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Minimizing the Expected Market Time to Reach a Certain Wealth Level*

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Abstract. In a financial market model, we consider variations of the problem of minimizing the expected time to upcross a certain wealth level. For exponential Lévy markets, we show the asymptotic optimality of the growth-optimal portfolio for the above problem and obtain tight bounds for the value function for any wealth level. In an Itô market, we employ the concept of *market time*, which is a clock that runs according to the underlying market growth. We show the optimality of the growth-optimal portfolio for minimizing the expected market time to reach any wealth level. This reveals a general definition of market time which can be useful from an investor's point of view. We utilize this last definition to extend the previous results in a general semimartingale setting.

Key words. numéraire portfolio, growth-optimal portfolio, market time, upcrossing, overshoot, exponential Lévy markets, Itô markets, semimartingale markets

AMS subject classifications. 60H99, 60G44, 91B28, 91B70

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1. Introduction.

The problem of *quickly* reaching certain goals in wealth management is one of the most fundamental tasks in the theory and practice of finance. However, making this idea mathematically precise has been a challenge. In particular, this would require a quantification of what is meant by achieving goals "quickly" in a model-independent manner, or, even better, coming endogenously from the description of the market as perceived by its participants. Such a mathematically precise description of the flow of time, as well as the corresponding optimal investment strategy, is clearly valuable. If a robust, model-independent answer to the previous questions can be given, it would go a long way towards a better understanding of the problem, as its statement should provide a deep insight into key quantitative characteristics of the market. Our aim in this paper is to present a way of addressing the aforementioned issues.

We proceed with a more thorough description of the problem. Imagine an investor holding some minute capital-in-hand, aiming to reach as quickly as possible a substantial wealth level by optimally choosing an investment opportunity in an active market. No matter what the mathematical formalization of the objective is, as long as it reasonably describes the above informal setting, intuition suggests that the investor should pick an aggressive strategy that provides ample wealth growth. The most famous wealth-optimizing strategy that could

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potentially achieve this is the *growth-optimal* strategy, which is sometimes also called the *Kelly* strategy, as the latter was introduced in [16]. Therefore, the portfolio generated by the growth-optimal strategy is a strong candidate for solving the aforementioned problem, at least in an approximate sense. This last point is augmented by the long line of research on the importance and optimality properties of the growth-optimal portfolio; we mention, for example, the following very incomplete list: [17], [1], [3], [18], [7], [12]. Note also that minimizing expected time to reach a wealth level is not the only interesting objective that one can seek. For example, maximizing the probability that a wealth level will be reached before some future time is also interesting; in this respect, see [6], [9].

Here, we shall identify a variant of the "quickest goal reach" problem for *continuous-time* models where the growth-optimal portfolio is indeed the best. The problem we consider then is that of minimizing the expected *market time* that it will take to reach a certain wealth level. Market time will be defined as a natural time scale which runs fast when the compensation for taking risk in the market is high and vice versa. In a market with continuous asset prices, this will be achieved by setting the slope of the market time equal to half the squared *risk premium*. In this case, it equals the growth rate of the corresponding growth-optimal portfolio, which leads to the interpretation of market time as integrated maximum growth rate.

The first attempt to minimize the expected upcrossing time in a discrete-time gambling-system model was described in [5], where indeed the *near* optimal wealth process was found to be characterized by Kelly's growth-optimal strategy. Models of gambling systems, as considered in [5], could be interpreted as discrete-time financial markets where the log-asset-price processes are random walks with a finite number of possible values for the increment of each step. The natural continuous-time generalization of the above setting is to consider *exponential Lévy markets*, i.e., markets where the log-asset-price processes have independent and stationary increments. For these markets, we establish here the exact analogues of the results in [5].

A continuous-time problem in the context of a Black–Scholes market was treated in [11], and then as an application of a more abstract problem in [10], essentially using methods of dynamic programming. In this case, the *numéraire portfolio* of the market, which was introduced in [17] and is also called the *growth-optimal portfolio*, as it is generated by the analogue of Kelly's growth-optimal strategy, is truly optimal for minimizing the expected calendar time to reach any wealth level. Unfortunately, the moment that one considers more complex Itô-process models, for example ones that are modelling feedback effects, such as the leverage effect in [4], the growth-optimal portfolio is no longer optimal for the problem of minimizing expected calendar time for upcrossing a certain wealth level. In fact, for general non-Markovian models there does not seem to be any hope in identifying what the optimal strategy and wealth process are when minimizing expected calendar time. We note, however, that for *Markovian* models one can still characterize the optimal strategy and portfolio in terms of a Hamilton–Jacobi–Bellman equation, which will most likely then have to be solved numerically.

We introduce in this paper a *market clock* which does not count time according to the natural calendar flow but rather according to the overall market growth. Under the objective that one minimizes expected market time, we show here that the solution again yields the growth-optimal portfolio as nearly optimal. There is a slight problem that results in the nonoptimality of the growth-optimal portfolio, if for finite wealth levels some *overshoot* is possible over the targeted wealth level at the time of the upcrossing. If there is no overshoot, which happens in particular in models with continuous asset prices, then the growth-optimal portfolio is indeed optimal. In [2], the author considers a ramification of the problem by offering a *rebate* for the overshoot that results in the growth-optimal portfolio again being optimal. Of course, we could do this even in the most general case. Since this rebate inclusion is somewhat arbitrary, we shall refrain from using it in our own analysis.

The optimality of the growth-optimal portfolio for minimizing expected time according to a clock counting time according to the overall market growth sounds a bit like a tautological statement. However, we shall make a conscious effort to convey that the concept of market time is very natural, by taking a stepwise approach in the model generality that we consider. The exponential Lévy process case is considered first. There, the market-time flow coincides with the calendar-time flow up to a multiplicative constant, since the model coefficients remain constant through time. As soon as the model coefficients are allowed to randomly change, one can regard the passage of time in terms of the *opportunities* for profit that are available. We first discuss this in the realm of markets where asset-prices are modeled via Itô processes, where the arguments are more intuitive. As soon as the natural candidate for the market time is understood, we proceed to discuss the results in the very general semimartingale model.

