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On the semimartingale property of discounted asset-price processes

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This work is dedicated to the memory of our colleague and dear friend Nicola Bruti Liberati, who died tragically on the 28th of August, 2007.

Abstract

A financial market model where agents trade using realistic combinations of *simple* (i.e., finite combinations of buy-and-hold) *no-short-sales* strategies is considered. Minimal assumptions are made on the discounted asset-price process — in particular, the semimartingale property is *not* assumed. Via a natural market viability assumption, namely, absence of arbitrage of the first kind, we establish that discounted asset-prices *have* to be semimartingales. Our main result can also be regarded as reminiscent of the Fundamental Theorem of Asset Pricing.

Keywords: Numéraire portfolio; Semimartingales; Buy-and-hold strategies; No-short-sales constraints; Arbitrage of the first kind; Supermartingale deflators

0. Introduction

In the mathematical modeling of discounted asset-price processes in frictionless financial markets, semimartingales play a central role. The main reason is the celebrated general version of the Fundamental Theorem of Asset Pricing (FTAP) in [13]; there, the powerful tool of

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stochastic integration with respect to general predictable integrands, that semimartingales are exactly tailored for, played a crucial role. The FTAP connects the economical notion of *No Free Lunch with Vanishing Risk* (NFLVR) with the mathematical concept of existence of an *Equivalent Martingale Measure* (EMM), i.e., an auxiliary probability, equivalent to the original (in the sense that they have the same impossibility events), that makes the discounted asset-price processes have some kind of martingale property. For the above approach to work one has to utilize stochastic integration using *general* predictable integrands, which translates to allowing for *continuous-time* trading in the market. Even though continuous-time trading is of vast theoretical importance, in practice it is only an ideal approximation; the only feasible way of trading is via *simple*, i.e., combinations of *buy-and-hold*, strategies.

Recently, it has been argued that existence of an EMM is *not* necessary for viability of the market; to this effect, see [23,25,14]. Even in cases where classical arbitrage opportunities are present in the market, credit constraints will not allow for the arbitrage to be scaled to any desired degree. It is rather the existence of a *strictly positive supermartingale deflator*, a concept weaker than existence of an EMM, that allows for a consistent theory to be developed.

The purpose of this work is to provide an answer to the following question:

Why are semimartingales important in modeling discounted asset-price processes?

A partial reason pinpointing the importance of semimartingales in modeling discounted asset-price processes is already present in [13]: market viability, formulated by requiring the NFLVR property for simple trading, already imposes the semimartingale property on discounted asset-price processes, as long as the latter processes are locally bounded. In this paper, we elaborate on the previous idea, undertaking a different approach, which ultimately leads to an improved result.[1] (There are papers dealing with market viability when only simple trading is involved and which allow for non-semimartingale discounted asset-price processes; see, for example, [2,5].) In [1,6,22], the semimartingale property of discounted asset-price processes is obtained via the finite value of a utility maximization problem; this approach will also be revisited here.

All the conditions that have appeared previously in the literature are only *sufficient* to ensure that discounted asset-price processes are semimartingales. Here, we shall also discuss a necessary and sufficient condition in terms of an extremely weak market viability notion that only involves simple, no-short-sales trading, under minimal structural assumptions on the discounted asset-price processes themselves. Our main result is reminiscent of (but involves much weaker notions, both on the economics as well as on the mathematics side, than) the FTAP, and can actually be regarded as a "simple, no-short-sales trading" version of [18, Theorem 4.12].

The structure of the paper is as follows. In Section 1, we introduce the market model, simple trading under no-short-sales constraints. Then, we discuss the market viability condition of *absence of arbitrage of the first kind* for such processes, as well as the concept of *strictly positive supermartingale deflators*. After this, our main result, Theorem 1.3, is formulated and proved, which underlines once more the importance of semimartingales in financial modeling. Section 2 deals with remarks on, and ramifications of, Theorem 1.3. We note that, though hidden in the background, the proofs of our results depend heavily on the notion of the *numéraire portfolio* (also called *growth-optimal*, *log-optimal* or *benchmark portfolio*), as it appears in a series of works: [21,24,3,16,25,26,18,11], to mention a few.

[1] After the present work was completed, the very interesting paper [4] appeared, which contains considerably more precise results than [13]. However, the approaches of the two papers are different.

