{\phi}_\tau + c,$$

for some constant c. But, as the right hand side above is permutation invariant, so is the left hand side. It remains to prove (B.1), which will follow by straight-forward computations, and which uses the following identities for $\tau \in \mathcal{T}$:

$$\begin{aligned} &f(x) = g(x_\tau) \quad \Longrightarrow \quad \partial_i f(x) = \partial_{\tau^{-1}(i)} g(x_\tau); \\ &p(x) = p(x_\tau), \end{aligned} \tag{B.2}$$

and

$$\begin{aligned} c^{ij}(x) &= c^{\tau^{-1}(i)\tau^{-1}(j)}(x_\tau); \\ \partial_j c^{ij}(x) &= \partial_{\tau^{-1}(j)} c^{\tau^{-1}(i)\tau^{-1}(j)}(x_\tau). \end{aligned} \tag{B.3}$$

Showing (B.2) is straight-forward. As for the first equality in (B.3), we have

$$\begin{aligned} c^{\tau^{-1}(i)\tau^{-1}(j)}(x_\tau) &= \kappa^{r(x(\tau(\tau^{-1}(i))))r(x(\tau(\tau^{-1}(j))))}(x^{()}) \\ &= \kappa^{r(x^i)r(x^j)}(x^{()}) \\ &= c^{ij}(x). \end{aligned}$$

The second equality in (B.3) follows from the first as well as (B.2). Now, plugging in for ℓ from (1.8) we have

$$\begin{aligned} J(\phi) - \int_{\Delta_+^{d-1}} p\ell' c\ell &= \int_{\Delta_+^{d-1}} p\nabla\phi' c\nabla\phi - 2\int_{\Delta_+^{d-1}} \nabla p' c\nabla\phi - 2\int_{\Delta_+^{d-1}} p\nabla\phi' \operatorname{div}(c) \\ &:= \mathbf{A}(\phi) + \mathbf{B}(\phi) + \mathbf{C}(\phi). \end{aligned}$$

We handle the three terms separately and repeatedly use (B.2), (B.3). Also, we will omit the summation symbols. As for $\mathbf{A}$, assume $x \in R_\tau$ so that $x^{()} = x_\tau$. Then

$$\begin{aligned} &\int_{\Delta_+^{d-1}} p(x)\partial_i(\phi_\tau)(x)c^{ij}(x)\partial_j(\phi_\tau)(x)\,dx \\ &= \int_{\Delta_+^{d-1}} p(x_\tau)\partial_{\tau^{-1}(i)}\phi(x_\tau)c^{\tau^{-1}(i)\tau^{-1}(j)}(x_\tau)\partial_{\tau^{-1}(j)}\phi(x_\tau)\,dx \\ &= \int_{\Delta_+^{d-1}} p(y)\partial_a\phi(y)c^{ab}(y)\partial_b\phi(y)\,dy, \end{aligned} \tag{B.4}$$

where to get the last equality we let $y = x_\tau$ and noted that $dy = dx$; and set $a = \tau^{-1}(i)$, $b = \tau^{-1}(j)$. This shows $\mathbf{A}(\phi_\tau) = \mathbf{A}(\phi)$. As for $\mathbf{B}$:

$$\begin{aligned}
&\int_{\Delta_+^{d-1}} \partial_i(p)(x)c^{ij}(x)\partial_j(\phi_\tau)(x)\,dx \\
&\quad = \int_{\Delta_+^{d-1}} \partial_{\tau^{-1}(i)} p(x_\tau)c^{\tau^{-1}(i)\tau^{-1}(j)}(x_\tau)\partial_{\tau^{-1}(j)}\phi(x_\tau)\,dx \\
&\quad = \int_{\Delta_+^{d-1}} \partial_a p(y)c^{ab}(y)\partial_b\phi(y)\,dy.
\end{aligned} \tag{B.5}$$