The results presented in this work are generalizations of the constant-coefficient result in [11]. The use of martingale methods and a natural definition of market time that we utilize make the proof of our claims more transparent and widens the scope and validity of the corresponding statements.

The structure of the paper is as follows. In section 2 we introduce the general financial market model, we define the problem of minimizing expected market time, and we present the standing assumptions, which are basically the existence of the numéraire portfolio. In section 3 we specialize in the case of exponential Lévy market models, where market time and calendar time coincide up to a multiplicative constant. Our first main result gives tight bounds for the near optimal performance of the growth-optimal portfolio for any wealth level that also result in its asymptotic optimality for increasing wealth levels. In section 4 we use Itô processes to model the market. After some discussion on the concept of market time, our second main result also shows here the optimality of the growth-optimal portfolio. In section 5, the concept of market time in a general semimartingale setting is introduced and a general result that covers all previous cases is presented. Finally, section 6 contains the proofs of the results in the previous sections.

2. Description of the problem.

In the following general remarks we fix some notation that will be used throughout.

By $\mathbb{R}_+$ we shall denote the positive real line, $\mathbb{R}^d$ the d-dimensional Euclidean space, and $\mathbb{N}$ the set of natural numbers $\{1, 2, \ldots\}$. Superscripts will be used to indicate coordinates, both for vectors and for processes; for example $z \in \mathbb{R}^d$ is written $z = (z^1, \ldots, z^d)$. On $\mathbb{R}^d$, $\langle \cdot, \cdot \rangle$ will denote the *usual* inner product: $\langle y, z \rangle := \sum_{i=1}^d y^i z^i$ for y and z in $\mathbb{R}^d$. Also $|\cdot|$ will denote the usual norm: $|z| := \sqrt{\langle z, z \rangle}$ for $z \in \mathbb{R}^d$.

On $\mathbb{R}_+$ equipped with the Borel σ-field $\mathcal{B}(\mathbb{R}_+)$, Leb will denote the *Lebesgue measure*.

All stochastic processes appearing in what follows are defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$. Here, $\mathbb{P}$ is a probability on $(\Omega, \mathcal{F})$, where $\mathcal{F}$ is a σ-algebra that will make all involved random variables measurable. The filtration $\mathbf{F} = (\mathcal{F}_t)_{t \in \mathbb{R}_+}$ is assumed to satisfy the *usual hypotheses* of right-continuity and saturation by $\mathbb{P}$-null sets. It will be assumed throughout that $\mathcal{F}_0$ is trivial modulo $\mathbb{P}$.

For a càdlàg (right-continuous with left limits) stochastic process $X = (X_t)_{t \in \mathbb{R}_+}$, define $X_{t-} := \lim_{s \uparrow t} X_s$ for $t > 0$ and $X_{0-} := 0$. The process X_- will denote this last left-continuous version of X, and $\Delta X := X - X_-$ will be the jump process of X.

2.1. Assets and wealth processes. The d-dimensional semimartingale $S = (S^1, \ldots, S^d)$ will be denoting the *discounted*, with respect to the savings account, price process of d financial assets.

Starting with initial capital $x \in \mathbb{R}_+$, and investing according to some predictable and S-integrable strategy ϑ, an investor's *discounted* total wealth process is given by

$$X^{x,\vartheta} := x + \int_0^\cdot \langle \vartheta_t, \mathrm{d}S_t \rangle . \tag{2.1}$$

Reflecting the investor's ability to hold only a portfolio of nonnegative total tradeable wealth, we then define the set of all nonnegative wealth processes starting from initial capital $x \in \mathbb{R}_+$:

$$\mathcal{X}(x) := \left\{ X^{x,\vartheta} \text{ as in (2.1)} \;\middle|\; \vartheta \text{ is predictable and } S\text{-integrable, and } X^{x,\vartheta} \geq 0 \right\}.$$

It is straightforward that $\mathcal{X}(x) = x\mathcal{X}(1)$ and that $x \in \mathcal{X}(x)$ for all $x \in \mathbb{R}_+$. We also set $\mathcal{X} := \bigcup_{x \in \mathbb{R}_+} \mathcal{X}(x)$.

2.2. The problem. We shall be concerned with the problem of *quickly reaching a wealth level ℓ starting from capital x*. This, of course, is nontrivial only when $x < \ell$, which will be tacitly assumed throughout. The challenge is now to rigorously define what is meant by "quickly." Take $\mathcal{O} = (\mathcal{O}_t)_{t \in \mathbb{R}_+}$ to be an increasing and adapted process such that, $\mathbb{P}$-a.s., $\mathcal{O}_0 = 0$ and $\mathcal{O}_\infty = +\infty$. $\mathcal{O}$ will be representing some kind of internal clock of the market, which we shall call *market time*. In the following sections we shall be more precise on choosing $\mathcal{O}$, guided by what we shall learn when identifying the consequences of applying the growth-optimal strategy.

For any càdlàg process X and $\ell \in \mathbb{R}_+$, define the *first upcrossing market time of X at level ℓ*:

$$\mathcal{T}(X; \ell) := \inf \{ \mathcal{O}_t \in \mathbb{R}_+ \mid X_t \geq \ell \} . \tag{2.2}$$

Of course, if $\ell \leq x$, then $\mathcal{T}(X; \ell) = 0$ for all $X \in \mathcal{X}(x)$. With the aforementioned inputs, define for all $x < \ell$ the value function

$$v(x; \ell) := \inf_{X \in \mathcal{X}(x)} \mathbb{E} [\mathcal{T}(X; \ell)] . \tag{2.3}$$

Our aims in this work are to

- identify a natural definition for the market time $\mathcal{O}$,
- obtain an explicit formula, or at least some useful tight bounds, for the value function $v(x; \ell)$ of (2.3), and
- find the optimal, or perhaps *near* optimal, portfolio for the above problem.

2.3. Standing assumptions. In order to make headway with the problem described in section 2.2, we shall make two natural and indispensable assumptions regarding the financial market that will be in force throughout.