1. The semimartingale property of discounted asset-price processes

1.1. The financial market model and trading via simple, no-short-sales strategies

The random movement of $d \in \mathbb{N}$ risky assets in the market is modeled via càdlàg, nonnegative stochastic processes S^i, where $i \in \{1, \dots, d\}$. We assume that all wealth processes are discounted by another special asset which is considered a "baseline". The above process $S = (S^i)_{i=1,\dots,d}$ is defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, \mathbb{P})$, where $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ is a filtration satisfying $\mathcal{F}_t \subseteq \mathcal{F}$ for all $t \in \mathbb{R}_+$, as well as the usual assumptions of right-continuity and saturation by $\mathbb{P}$-null sets of $\mathcal{F}$.

Observe that there is no *a priori* assumption on S being a semimartingale. This property will come as a consequence of a natural market viability assumption.

In the market described above, economic agents can trade in order to reallocate their wealth. Consider a *simple* predictable process $\theta := \sum_{j=1}^{n} \vartheta_j \mathbb{I}_{]\!]\tau_{j-1}, \tau_j]\!]}$. Here, $\tau_0 = 0$, and for all $j \in \{1, \dots, n\}$ (where n ranges in $\mathbb{N}$), τ_j is a *finite* stopping time and $\vartheta_j = (\vartheta^i_j)_{i=1,\dots,d}$ is $\mathcal{F}_{\tau_{j-1}}$-measurable. Each τ_{j-1}, $j \in \{1, \dots, n\}$, is an instance when some given economic agent may trade in the market; then, ϑ^i_j is the number of units from the ith risky asset that the agent will hold in the trading interval $]\tau_{j-1}, \tau_j]$. This form of trading is called *simple*, as it comprises of a finite number of *buy-and-hold* strategies, in contrast to *continuous* trading where one is able to change the position in the assets in a continuous fashion. This last form of trading is only of theoretical value, since it cannot be implemented in reality, even if one ignores market frictions. Starting from initial capital $x \in \mathbb{R}_+$ and following the strategy described by the simple predictable process $\theta := \sum_{j=1}^{n} \vartheta_j \mathbb{I}_{]\!]\tau_{j-1}, \tau_j]\!]}$, the agent's discounted wealth process is given by

$$X^{x,\theta} = x + \int_0^{\cdot} \langle \theta_t, \mathrm{d}S_t \rangle := x + \sum_{j=1}^{n} \left\langle \vartheta_j, S_{\tau_j \wedge \cdot} - S_{\tau_{j-1} \wedge \cdot} \right\rangle. \tag{1.1}$$

Note that "$\langle \cdot, \cdot \rangle$" is used throughout to denote the usual Euclidean inner product on $\mathbb{R}^d$. This should not be confused with the angle-bracket process, which is not used at all in this paper.[2]

The wealth process $X^{x,\theta}$ of (1.1) is càdlàg and adapted, but could in principle become negative. In real markets, some economic agents, for instance pension funds, face several institution-based constraints when trading. The most important constraint is prevention of having negative positions in the assets; we plainly call this *no-short-sales* constraints. In order to ensure that no short sales are allowed in the risky assets, which also include the baseline asset used for discounting, we define $\mathcal{X}_{\mathsf{s}}(x)$ to be the set of all wealth processes $X^{x,\theta}$ given by (1.1), where $\theta = \sum_{j=1}^{n} \vartheta_j \mathbb{I}_{]\!]\tau_{j-1}, \tau_j]\!]}$ is simple and predictable and such that $\vartheta^i_j \geq 0$ and $\left\langle \vartheta_j, S_{\tau_{j-1}} \right\rangle \leq X^{x,\theta}_{\tau_{j-1}}$ hold for all $i \in \{1, \dots, d\}$ and $j \in \{1, \dots, n\}$. (The subscript "s" in $\mathcal{X}_{\mathsf{s}}(x)$ is a mnemonic for "simple, no-short-sales"; the same is true for all subsequent definitions where this subscript appears.) Note that the previous no-short-sales restrictions, coupled with the nonnegativity of S^i, $i \in \{1, \dots, d\}$, imply that $\theta^i \geq 0$ for all $i = 1, \dots, d$ and $\langle \theta, S_- \rangle \leq X^{x,\theta}_-$. (The subscript "$_-$" is used to denote the left-continuous version of a càdlàg process.) It is clear that $\mathcal{X}_{\mathsf{s}}(x)$ is a convex set for all $x \in \mathbb{R}_+$. Observe also that $\mathcal{X}_{\mathsf{s}}(x) = x \mathcal{X}_{\mathsf{s}}(1)$ for all $x \in \mathbb{R}_+ \setminus \{0\}$. Finally, define $\mathcal{X}_{\mathsf{s}} := \bigcup_{x \in \mathbb{R}_+} \mathcal{X}_{\mathsf{s}}(x)$.