Thus, $\mathbf{B}(\phi_\tau) = \mathbf{B}(\phi)$. Lastly, for $\mathbf{C}$:

$$\begin{aligned}
&\int_{\Delta_+^{d-1}} p(x)\partial_i(\phi_\tau)(x)\partial_j c^{ij}(x)\,dx \\
&\quad = \int_{\Delta_+^{d-1}} p(x_\tau)\partial_{\tau^{-1}(i)}\phi(x_\tau)\partial_{\tau^{-1}(j)}c^{\tau^{-1}(i)\tau^{-1}(j)}(x_\tau)\,dx \\
&\quad = \int_{\Delta_+^{d-1}} p(y)\partial_a\phi(y)\partial_b c^{ab}(y)\,dy.
\end{aligned} \tag{B.6}$$

Thus, $\mathbf{C}(\phi_\tau) = \mathbf{C}(\phi)$ and hence (B.1) holds.

The third (last) equality in (3.9) holds in view of Theorem 1.8. Next, we show the second equality in (3.9). To do so, we will prove three equalities, analogous to (B.4), (B.5) and (B.6), which all follow by construction of p, c, since $\phi(x) = \phi(x_\tau)$, and (B.2), (B.3). Proceeding, let $\tau \in \mathcal{T}$ and $x \in R_\tau$. We first have

$$\begin{aligned}
\partial_i\phi(x)c^{ij}(x)\partial_j\phi(x) &= \partial_{\tau^{-1}(i)}\phi(x_\tau)c^{\tau^{-1}(i)\tau^{-1}(j)}(x_\tau)\partial_{\tau^{-1}(j)}\phi(x_\tau) \\
&= \partial_a\phi(x_\tau)\kappa^{ab}(x_\tau)\partial_b\phi(x_\tau) \\
&= \partial_a\phi(x^{()})\kappa^{ab}(x^{()})\partial_b\phi(x^{()}).
\end{aligned} \tag{B.7}$$

Next, we have

$$\begin{aligned}
&\partial_i\phi(x)\left(c^{ij}(x)\frac{\partial_j p(x)}{p(x)} + \partial_j c^{ij}(x)\right) \\
&= \partial_{\tau^{-1}(i)}\phi(x_\tau)\left(c^{\tau^{-1}(i)\tau^{-1}(j)}(x_\tau)\frac{\partial_{\tau^{-1}(j)}p(x_\tau)}{p(x_\tau)} + \partial_{\tau^{-1}(j)}c^{\tau^{-1}(i)\tau^{-1}(j)}(x_\tau)\right) \\
&= \partial_a\phi(x_\tau)\left(c^{ab}(x_\tau)\frac{\partial_b p(x_\tau)}{p(x_\tau)} + \partial_b c^{ab}(x_\tau)\right) \\
&= \partial_a\phi(x^{()})\left(\kappa^{ab}(x^{()})\frac{\partial_b q(x^{()})}{q(x^{()})} + \partial_b\kappa^{ab}(x^{()})\right),
\end{aligned} \tag{B.8}$$

where the last equality holds because $c(x_\tau) = \kappa(x_\tau) = \kappa(x^{()})$ and $p(x_\tau) = (1/d!)q(x_\tau) = (1/d!)q(x^{()})$ in R_τ. Last, define

$$\ell_{\leq}(x^{()}) := \left(\frac{\nabla q}{q} + \kappa^{-1}\,\mathsf{div}(\kappa)\right)(x^{()}); \quad x^{()} \in \Delta_{+,\leq}^{d-1}. \tag{B.9}$$