Assumptions 2.1. In our financial market model, we assume the following:

(1) There exists $\widehat{X} \in \mathcal{X}(1)$ such that $X/\widehat{X}$ is a supermartingale for all $X \in \mathcal{X}$.

(2) For every $\ell \in \mathbb{R}_+$, there exists $X \in \mathcal{X}(1)$, possibly depending on ℓ, such that, $\mathbb{P}$-a.s., $\mathcal{T}(X; \ell) < +\infty$.

A process $\widehat{X}$ with the properties described in Assumption 2.1(1) is unique and is called the *numéraire portfolio*. Existence of the numéraire portfolio is a *minimal* assumption for the viability of the financial market. It is essentially equivalent to the boundedness in probability of the set $\{X_T \mid X \in \mathcal{X}(1)\}$ of all possible discounted wealth starting from unit capital and observed at any time $T \in \mathbb{R}_+$. We refer the interested reader to [7], [12], and [15] for more information in this direction. We shall frequently refer to the numéraire portfolio as the *growth-optimal portfolio*, as the two notions coincide.

Assumption 2.1(2) constitutes what has been coined a "favorable game" in [5], and it is *necessary* in order for the problem described in (2.3) to have finite value and therefore to be well-posed. Under Assumption 2.1(2), and in view of the property $\mathcal{X}(x) = x\mathcal{X}(1)$ for $x \in \mathbb{R}_+$, it is obvious that for all $x \in \mathbb{R}_+$ and $\ell \in \mathbb{R}_+$ there exists $X \in \mathcal{X}(x)$ such that $\mathbb{P}[\mathcal{T}(X; \ell) < +\infty] = 1$.

Actually, if Assumption 2.1(1) is in force, Assumption 2.1(2) has a convenient equivalent.

Proposition 2.2. *Under Assumption* 2.1(1)*, Assumption* 2.1(2) *is equivalent to*

(2′) $\lim_{t \to +\infty} \widehat{X}_t = +\infty$, *$\mathbb{P}$-a.s.*

This last result enables one to easily check the validity of Assumptions 2.1 by looking only at the numéraire portfolio. In each of the specific cases we shall consider in what follows, equivalent characterizations of Assumptions 2.1 will be given in terms of the model under consideration.

3. Exponential Lévy markets.

3.1. The setup. For this section we assume that the discounted asset-price processes satisfy $\mathrm{d}S^i_t = S^i_{t-}\,\mathrm{d}R^i_t$ for $t \in \mathbb{R}_+$, where, for all $i = 1, \ldots, d$, R^i is a Lévy process on $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$. Each R^i for $i = 1, \ldots, d$ is the *total returns process* associated with S^i.

In order to make sure that the asset-price processes remain nonnegative, it is necessary and sufficient that $\Delta R^i \geq -1$ for all $i = 1, \ldots, d$. We shall actually impose a further restriction on the structure of the jumps of the returns processes, also bounding them from above. This is mostly done in order to obtain later in Theorem 3.3 a statement which parallels the result in [5]. For the asymptotic result that will be presented in section 3.6 this bounded-jump assumption will be dropped.

Assumption 3.1. For all $i = 1, \ldots, d$ we have $-1 \leq \Delta R^i \leq \kappa$ for some $\kappa \in \mathbb{R}_+$.

Denote by R the d-dimensional Lévy process $(R^1, \ldots, R^d)$. In view of the boundedness of the jumps of R, as stated in Assumption 3.1, we can write

$$R_T = aT + \sigma W_T + \int_{[0,T] \times \mathbb{R}^d} z\,(\mu(\mathrm{d}z, \mathrm{d}t) - \nu(\mathrm{d}z)\,\mathrm{d}t) \tag{3.1}$$

for all $T \in \mathbb{R}_+$. In view of Assumption 3.1, the elements in the above representation satisfy the following:

- $a \in \mathbb{R}^d$.
- σ is a $(d \times m)$-matrix, where $m \in \mathbb{N}$.
- W is a standard m-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$.
- μ is the *jump measure* of R, i.e., the random counting measure on $\mathbb{R}_+ \times \mathbb{R}^d$ defined via $\mu([0,T] \times E) := \sum_{0 \le t \le T} \mathbb{I}_{E \setminus \{0\}}(\Delta R_t)$ for $T \in \mathbb{R}_+$ and $E \subseteq \mathbb{R}^d$.
- ν, the *compensator* of μ, is a *Lévy measure* on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, where $\mathcal{B}(\mathbb{R}^d)$ is the Borel σ-field on $\mathbb{R}^d$. More precisely, ν is a measure with $\nu[\{0\}] = 0$, $\nu\left[\mathbb{R}^d \setminus [-1, \kappa]\right] = 0$, and $\int_{\mathbb{R}^d} |x|^2 \nu[\mathrm{d}x] < +\infty$.

For more information on Lévy processes one can check, for example, [19].

Define the $(d \times d)$ matrix $c := \sigma\sigma^\top$, where "$\top$" denotes matrix transposition. The triplet $(a,\, c,\, \nu)$ will play a crucial role in the discussion below.

In the notation of (2.1), let $X^{x,\vartheta} \in \mathcal{X}(x)$. The nonnegativity requirement $X^{x,\vartheta} \ge 0$ is equivalent to $\Delta X^{x,\vartheta} \ge X^{x,\vartheta}_{-}$, or further to $\langle \vartheta,\, \Delta S \rangle \ge X^{x,\vartheta}_{-}$. Since $\Delta S^i = S^i_{-} \Delta R^i$ for each $i = 1, \ldots, d$, and recalling that ν is the Lévy measure of R, we conclude that $X^{x,\vartheta} \ge 0$ if and only if

$$\left(\vartheta^i_t(\omega) S^i_{t-}(\omega)\right)_{i=1,\ldots,d} \in X^{x,\vartheta}_{t-}(\omega)\mathfrak{C} \quad \text{for all } (\omega, t) \in \Omega \times \mathbb{R}_+,$$

where $\mathfrak{C}$ is the set of *natural constraints* defined via

$$\mathfrak{C} := \left\{ \eta \in \mathbb{R}^d \;\middle|\; \nu\left[z \in \mathbb{R}^d \mid \langle \eta, z \rangle < -1\right] = 0 \right\}.$$

It is easy to see that $\mathfrak{C}$ is convex; it is also closed, as follows from Fatou's lemma.