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[2] The angle-bracket process is defined as the predictable compensator of the quadratic covariation process between two semimartingales, whenever this is well defined. At any rate, and as we already noted, we are not assuming any *a priori* semimartingale property for the involved processes.

1.2. Market viability

We now aim at defining the essential "no-free-lunch" concept to be used in our discussion. In a market where only simple, no-short-sales trading is allowed, we shall say that there are opportunities for arbitrage of the first kind if there exist $T \in \mathbb{R}_+$ and an $\mathcal{F}_T$-measurable random variable ξ such that:

- $\mathbb{P}[\xi \geq 0] = 1$ and $\mathbb{P}[\xi > 0] > 0$;
- for all $x > 0$ there exists $X \in \mathcal{X}_\mathsf{s}(x)$, which may depend on x, with $\mathbb{P}[X_T \geq \xi] = 1$.

If there are *no* opportunities for arbitrage of the first kind, we shall say that condition $\mathrm{NA1}_\mathsf{s}$ holds.

It is immediate to see that an arbitrage of the first kind in a market when only simple, no-short-sales trading is allowed gives rise to a free lunch with vanishing risk, as the latter is defined for simple trading in [13, Section 7]—one has to simply rescale the wealth processes involved in the definition of an arbitrage of the first kind to start from zero initial wealth. Therefore, condition $\mathrm{NA1}_\mathsf{s}$ is weaker than condition NFLVR stated for simple trading. (Given Proposition 1.1 below, one can also show this fact using the same ideas as in [18, Proposition 4.2].)

The next result describes an equivalent reformulation of condition $\mathrm{NA1}_\mathsf{s}$ in terms of boundedness in probability of the set of outcomes of wealth processes. This is essentially condition "*No Unbounded Profit with Bounded Risk*" of [18] for all finite time-horizons in our setting of simple, no-short-sales trading.

Proposition 1.1. *Condition* $\mathrm{NA1}_\mathsf{s}$ *holds if and only if, for all* $T \in \mathbb{R}_+$*, the set* $\{X_T \mid X \in \mathcal{X}_\mathsf{s}(1)\}$ *is bounded in probability, i.e.,* $\downarrow \lim_{\ell \to \infty} \sup_{X \in \mathcal{X}_\mathsf{s}(1)} \mathbb{P}[X_T > \ell] = 0$ *holds for all* $T \in \mathbb{R}_+$.

Proof. Using the fact that $\mathcal{X}_\mathsf{s}(x) = x\mathcal{X}_\mathsf{s}(1)$ for all $x > 0$, it is straightforward to check that if an arbitrage of the first kind exists on $[0, T]$ for some $T \in \mathbb{R}_+$ then $\{X_T \mid X \in \mathcal{X}_\mathsf{s}(1)\}$ is not bounded in probability. Conversely, assume the existence of $T \in \mathbb{R}_+$ such that $\{X_T \mid X \in \mathcal{X}_\mathsf{s}(1)\}$ is not bounded in probability. As $\{X_T \mid X \in \mathcal{X}_\mathsf{s}(1)\}$ is further convex, [8, Lemma 2.3] implies the existence of $\Omega_u \in \mathcal{F}_T$ with $\mathbb{P}[\Omega_u] > 0$ such that, for all $n \in \mathbb{N}$, there exists $\widetilde{X}^n \in \mathcal{X}_\mathsf{s}(1)$ with $\mathbb{P}[\{\widetilde{X}^n_T \leq n\} \cap \Omega_u] \leq \mathbb{P}[\Omega_u]/2^{n+1}$. For all $n \in \mathbb{N}$, let $A^n = \left\{\widetilde{X}^n_T > n\right\} \cap \Omega_u \in \mathcal{F}_T$. Then, set $A := \bigcap_{n \in \mathbb{N}} A^n \in \mathcal{F}_T$ and $\xi := \mathbb{I}_A$. It is clear that ξ is $\mathcal{F}_T$-measurable and that $\mathbb{P}[\xi \geq 0] = 1$. Furthermore, since $A \subseteq \Omega_u$ and