We then have

$$
\begin{aligned}
\frac{1}{p(x)}\partial_i(pc\ell)^i(x) &= c^{ij}(x)\frac{\partial_{ij}p(x)}{p(x)} + 2\frac{\partial_i p(x)}{p(x)}\partial_j c^{ij}_j(x) + \partial_{ij}c^{ij}(x) \\
&= c^{\tau^{-1}(i)\tau^{-1}(j)}(x_\tau)\frac{\partial_{\tau^{-1}(i)\tau^{-1}(j)}p(x_\tau)}{p(x_\tau)} \\
&\quad + 2\frac{\partial_{\tau^{-1}(i)}p(x_\tau)}{p(x_\tau)}\partial_{\tau^{-1}(j)}c^{\tau^{-1}(i)\tau^{-1}(j)}(x_\tau) \\
&\quad + \partial_{\tau^{-1}(i)\tau^{-1}(j)}c^{\tau^{-1}(i)\tau^{-1}(j)}(x_\tau) \\
&= c^{ab}(x_\tau)\frac{\partial_{ab}p(x_\tau)}{p(x_\tau)} + 2\frac{\partial_a p(x_\tau)}{p(x_\tau)}\partial_b c^{ab}(x_\tau) + \partial_{ab}c^{ab}(x_\tau) \\
&= \kappa^{ab}(x^{()})\frac{\partial_{ab}q(x^{()})}{p(x^{()})} + 2\frac{\partial_a q(x^{()})}{q(x^{()})}\partial_b\kappa^{ab}(x^{()}) + \partial_{ab}\kappa^{ab}(x^{()}) \\
&= \frac{1}{q(x^{()})}\partial_a(q\kappa\ell_{\le})^a(x^{()}).
\end{aligned}
\tag{B.10}
$$

Since (B.7), (B.8), (B.10) hold for $x \in R_\tau$ for any $\tau \in \mathcal{T}$, they in fact hold for all $x \in \Delta^{d-1}_+$. Thus, from (B.7) we obtain

$$
\int_{\Delta^{d-1}_+}(p\nabla\phi' c\nabla\phi)(x) = \frac{1}{d!}\int_{\Delta^{d-1}_+}(q\nabla\phi'\kappa\nabla\phi)(x^{()}) = \int_{\Delta^{d-1}_{+,\le}}(q\nabla\phi'\kappa\nabla\phi)(x^{()}),
$$

which is the second equality in (3.9).

Continuing, we show (3.10). Let $\mathbb{P} \in \Pi_{\le}$. Itô's formula, (3.3) and (3.6) give

$$
\begin{aligned}
\frac{1}{T}\log(V_T^{\hat\vartheta}) &= \frac{1}{T}\log\left(\frac{\hat u(X_T)}{\hat u(X_0)}\right) - \frac{1}{2T}\int_0^T \frac{1}{\hat u(X_t)}\sum_{i,j=1}^d \partial^2_{ij}\hat u(X_t)\kappa^{r(X_t^i)r(X_t^j)}(X_t^{()})\,dt \\
&= \frac{1}{T}\log\left(\frac{\hat u(X_T)}{\hat u(X_0)}\right) - \frac{1}{T}\int_0^T \frac{L^c\hat u}{\hat u}(X_t)\,dt.
\end{aligned}
$$

From (A.25) and $\hat u = e^{(1/2)\hat\phi}$ we obtain

$$
\begin{aligned}
\frac{1}{T}\log(V_T^{\hat\vartheta}) &= \frac{1}{2T}\hat\phi(X_T) - \frac{1}{2T}\hat\phi(X_0) - \frac{1}{4T}\int_0^T \frac{1}{p}\nabla\cdot(pc\ell)(X_t)\,dt \\
&\quad - \frac{1}{8T}\int_0^T \nabla\hat\phi' c\nabla\hat\phi(X_t)\,dt + \frac{1}{4T}\int_0^T \nabla\hat\phi'\left(c\frac{\nabla p}{p} + \mathsf{div}(c)\right)(X_t)\,dt \\
&= \frac{1}{2T}\hat\phi(X_T) - \frac{1}{2T}\hat\phi(X_0) - \frac{1}{4T}\int_0^T \frac{1}{q}\nabla\cdot(q\kappa\ell_{\le})(X_t^{()})\,dt \\
&\quad - \frac{1}{8T}\int_0^T \nabla\hat\phi'\kappa\nabla\hat\phi(X_t^{()})\,dt + \frac{1}{4T}\int_0^T \nabla\hat\phi'\left(\kappa\frac{\nabla q}{q} + \mathsf{div}(\kappa)\right)(X_t^{()})\,dt \\
&= \frac{1}{2T}\hat\phi(X_T) - \frac{1}{2T}\hat\phi(X_0) - \frac{1}{4T}\int_0^T \frac{1}{q}\nabla\cdot(q\kappa\ell_{\le})(X_t^{()})\,dt \\
&\quad - \frac{1}{8T}\int_0^T (\nabla\hat\phi - \ell_{\le})'\kappa(\nabla\hat\phi - \ell_{\le})(X_t^{()})\,dt + \frac{1}{8T}\int_0^T \ell_{\le}'\kappa\ell_{\le}(X_t^{()})\,dt,
\end{aligned}
$$