3.2. Growth rate.

For any $\pi \in \mathfrak{C}$, define

$$\mathfrak{g}(\pi) := \langle \pi, a \rangle - \frac{1}{2} \langle \pi, c\pi \rangle - \int_{\mathbb{R}^d} \left[\langle \pi, z \rangle - \log(1 + \langle \pi, z \rangle)\right] \nu[\mathrm{d}z]. \tag{3.2}$$

For $\pi \in \mathfrak{C}$, $\mathfrak{g}(\pi)$ is the drift rate of the logarithm of the wealth process $X \in \mathcal{X}(1)$ that satisfies $\mathrm{d}X_t = X_{t-} \langle \pi,\, \mathrm{d}R_t \rangle = X_{t-}\, \mathrm{d}\langle \pi, R_t \rangle$ for all $t \in \mathbb{R}_+$; for this reason, $\mathfrak{g}(\pi)$ is also called the *growth rate* of the last wealth process.

Define $\mathfrak{g}^* := \sup_{\pi \in \mathfrak{C}} \mathfrak{g}(\pi)$ to be the maximum growth rate. Since $0 \in \mathfrak{C}$, we certainly have $\mathfrak{g}^* \ge \mathfrak{g}(0) = 0$. Actually, under the bounded-jump Assumption 3.1, the standing Assumptions 2.1 are equivalent to $0 < \mathfrak{g}^* < \infty$. In order to achieve this last claim, we shall connect the viability of the market with the concept of immediate arbitrage opportunities, as will now be introduced.

3.3. Market viability.

Define the set $\mathfrak{I}$ of *immediate arbitrage opportunities* to consist of all vectors $\xi \in \mathbb{R}^d$ such that $c\xi = 0$, $\nu\left[z \in \mathbb{R}^d \mid \langle \xi, z \rangle < 0\right] = 0$, and $\langle \xi,\, a \rangle \ge 0$ and where further at least one of $\nu\left[z \in \mathbb{R}^d \mid \langle \xi, z \rangle > 0\right] > 0$ or $\langle \xi,\, a \rangle > 0$ holds. As part of the next result, we get that the previously described exponential Lévy market is viable if and only if the intersection of $\mathfrak{I}$ with the *recession cone* of $\mathfrak{C}$, defined as $\check{\mathfrak{C}} := \bigcap_{u > 0} u\mathfrak{C}$, is empty.

Proposition 3.2. *Assumptions 2.1 are equivalent to requiring both* $\mathfrak{I} \cap \check{\mathfrak{C}} = \emptyset$ *and* $\mathfrak{g}^* > 0$.

Suppose now that the above is true, as well as that Assumption 3.1 is in force. Then, $\mathfrak{g}^* < \infty$ *and there exists* $\rho \in \mathfrak{C}$ *such that* $\mathfrak{g}(\rho) = \mathfrak{g}^*$. *Furthermore, the numéraire portfolio* $\widehat{X}$ *satisfies the dynamics* $\mathrm{d}\widehat{X}_t = \widehat{X}_{t-} \langle \rho, \mathrm{d}R_t \rangle = \widehat{X}_{t-} \mathrm{d} \langle \rho, R_t \rangle$. *In other words, for* $T \in \mathbb{R}_+$,

$$\log\left(\widehat{X}_T\right) = \langle \rho, R_T \rangle - \frac{1}{2} \langle \rho, c\rho \rangle T - \sum_{0 \le t \le T} \left(\langle \rho, \Delta R_t \rangle - \log\left(1 + \langle \rho, \Delta R_t \rangle\right) \right). \tag{3.3}$$

Instead of using the general Assumptions 2.1 in this section, we shall use the equivalent conditions $\mathfrak{I} \cap \check{\mathfrak{C}} = \emptyset$ and $\mathfrak{g}^* > 0$. We also note that the vector $\rho \in \mathfrak{C}$ in the statement of Proposition 3.2 that leads to the numéraire portfolio is *essentially* unique, modulo any degeneracies that might be present in the market and lead to nonzero portfolios having zero returns.

3.4. The main result. Since Lévy processes have stationary and independent increments, the natural candidate for market time is to consider calendar time up to a multiplicative constant $\gamma > 0$, i.e., to set $\mathcal{O}_t = \gamma t$ for $t \in \mathbb{R}_+$. In Theorem 3.3, we shall actually choose $\gamma = \mathfrak{g}^*$. This turns out to be the appropriate choice of market velocity that reflects a universal characteristic of the market and will result in the bounds (3.4) for the optimal upcrossing time in Theorem 3.3 not depending on the actual model under consideration.

Theorem 3.3. *We work under Assumption 3.1 and also assume that* $\mathfrak{I} \cap \check{\mathfrak{C}} = \emptyset$ *and* $\mathfrak{g}^* > 0$. *Define the finite nonnegative constant* $\alpha := \inf\left\{\beta \in \mathbb{R}_+ \,|\, \nu\left[z \in \mathbb{R}^d \,|\, \langle \rho, z \rangle > \beta\right] = 0\right\}$. *Let the market time* $\mathcal{O}$ *be defined via* $\mathcal{O}_t = \mathfrak{g}^* t$ *for all* $t \in \mathbb{R}_+$. *With* $\widehat{X}(x) := x\widehat{X}$, *we have the inequalities*

$$\log\left(\frac{\ell}{x}\right) \le v(x;\ell) \le \mathbb{E}\left[\mathcal{T}(\widehat{X}(x);\ell)\right] \le \log\left(\frac{\ell}{x}\right) + \log(1+\alpha). \tag{3.4}$$

Actually, Theorem 3.3 is an instance of a more general statement that will be presented in section 5. We note that the bounds (3.4) are in complete accordance with the discrete-time result in [5] and that the nonnegative constant $\log(1+\alpha)$ does not involve x or ℓ.