$$\mathbb{P}\left[\Omega_u \setminus A\right] = \mathbb{P}\left[\bigcup_{n \in \mathbb{N}} \left(\Omega_u \setminus A^n\right)\right] \leq \sum_{n \in \mathbb{N}} \mathbb{P}\left[\Omega_u \setminus A^n\right] = \sum_{n \in \mathbb{N}} \mathbb{P}\left[\{\widetilde{X}^n_T \leq n\} \cap \Omega_u\right] \leq \sum_{n \in \mathbb{N}} \frac{\mathbb{P}[\Omega_u]}{2^{n+1}} = \frac{\mathbb{P}[\Omega_u]}{2},$$

we obtain $\mathbb{P}[A] > 0$, i.e., $\mathbb{P}[\xi > 0] > 0$. For all $n \in \mathbb{N}$ set $X^n := (1/n)\widetilde{X}^n$, and observe that $X^n \in \mathcal{X}_\mathsf{s}(1/n)$ and $\xi = \mathbb{I}_A \leq \mathbb{I}_{A^n} \leq X^n_T$ hold for all $n \in \mathbb{N}$. It follows that the market allows for opportunities for arbitrage of the first kind, which finishes the proof. □

Remark 1.2. The constant wealth process $X \equiv 1$ belongs to $\mathcal{X}_\mathsf{s}(1)$. Then, Proposition 1.1 implies that condition $\mathrm{NA1}_\mathsf{s}$ is also equivalent to the requirement that the set $\{X_T \mid X \in \mathcal{X}_\mathsf{s}(1)\}$ is bounded in probability for all finite stopping times T.

1.3. Strictly positive supermartingale deflators

Define the set $\mathcal{Y}_{\mathsf{s}}$ of *strictly positive supermartingale deflators for simple, no-short-sales trading* to consist of all càdlàg processes Y such that $\mathbb{P}[Y_0 = 1, \text{ and } Y_t > 0 \ \forall t \in \mathbb{R}_+] = 1$, and YX is a supermartingale for all $X \in \mathcal{X}_{\mathsf{s}}$. Note that existence of a strictly positive supermartingale deflator is a condition closely related, but strictly weaker, to existence of equivalent (super)martingale probability measures. (See Section 2.3 for a quite trivial example in this respect.)

1.4. The main result

Condition $\mathrm{NA1}_{\mathsf{s}}$, existence of strictly positive supermartingale deflators and the semimartingale property of S are immensely tied to each other, as will be revealed below.

Define the (first) *bankruptcy time* of $X \in \mathcal{X}_{\mathsf{s}}$ to be $\zeta^X := \inf\{t \in \mathbb{R}_+ \mid X_{t-} = 0 \text{ or } X_t = 0\}$. We shall say that $X \in \mathcal{X}_{\mathsf{s}}$ *cannot revive from bankruptcy* if $X_t = 0$ holds for all $t \geq \zeta^X$ on $\{\zeta^X < \infty\}$. As $S^i \in \mathcal{X}_{\mathsf{s}}$ for $i \in \{1, \ldots, d\}$, the previous definitions apply in particular to each S^i, $i \in \{1, \ldots, d\}$.

Before stating our main Theorem 1.3, recall that S^i, $i \in \{1, \ldots, d\}$, is an *exponential semimartingale* if there exists a semimartingale R^i such that $S^i = S^i_0 \mathcal{E}(R^i)$, where "$\mathcal{E}$" denotes the *stochastic exponential* operator.

Theorem 1.3. *Let $S = (S^i)_{i=1,\ldots,d}$ be an adapted, càdlàg stochastic process such that S^i is nonnegative for all $i \in \{1, \ldots, d\}$. Consider the following four statements:*

(i) *Condition* $\mathrm{NA1}_{\mathsf{s}}$ *holds in the market.*
(ii) $\mathcal{Y}_{\mathsf{s}} \neq \emptyset$.
(iii) *S is a semimartingale, and S^i cannot revive from bankruptcy for all $i \in \{1, \ldots, d\}$.*
(iv) *For all $i \in \{1, \ldots, d\}$, S^i is an exponential semimartingale.*

Then, we have the following:

(1) *It holds that* (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii)*, as well as* (iv) $\Rightarrow$ (i).
(2) *Assume further that $S^i_{\zeta^{S^i}-} > 0$ holds on $\{\zeta^{S^i} < \infty\}$ for all $i \in \{1, \ldots, d\}$. Then, we have the equivalences* (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv).