where the second to last equality follows from (B.4), (B.5), (B.6). These equalities, in conjunction with the integrability assumptions of Assumption 1.5, and $\mathbb{P} \in \Pi_{\leq}$ allow us to deduce

$$
\begin{aligned}
G(V^{\hat{\vartheta}}, \mathbb{P}) &= \frac{1}{4}\int_{\Delta^{d-1}_{+,\leq}} \nabla \cdot (q\kappa\ell_{\leq})(x^{()}) - \frac{1}{8}\int_{\Delta^{d-1}_{+,\leq}} ((\nabla\hat{\phi} - \ell_{\leq})'\kappa(\nabla\hat{\phi} - \ell_{\leq})q)(x^{()}) \\
&\quad + \frac{1}{8}\int_{\Delta^{d-1}_{+,\leq}} (\ell_{\leq}'\kappa\ell_{\leq}q)(x^{()}) \\
&= \frac{1}{4}\int_{\Delta^{d-1}_{+,\leq}} \nabla \cdot (q\kappa\ell_{\leq})(x^{()}) - \frac{1}{8}\int_{\Delta^{d-1}_{+,\leq}} (\nabla\hat{\phi}'\kappa\nabla\hat{\phi}q)(x^{()}) \\
&\quad + \frac{1}{4}\int_{\Delta^{d-1}_{+,\leq}} (\nabla\hat{\phi}'(\kappa\nabla q + q\,\mathsf{div}(\kappa)))(x^{()}) \\
&= \frac{1}{4}\int_{\Delta^{d-1}_{+}} \nabla \cdot (pc\ell)(x) - \frac{1}{8}\int_{\Delta^{d-1}_{+}} (\nabla\hat{\phi}'c\nabla\hat{\phi}p)(x) \\
&\quad + \frac{1}{4}\int_{\Delta^{d-1}_{+}} (\nabla\hat{\phi}'(c\nabla p + p\,\mathsf{div}(c)))(x) \\
&= \frac{1}{4}\int_{\Delta^{d-1}_{+}} \nabla \cdot (pc\ell)(x) - \frac{1}{8}\int_{\Delta^{d-1}_{+}} ((\nabla\hat{\phi} - \ell)'c(\nabla\hat{\phi} - \ell)p)(x) \\
&\quad + \frac{1}{8}\int_{\Delta^{d-1}_{+}} (\ell'c\ell p)(x) \\
&= \frac{1}{8}\int_{\Delta^{d-1}_{+}} (\nabla\hat{\phi}'c\nabla\hat{\phi}p)(x).
\end{aligned}
$$

Above, the third equality holds again because of (B.7), (B.8), (B.10). The fifth inequality follows because of Lemma A.5 and (A.28) in the proof of Theorem 1.8. This, and the fact we have already proved the second and third equalities in (3.9), yields (3.10) since $\hat{u} = e^{(1/2)\hat{\phi}}$.