Remark 3.4. Under a mild condition, namely that the marginal one-dimensional distributions of $\log(\widehat{X})$ are nonlattice, the overshoot of $\log(\widehat{X})$ over the level $\log(\ell)$ actually has a limiting distribution as $\ell \to \infty$ that is supported on $[0, \log(1+\alpha)]$. In that case,

$$\lim_{\ell \to \infty} \left(\mathbb{E}\left[\mathcal{T}(\widehat{X}(x);\ell)\right] - \log\left(\frac{\ell}{x}\right) \right)$$

exists and is exactly equal to the mean of that limiting distribution.

3.5. True optimality. There is a special case when the growth-optimal portfolio is indeed optimal for all levels ℓ, which covers in particular the Black–Scholes market result in [11]. The following result directly stems out of the statement of Theorem 3.3.

Corollary 3.5. *Suppose that the numéraire portfolio* $\widehat{X}$ *of* (3.3) *has no positive jumps:* $\langle \rho, \Delta R \rangle \le 0$. *Then,*

$$v(x;\ell) = \log\left(\frac{\ell}{x}\right) = \mathbb{E}\left[\mathcal{T}(\widehat{X}(x);\ell)\right].$$

For an easy example where the last equality occurs, consider in (3.1) the case where $d = 1$, $\kappa = 0$, and $a = a^1 > 0$. This is a reasonable model where the excess rate of return is strictly positive and only negative jumps are present in the dynamics of the discounted asset-price process.

3.6. Asymptotic optimality without the bounded-jump assumption. Theorem 3.3 gives the asymptotic (for large ℓ) optimality of the growth-optimal portfolio, since, by (3.4),

$$\lim_{\ell \to \infty} \frac{v(x;\ell)}{\log(\ell)} \;=\; 1 \;=\; \lim_{\ell \to \infty} \frac{\mathbb{E}\left[\mathcal{T}(\widehat{X}(x);\ell)\right]}{\log(\ell)}. \tag{3.5}$$

The validity of the asymptotic optimality in (3.5) goes well beyond the bounded-jump Assumption 3.1, as we shall describe now. For the total returns process $R = (R^1, \dots, R^d)$, we can write the canonical representation (3.1) if and only if the Lévy measure ν is such that $\int_{\mathbb{R}^d} \left(|x| \wedge |x|^2\right) \nu[\,\mathrm{d}x] < +\infty$. In that case, the definition in (3.2) of the growth rate is still the same, even without the validity of Assumption 3.1. We then have the following result.

Proposition 3.6. *Suppose that the canonical representation* (3.1) *is valid. Then, if* $\mathfrak{I} \cap \check{\mathfrak{C}} = \emptyset$ *and* $\mathfrak{g}^* > 0$ *hold, we have* $\mathfrak{g}^* < \infty$ *and there exists* $\rho \in \mathfrak{C}$ *such that* $\mathfrak{g}(\rho) = \mathfrak{g}^*$*. One can then define the growth-optimal portfolio* $\widehat{X}$ *using* (3.3)*. Defining* $\mathcal{O}$ *via* $\mathcal{O}_t = \mathfrak{g}^* t$*, and with* $\widehat{X}(x) := x\widehat{X}$*, the asymptotics* (3.5) *hold.*

4. Itô markets and market time. As already mentioned in the introduction, the growth-optimal portfolio is *not* optimal for the problem of minimizing the expected calendar time to reach a wealth level when considering models where the coefficients may change randomly through time. If the objective is somewhat altered into minimizing expected market time, as we shall define below, then the growth-optimal portfolio is indeed optimal. It is our belief that the notion of market time, as it naturally emerges in our paper, has a very clear and natural interpretation and makes deep sense, and is therefore worth studying beyond the context of the questions raised.

To keep the technical details simple, in this section we assume that S is an Itô process. Later, in section 5, we shall see how to relax this assumption to more complex models and still keep the main result holding.

4.1. The setup. The dynamics of the discounted asset-prices are

$$\mathrm{d}S^i_t = S^i_t \left(a^i_t \,\mathrm{d}t + \sum_{j=1}^{m} \sigma^{ij}_t \,\mathrm{d}W^j_t \right) \tag{4.1}$$

for each $i = 1, \dots, d$ and $t \in \mathbb{R}_+$. Here $a = (a^i)_{i=1,\dots,d}$ is the predictable d-dimensional process of excess *appreciation rates*, $\sigma = (\sigma^{ij})_{i=1,\dots,d,\, j=1,\dots,m}$ is a predictable $(d \times m)$-matrix-valued process of *volatilities*, and $W = (W^j)_{j=1,\dots,m}$ is a standard m-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$. We let $c := \sigma\sigma^\top$ denote the $(d \times d)$-matrix-valued process of *local covariances.*

4.2. Assumptions. The general Assumptions 2.1 have a well-described equivalent for the Itô market we are considering.

Proposition 4.1. *Assumptions* 2.1 *are equivalent to the following:*

(1) *There exists a d-dimensional predictable process ρ such that, $(\mathbb{P} \otimes \mathsf{Leb})$-a.e., $c\rho = a$. (In that case, $\rho = c^\dagger a$, where $c^\dagger$ is the Moore–Penrose pseudoinverse of c.)*
(2) *$\int_0^T |\lambda_t|^2 \,\mathrm{d}t < \infty$ for all $T \in \mathbb{R}_+$, where $\lambda := \sigma^\top c^\dagger a$ is the m-dimensional risk premium process. (Then, $|\lambda|^2 = \langle a, c^\dagger a \rangle = \langle \rho, c\rho \rangle$.)*
(3) *$\int_0^\infty |\lambda_t|^2 \,\mathrm{d}t = \infty$, $\mathbb{P}$-a.s.*

In this case, it follows that the logarithm of the numéraire portfolio $\widehat{X}$ is given by

$$\log(\widehat{X}) = \frac{1}{2}\int_0^\cdot |\lambda_t|^2 \,\mathrm{d}t + \int_0^\cdot \lambda_t \,\mathrm{d}W_t. \tag{4.2}$$

It follows from (4.2) that $\mathfrak{g}_t^* := (1/2)|\lambda_t|^2$ equals the maximum growth rate at time $t \in \mathbb{R}_+$ in the given Itô market.