It remains to prove that $\lambda_{\leq} = \lambda$, which will establish all the equalities in (3.9). Recall that $\lambda_{\leq} \leq \lambda$ holds from (3.8). Furthermore, using (3.10) and the last equality in (3.9),

$$
\lambda_{\leq} \geq \inf_{\mathbb{P}\in\Pi_{\leq}} G(V^{\hat{\vartheta}}, \mathbb{P}) = \frac{1}{2}\int_{\Delta^{d-1}_{+}} \left(\frac{\nabla\hat{u}}{\hat{u}}\right)' c \left(\frac{\nabla\hat{u}}{\hat{u}}\right) p = \lambda.
$$

Thus, $\lambda = \lambda_{\leq}$ and the proof is finished.

B.2. Proof of Proposition 3.8. We start with the construction of a particular matrix valued function which works well with Assumption 3.3. To state the following auxiliary result, define

$$
\text{(B.11)} \qquad \overline{x} := \max\{x^1, \dots, x^d\}, \qquad \underline{x} := \min\{x^1, \dots, x^d\}; \quad x \in \Delta^{d-1}_{+}.
$$

LEMMA B.1. *Let* $A, B, C \in \mathbb{R}$ *be such that* (1) $C \geq 0$, (2) $B \leq A < 2B$ *and* (3) $A + C \geq 2$. *For* $x \in \Delta^{d-1}_{+}$ *define the matrix* θ *via*

$$
\text{(B.12)} \qquad
\begin{aligned}
\theta^{ij}(x) &:= 1_{i=j}\left((x^i)^A \prod_{l=1}^{d} (x^l)^C\right) \\
&\quad + 1_{i\neq j}\left((x^i)^B (x^j)^B \prod_{l=1}^{d} (x^l)^{A+C-B}\right); \quad i, j = 1, \dots, d.
\end{aligned}
$$

Then:

(1) *For any* $\tau \in \mathcal{T}$, *we have* $\theta^{\tau^{-1}(i)\tau^{-1}(j)}(x_\tau) = \theta^{ij}(x)$.

(2) *For every* $\xi \in \mathbb{R}^d$ *we have*

$$\xi'\theta(x)\xi \geq k(x)\xi'\xi; \qquad k(x) := \left(\prod_{l=1}^d (x^l)^C\right)(1 - \overline{x}^{2B-A}) \min\{1, \underline{x}^A\}.$$

(3) θ *is smooth in* Δ_+^{d-1} *and the diffusion*

$$dX_t = \frac{1}{2}\operatorname{div}(\theta)(X_t)\,dt + \sqrt{\theta(X_t)}\,dW_t,$$

does not explode to $\partial\Delta_+$.

(4) $\int_{\Delta_+} |\nabla \cdot (\operatorname{div}(\theta))| < \infty$.

(5) $\int_{\Delta_+} \operatorname{div}(\theta)'\theta^{-1}\operatorname{div}(\theta) < \infty$.

(6) $\theta^{-1}\operatorname{div}(\theta) = \nabla H$ *where* $H(x) = (A + C)\log(\prod_{l=1}^d x^l)$.

PROOF. We tackle each point below.

(1) We have

$$\begin{aligned}
\theta^{\tau^{-1}(i)\tau^{-1}(j)}(x_\tau) &= 1_{i=j}\left(x_\tau(\tau^{-1}(i))^A \prod_{l=1}^d x_\tau(l)^C\right) \\
&\quad + 1_{i\neq j}\left(x_\tau(\tau^{-1}(i))^B x_\tau(\tau^{-1}(j))^B \prod_{l=1}^d x_\tau(l)^{A+C-B}\right) \\
&= 1_{i=j}\left((x^i)^A \prod_{l=1}^d (x^l)^C\right) + 1_{i\neq j}\left((x^i)^B (x^j)^B \prod_{l=1}^d (x^l)^{A+C-B}\right) \\
&= \theta^{ij}(x).
\end{aligned}$$