As we did in the case of exponential Lévy markets, we shall use statements (1), (2), and (3) of Proposition 4.1 in place of the general Assumptions 2.1 in what follows.

4.3. Market time. With the above notation define now, similar to the previous section, the *market time* process $\mathcal{O} = (\mathcal{O}_t)_{t\in\mathbb{R}_+}$ by setting it equal to the integral over the maximum growth rate, i.e.,

$$\mathcal{O}_t := \int_0^t \mathfrak{g}_s^* \,\mathrm{d}s = \frac{1}{2}\int_0^t |\lambda_s|^2 \,\mathrm{d}s$$

for $t \in \mathbb{R}_+$. Observe that, under the validity of statements (1), (2), and (3) of Proposition 4.1, we have $\mathbb{P}[\mathcal{O}_\infty = \infty] = 1$ as follows from Proposition 4.1(3). As explained in section 2.2, for given $x < \ell$, our aim is to find the wealth process $X \in \mathcal{X}(x)$ that minimizes $\mathbb{E}[\mathcal{T}(X;\ell)]$.

We briefly explain why the problem of minimizing expected market time to reach a wealth level using such a random clock and not calendar time is natural and worth studying. Consider for simplicity the one-asset case $d = 1$. Then, at any time $t \in \mathbb{R}_+$, $|\lambda_t|^2 = |a_t/\sigma_t|^2$ is the "squared signal to noise ratio" of the asset-price process or more precisely the squared risk premium. When this quantity is small, the opportunities for making profits over those obtainable from the savings account are rather small; on the other hand, when $|\lambda_t|^2$ is large, at time $t \in \mathbb{R}_+$ an investor has a lot of opportunities to use the favorable fact that the premium for taking risk is high. Stalling to reach the wealth level ℓ when opportunities are favorable should be punished more severely, especially for fund managers, and this is exactly what the market time $\mathcal{O}$ does. From an economic point of view, market time simply conforms with the underlying growth of the market.

4.4. The main result. We are ready to present the solution to the optimization problem of section 2.2, both giving an expression for the value function v and again showing that the growth-optimal portfolio is optimal.

Theorem 4.2. *Under the validity of statements (1), (2), and (3) of Proposition 4.1 for an Itô market, and with $\widehat{X}(x) := x\widehat{X} \in \mathcal{X}(x)$, for $x < \ell$ we have*

$$v(x;\ell) = \log\left(\frac{\ell}{x}\right) = \mathbb{E}\big[\mathcal{T}(\widehat{X}(x);\ell)\big].$$

Once again, this last result is a special case of Theorem 5.3 that will be presented in the next section.

5. Market time in general semimartingale markets. The purpose of this section is to give a wide-encompassing definition of market time for semimartingale financial markets and to present a general result on the expected market time to reach a given wealth level, of which both Theorems 3.3 and 4.2 are special cases. We are now in the very general market model described in section 2.

5.1. Market time. Guided by the discussions and results in both the exponential Lévy market case of section 3 and the Itô market case of section 4, it makes sense to define market time as the underlying optimal growth of the market, i.e., the drift part of the logarithm of the growth-optimal portfolio. We shall have to make minimal assumptions for market time to be well defined, namely, that the drift part of the logarithm of the growth-optimal portfolio *does* exist. The following result, which is a refined version of Proposition 2.2, ensures that the discussions that follow make sense.

Proposition 5.1. *Under the validity of Assumption* 2.1(1)*, further assume that the logarithm of the numéraire portfolio* $\widehat{X}$ *is a special semimartingale and write* $\log(\widehat{X}) = \mathcal{O} + M$ *for its canonical decomposition, where* $\mathcal{O}$ *is a predictable nondecreasing process and* M *is a local martingale. Then, Assumption* 2.1(2) *is equivalent to*

(2″) $\lim_{t\to+\infty} \mathcal{O}_t = +\infty$, $\mathbb{P}$*-a.s.*

The following slightly strengthened version of Assumptions 2.1 will enable us to state our general result in Theorem 5.3.

Assumption 5.2. With Assumptions 2.1 in force, we further postulate that the logarithm of the numéraire portfolio $\widehat{X}$ is a special semimartingale.

Under Assumption 5.2, we can write $\log(\widehat{X}) = \mathcal{O}+M$, where $\mathcal{O}$ is a predictable nondecreasing process and M is a local martingale. We then *define* market time to be the nondecreasing predictable process $\mathcal{O}$. According to Proposition 5.1, we have, $\mathbb{P}$-a.s., $\mathcal{O}_0 = 0$ and $\mathcal{O}_\infty = \infty$. This makes $\mathcal{O}$ a bona fide clock.

5.2. A general result. In what follows, α will denote a nonnegative, possibly infinite-valued random variable such that

$$\frac{\Delta \widehat{X}}{\widehat{X}_-} \leq \alpha. \tag{5.1}$$

Of course, α can be chosen in a minimal way as $\alpha := \sup_{t\in\mathbb{R}_+}(\Delta\widehat{X}_t/\widehat{X}_{t-})$.

Theorem 5.3. *Let Assumption* 5.2 *be in force. With the above definition of the market time* $\mathcal{O}$ *and a random variable* α *satisfying* (5.1)*, we have*

$$\log\left(\frac{\ell}{x}\right) \;\leq\; v(x;\ell) \;\leq\; \mathbb{E}\big[\mathcal{T}(\widehat{X}(x);\ell)\big] \;\leq\; \log\left(\frac{\ell}{x}\right) + \mathbb{E}\left[\log(1+\alpha)\right]. \tag{5.2}$$

It is straightforward that Theorem 5.3 covers both Theorem 3.3 and Theorem 4.2 as special cases. For Theorem 3.3, α is the constant defined in its statement, while for Theorem 4.2 we have $\alpha = 0$.

Dividing the inequalities (5.2) with $\log(\ell)$ throughout, we get the following corollary of Theorem 5.3.

Corollary 5.4. *In the setting of Theorem 5.3, suppose that* $\mathbb{E}[\log(1+\alpha)] < \infty$. *Then,*

$$\lim_{\ell \to \infty} \frac{v(x;\ell)}{\log(\ell)} \;=\; 1 \;=\; \lim_{\ell \to \infty} \frac{\mathbb{E}\big[\mathcal{T}(\widehat{X}(x);\ell)\big]}{\log(\ell)}.$$

This last result shows that, under some integrability condition on the possible size of the jumps of the logarithm of the growth-optimal portfolio, the problem of possible overshoots vanishes asymptotically when considering increasing wealth levels ℓ.

6. Proofs. Before we embark on proving all the results of the previous sections, we define, in accordance to (2.2), for any càdlàg process X and $\ell \in \mathbb{R}_+$,

$$\tau(X;\ell) \;:=\; \inf\{t \in \mathbb{R}_+ \,|\, X_t \geq \ell\}$$

to be the *first upcrossing calendar time of* X *at level* ℓ. It is clear that $\tau(X;\ell)$ is a stopping time and that $\mathcal{O}_{\tau(X;\ell)} = \mathcal{T}(X;\ell)$ for all càdlàg processes X and $\ell \in \mathbb{R}_+$.

6.1. Proof of Proposition 2.2. Recall that the clock $\mathcal{O}$ satisfies $\mathbb{P}[\mathcal{O}_\infty = \infty] = 1$. Therefore, for any $X \in \mathcal{X}$ and $\ell \in \mathbb{R}_+$, $\mathbb{P}[\tau(X;\ell) < \infty] = 1$ is equivalent to $\mathbb{P}[\mathcal{T}(X;\ell) < \infty] = 1$.

Condition (2′) of Proposition 2.2 obviously implies Assumption 2.1(2). Conversely, assume that Assumptions 2.1 are in force. For any $n \in \mathbb{N}$, pick $X \in \mathcal{X}(1)$ such that $\mathbb{P}[\tau^n < \infty] = 1$, where $\tau^n := \tau(X;n)$. Since $X/\widehat{X}$ is a nonnegative supermartingale, the optional sampling theorem (see, for example, section 1.3.C of [13]) gives

$$1 \geq \mathbb{E}\left[\frac{X_{\tau^n}}{\widehat{X}_{\tau^n}}\right] \geq n\mathbb{E}\left[\frac{1}{\widehat{X}_{\tau^n}}\right].$$

It follows that $(1/\widehat{X}_{\tau^n})_{n\in\mathbb{N}}$ converges to zero in probability. As $1/\widehat{X}$ is a nonnegative supermartingale, this implies that $\lim_{t\to\infty}(1/\widehat{X}_t) = 0$, $\mathbb{P}$-a.s., which establishes the result.

6.2. Proof of Proposition 5.1. Under the assumption that the numéraire portfolio $\widehat{X}$ is a special semimartingale with canonical decomposition $\widehat{X} = \mathcal{O} + M$, the event equality

$$\left\{\lim_{t\to\infty} \widehat{X}_t = +\infty\right\} = \left\{\lim_{t\to\infty} \mathcal{O}_t = +\infty\right\},$$

which is to be understood in a modulo $\mathbb{P}$ sense, is a consequence of Proposition 3.21 in [12]. Then, the result of Proposition 5.1 readily follows in view of Proposition 2.2.

6.3. Proof of Proposition 3.2. The fact that $\mathfrak{I} \cap \check{\mathfrak{C}} = \emptyset$ is equivalent to the existence of $\rho \in \mathfrak{C}$ such that $\mathfrak{g}(\rho) = \mathfrak{g}^* < \infty$, as well as that $\widehat{X}$ as defined in (3.3) is the numéraire portfolio, is a consequence of Lemma 4.1 in [14], as soon as one also uses the bounded-jump Assumption 3.1.

Now, it is straightforward to check that $\mathfrak{g}^* = 0$ is equivalent to $\widehat{X}$ being a positive local martingale, in which case we have that, $\mathbb{P}$-a.s., $\lim_{t\to\infty} \widehat{X}_t < \infty$. On the other hand, if $\mathfrak{g}^* > 0$, then the Lévy process $\log(\widehat{X})$ is integrable and has strictly positive drift $\mathfrak{g}^*$; therefore, $\mathbb{P}$-a.s., $\lim_{t\to\infty} \widehat{X}_t = \infty$. In view of Proposition 2.2, the result follows.

6.4. Proof of Proposition 4.1. The fact that (1) and (2) of Proposition 4.1 are equivalent to the existence of the numéraire portfolio $\widehat{X}$, as well as that $\widehat{X}$ is given by (4.2), is a special case of Theorem 3.15 in [12]—see also [8]. Under the validity of (1) and (2) of Proposition 4.1, it is straightforward to see that (3) of Proposition 4.1 is equivalent to $\lim_{t\to\infty} \widehat{X}_t = \infty$. Using Proposition 2.2, the result follows.

6.5. Proof of Theorem 5.3. Let $\widehat{L}(x) := \log(\widehat{X}(x))$. Observe that, since $\Delta\widehat{X} \le \alpha\widehat{X}_-$,

$$\Delta\widehat{L}(x) = \log\left(1 + \frac{\Delta\widehat{X}}{\widehat{X}_-}\right) \le \log(1+\alpha). \tag{6.1}$$

Write $\widehat{L}(x) = \log(x) + \mathcal{O} + M$, where M is a local martingale. Let $(\tau^n)_{n\in\mathbb{N}}$ be a localizing sequence for M. The estimate (6.1) gives, for all $n \in \mathbb{N}$,

$$\log(x) + \mathbb{E}\left[\mathcal{O}_{\tau^n\wedge\tau(\widehat{X}(x);\ell)}\right] = \mathbb{E}\left[\widehat{L}_{\tau^n\wedge\tau(\widehat{X}(x);\ell)}(x)\right] \le \log(\ell) + \mathbb{E}[\log(1+\alpha)].$$

Now letting n tend to infinity and using the monotone convergence theorem, we get

$$\mathbb{E}\big[\mathcal{T}(\widehat{X}(x);\ell)\big] \le \log(\ell/x) + \mathbb{E}[\log(1+\alpha)]. \tag{6.2}$$

Now take any $X \in \mathcal{X}(x)$. If $\mathbb{P}[\mathcal{T}(X,\ell) = \infty] > 0$, we have $\mathbb{E}[\mathcal{T}(X,\ell)] = \infty$ and $\log(\ell/x) \le \mathbb{E}[\mathcal{T}(X,\ell)]$ is trivial. It remains to consider the case $\mathbb{P}[\mathcal{T}(X,\ell) < \infty] = 1$, or equivalently $\mathbb{P}[\tau(X,\ell) < \infty] = 1$.