(2) We have

$$\begin{aligned}
\xi'\theta(x)\xi &= \sum_{i=1}^d \xi(i)^2 (x^i)^A \prod_{l=1}^d (x^l)^C + \sum_{i,j=1, i\neq j}^d \xi(i)\xi(j)(x^i)^B (x^j)^B \prod_{l=1}^d (x^l)^{A+C-B} \\
&= \sum_{i=1}^d \xi(i)^2 \left((x^i)^A \prod_{l=1}^d (x^l)^C - (x^i)^{2B}\prod_{l=1}^d (x^l)^{A+C-B}\right) \\
&\quad + \left(\prod_{l=1}^d (x^l)^{A+C-B}\right)\left(\sum_{i=1}^d \xi(i)(x^i)^B\right)^2 \\
&\geq \sum_{i=1}^d \xi(i)^2 \left((x^i)^A \prod_{l=1}^d (x^l)^C - (x^i)^{2B}\prod_{l=1}^d (x^l)^{A+C-B}\right) \\
&= \left(\prod_{l=1}^d (x^l)^C\right)\sum_{i=1}^d \xi(i)^2 (x^i)^A \left(1 - (x^i)^{2B-A}\prod_{l=1}^d (x^l)^{A-B}\right).
\end{aligned}$$

Since $A \geq B$ we have $\prod_{l=1}^d (x^l)^{A-B} \leq 1$. Since $2B - A > 0$ we have $(x^i)^{2B-A} \leq \overline{x}^{2B-A} < 1$. Last, we have $(x^i)^A \geq \min\{1, \underline{x}^A\}$. Putting these together gives the claim.

(3) We have

$$\theta_j^{ij} = 1_{i=j}\Big((A+C)(x^i)^{A+C-1}\prod_{l\neq i}(x^l)^C\Big) + 1_{i\neq j}\Big((x^i)^{A+C}(A+C)(x^j)^{A+C-1}\prod_{l\neq i,j}(x^l)^{A+C-B}\Big).$$

Thus, we see that

$$\begin{aligned}\mathsf{div}(\theta)^i &= \sum_j \theta_j^{ij}\\ &= (A+C)\Big((x^i)^{A+C-1}\prod_{l\neq i}(x^l)^C + (x^i)^{A+C}\sum_{j\neq i}(x^j)^{A+C-1}\prod_{l\neq i,j}(x^l)^{A+C-B}\Big)\\ &= x^i Y_i,\end{aligned}$$

where

$$Y_i := (A+C)(x^i)^{A+C-2}\Big(\prod_{l\neq i}(x^l)^C + x^i\sum_{j\neq i}(x^j)^{A+C-1}\prod_{l\neq i,j}(x^l)^{A+C-B}\Big).$$

Since $C\geq 0$ and $A+C\geq 2$ we see that

$$0\leq Y_i\leq d(A+C).$$

In a similar manner we have

$$\theta^{ii}(x) = (x^i)^2 Z_i^2; \qquad Z_i := (x^i)^{(A+C-2)/2}\prod_{l\neq i}(x^l)^{C/2}.$$

Again, the given hypotheses yield that $0\leq Z_i\leq 1$. Now, let $X(t)$ be a local solution (i.e., up to first exit time τ of some set compactly contained within Δ_+^{d-1}) to the above SDE. We have that for $t\leq\tau$ that

$$dX_t(i) = \frac{1}{2}X_t(i)Y_i(X_t)\,dt + X_t(i)Z_i(X_t)\,d\tilde{B}_t,$$

for a Brownian motion $\tilde{B}$. Now, since Y_i and Z_i are bounded on Δ_+ it is clear that $X(i)$ does not hit zero for any i. This gives the result.

(4) We have from (3) above that

$$\partial_i\Big(\sum_j\theta_j^{ij}\Big) = (A+C)\Big((A+C-1)(x^i)^{A+C-2}\prod_{l\neq i}(x^l)^C + (A+C)(x^i)^{A+C-1}\sum_{j\neq i}(x^j)^{A+C-1}\prod_{l\neq i,j}(x^l)^{A+C-B}\Big).$$

Since $A+C\geq 2$, $C\geq 0$ and $A\geq B$ we see that

$$|\nabla\cdot(\mathsf{div}(\theta))| = \Big|\sum_{i,j}\theta_{ij}^{ij}\Big| \leq (A+C)\big(d(A+C-1)+(d-1)(A+C)\big),$$

from which the result follows.