For all $\epsilon \in (0,1)$, define $X^\epsilon := (1-\epsilon)X + \epsilon x$. Then, $X^\epsilon \in \mathcal{X}(x)$ and $\tau(X^\epsilon, \epsilon x + (1-\epsilon)\ell) = \tau(X,\ell)$. The drift part of the process $L^\epsilon := \log(X^\epsilon)$ is bounded above by $\mathcal{O}$. Therefore,

$$L^\epsilon \le \log(x) + \mathcal{O} + M^\epsilon$$

for some local martingale M^ϵ. Let $(\tau^{\epsilon,n})_{n\in\mathbb{N}}$ be a localizing sequence for M^ϵ. Since the stopped process $M^\epsilon_{\tau(X,\ell)\wedge\tau^{\epsilon,n}\wedge\cdot}$ is a martingale, we have that

$$\mathbb{E}\left[L^\epsilon_{\tau(X,\ell)\wedge\tau^{\epsilon,n}}\right] \;\le\; \log(x) + \mathbb{E}\left[\mathcal{O}_{\tau(X,\ell)\wedge\tau^{\epsilon,n}}\right] \;=\; \log(x) + \mathbb{E}\left[\mathcal{T}(X,\ell)\wedge\mathcal{O}_{\tau^{\epsilon,n}}\right].$$

Now, L^ϵ is uniformly bounded from below by $\log(\epsilon x)$. Furthermore, $\uparrow \lim_{n\to\infty} \mathcal{O}_{\tau^n} = \infty$ holds in a $\mathbb{P}$-a.s. sense. Therefore, applications of Fatou's lemma and the monotone convergence theorem will give

$$\begin{aligned}
\log(\ell) + \log(1-\epsilon) \;\le\; \mathbb{E}\left[L^\epsilon_{\tau(X,\ell)}\right] &\le \liminf_{n\to\infty} \mathbb{E}\left[L^\epsilon_{\tau(X,\ell)\wedge\tau^n}\right] \\
&\le \log(x) + \liminf_{n\to\infty} \mathbb{E}\left[\mathcal{T}(X,\ell)\wedge\mathcal{O}_{\tau^n}\right] \\
&= \log(x) + \mathbb{E}\left[\mathcal{T}(X,\ell)\right].
\end{aligned}$$

Now sending ϵ to zero, we also get $\log(\ell/x) \le \mathbb{E}[\mathcal{T}(X,\ell)]$ for all $X \in \mathcal{X}(x)$ that satisfy $\mathbb{P}[\mathcal{T}(X,\ell) < \infty] = 1$. This, coupled with (6.2), finishes the proof.

6.6. Proof of Proposition 3.6. The existence of $\rho \in \mathfrak{C}$ such that $\mathfrak{g}(\rho) = \mathfrak{g}^* < \infty$ follows from Lemma 4.1 in [14] in view of $\mathfrak{I} \cap \check{\mathfrak{C}} \neq \emptyset$. Note that the finiteness of $\mathfrak{g}^*$ is straightforward from the defining equation (3.2) for $\mathfrak{g}$.

Call $\widehat{L} := \log(\widehat{X})$. For each $n \in \mathbb{N}$, let

$$\widehat{L}^n := \widehat{L} - \sum_{t \leq \cdot} (\Delta \widehat{L}_t) \mathbb{I}_{\{\Delta \widehat{L}_t > n\}}.$$

Then, $\widehat{L}^n$ is a Lévy process and we can write

$$\widehat{L}^n_t = \mathfrak{g}^n t + M^n_t$$

for all $t \in \mathbb{R}_+$, where M^n is a Lévy martingale and $\uparrow \lim_{n\to\infty} \mathfrak{g}^n = \mathfrak{g}^* > 0$. Then,

$$\mathbb{E}[\mathcal{T}(\widehat{X}(x);\ell)] = \mathfrak{g}^* \mathbb{E}[\tau(\widehat{X}(x);\ell)] \leq \mathfrak{g}^* \mathbb{E}\left[\tau\left(\widehat{L}^n(x); \log(\ell)\right)\right] \leq \frac{\mathfrak{g}^*}{\mathfrak{g}^n}\left(\log\left(\frac{\ell}{x}\right) + \log(1+n)\right)$$

holds for all $n \in \mathbb{N}$ such that $\mathfrak{g}^n > 0$, where the last inequality follows along the same lines of the proof of (6.2). It then follows that

$$\limsup_{\ell\to\infty} \frac{\mathbb{E}[\mathcal{T}(\widehat{X}(x);\ell)]}{\log(\ell)} \leq \frac{\mathfrak{g}^*}{\mathfrak{g}^n}$$

holds for all $n \in \mathbb{N}$ such that $\mathfrak{g}^n > 0$. Since $\uparrow \lim_{n\to\infty} \mathfrak{g}^n = \mathfrak{g}^* > 0$, sending n to infinity in the last inequality we get

$$\limsup_{\ell\to\infty} \frac{\mathbb{E}[\mathcal{T}(\widehat{X}(x);\ell)]}{\log(\ell)} \leq 1.$$

Of course, in view of the bounds (5.2) of Theorem 5.3, we always have

$$1 = \lim_{\ell\to\infty} \frac{v(x;\ell)}{\log(\ell)} \leq \liminf_{\ell\to\infty} \frac{\mathbb{E}[\mathcal{T}(\widehat{X}(x);\ell)]}{\log(\ell)},$$

which completes the proof.

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