(5) Write $\tau=\theta^{-1}\mathsf{div}(\theta)$ so that $\mathsf{div}(\theta)=\theta\tau$. Plugging in for θ, $\mathsf{div}(\theta)$ we see that

$$\mathsf{div}(\theta)^i = (A+C)\Big((x^i)^{A+C-1}\prod_{l\neq i}(x^l)^C + (x^i)^{A+C}\sum_{j\neq i}(x^j)^{A+C-1}\prod_{l\neq i,j}(x^l)^{A+C-B}\Big);$$

$$(\theta\tau)^i = (x^i)^{A+C}\prod_{l\neq i}(x^l)^C\ell^i + (x^i)^{A+C}\sum_{j\neq i}(x^j)^{A+C}\prod_{l\neq i,j}(x^l)^{A+C-B}\ell^j.$$

From here, it is clear that $\tau^i = (A+C)/x^i$. Therefore, we have

$$\begin{aligned}&\mathsf{div}(\theta)'\theta^{-1}\,\mathsf{div}(\theta)\\&= \tau' c\tau\\&= (A+C)^2\Big(\sum_i (x^i)^{A+C-2}\prod_{l\neq i}(x^l)^C + \sum_{i\neq j}(x^i)^{A+C-1}(x^j)^{A+C-1}\prod_{l\neq i,j}(x^l)^{A+C-B}\Big)\\&\le d^2(A+C)^2,\end{aligned}$$

and hence the result holds.

(6) We just showed that $(\theta^{-1}\,\mathsf{div}(\theta))^i = \tau^i = (A+C)/x^i$ for $i = 1,\dots,d$. Thus, the result follows since $\nabla(\prod_l x^l)^i = 1/x^i$. □

We are now in position to give the proof of Proposition 3.8.

PROOF OF PROPOSITION 3.8. Assume that V is an open subset of Δ_+^{d-1} such that $\bar V \subset W$ with W open and $\bar W \subseteq \Delta^{d-1}_{+,\le}$. As such $\mathrm{dist}(V, \partial\Delta^{d-1}_{+,\le}) > \delta > 0$ and we may find a C^∞ function χ on $\Delta^{d-1}_{+,\le}$ with $0 \le \chi \le 1$, $\chi = 1$ on V and $\chi(x) = 0$ if $\mathrm{dist}(x, \partial\Delta^{d-1}_{+,\le}) \le \delta/3$, for example. For θ as in Lemma B.1 we then set

$$\begin{aligned}&\kappa_V(x) := \chi(x)\kappa(x) + \big(1-\chi(x)\big)\theta(x);\\&q_V(x) := \chi(x)q(x) + \big(1-\chi(x)\big)\frac{1-\int_{\Delta^{d-1}_{+,\le}}\chi q}{\int_{\Delta^{d-1}_{+,\le}}(1-\chi)}.\end{aligned}\tag{B.13}$$

Now, create c_V, p_V as in (3.6), (3.7) respectively. By construction of θ in Lemma B.1, we see that for any $x \in \Delta_+^{d-1}$ such that $\chi(x^{()}) = 1$, with the τ such that $x^{()} = x_\tau$:

$$c_V^{ij}(x) = \theta^{r(x^i)r(x^j)}\big(x^{()}\big) = \theta^{\tau^{-1}(i)\tau^{-1}(j)}(x_\tau) = \theta^{ij}(x),$$

where the last equality follows from Lemma B.1. Thus, we see that c is smooth in Δ_+^{d-1}. The rest of the conditions in Assumptions 1.1, 1.5 readily follow from Lemma B.1 as q_v is constant near the boundary of $\partial\Delta^{d-1}_{+,\le}$. □

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