1.5. Proof of Theorem 1.3, statement (1)

Proof ((i) $\Rightarrow$ (ii))**.** We start the proof by stating and proving a result that is, in a certain sense, a "static" version of implication (i) $\Rightarrow$ (ii) of Theorem 1.3. Define the d-dimensional simplex

$$\Delta^d := \left\{ z = (z^i)_{i=1,\ldots,d} \in \mathbb{R}^d | z^i \geq 0 \text{ for } i = 1, \ldots, d, \text{ and } \sum_{i=1}^d z^i \leq 1 \right\}.$$

Lemma 1.4. *Let $\mathcal{F}' \subseteq \mathcal{F}$ be a σ-field over Ω. Also, let χ be some $[-1, \infty)^d$-valued and $\mathcal{F}'$-measurable random vector. Then, there exists some Δ^d-valued and $\mathcal{F}'$-measurable random vector ρ with the property that*

$$\mathbb{E}\left[\frac{1 + \langle \pi, \chi \rangle}{1 + \langle \rho, \chi \rangle} \middle| \mathcal{F}' \right] \leq 1$$

holds for any Δ^d-valued and $\mathcal{F}'$-measurable random vector π.

Proof. Define $\mathcal{C}$ as the set of nonnegative random variables of the form $1 + \langle \pi, \chi \rangle$, where π ranges through all Δ^d-valued and $\mathcal{F}'$-measurable random vectors. It is clear that $\mathcal{C}$ is a convex set. Furthermore, $\mathcal{C}$ is bounded in probability, since the finite-valued random variable $\sum_{i=1}^d (1 + \chi^i)$ dominates $\mathbb{P}$-a.s. every element of $\mathcal{C}$. We claim that $\mathcal{C}$ is also closed in probability. To wit, suppose that a $\mathcal{C}$-valued sequence $(1 + \langle \pi^n, \chi \rangle)_{n \in \mathbb{N}}$ converges in probability to a nonnegative random variable f. By passing to a subsequence if necessary, we may assume that the convergence actually holds in the $\mathbb{P}$-a.s. sense. Then, since Δ^d is a compact set, the randomized version of the Bolzano–Weierstrass theorem (see, for example, [17, Lemma 2]) implies that there exists an increasing sequence $(n_k)_{k \in \mathbb{N}}$ of $\mathbb{N}$-valued $\mathcal{F}'$-measurable random variables such that $\pi := \lim_{k \to \infty} \pi^{n_k}$ exists. Of course, π is a Δ^d-valued and $\mathcal{F}'$-measurable random vector. The fact that $\mathbb{P}\left[\lim_{n \to \infty} (1 + \langle \pi^n, \chi \rangle) = f\right] = 1$ implies $\mathbb{P}\left[\lim_{k \to \infty} (1 + \langle \pi^{n_k}, \chi \rangle) = f\right] = 1$; therefore, $\mathbb{P}[1 + \langle \pi, \chi \rangle = f] = 1$. The previous argument establishes that $\mathcal{C}$ is closed in probability.

Since $\mathcal{C}$ is convex, bounded in probability and closed in probability, [20, Theorem 1.1(4)] implies that there exists a Δ^d-valued and $\mathcal{F}'$-measurable random vector ρ with the property that

$$\mathbb{E}\left[\frac{1 + \langle \pi, \chi \rangle}{1 + \langle \rho, \chi \rangle}\right] \leq 1 \tag{1.2}$$

holds for all Δ^d-valued and $\mathcal{F}'$-measurable random vectors π. Now, for any Δ^d-valued and $\mathcal{F}'$-measurable random vector π and any $A \in \mathcal{F}'$, $\pi_A := \pi \mathbb{I}_A + \rho \mathbb{I}_{\Omega \setminus A}$ is itself a Δ^d-valued and $\mathcal{F}'$-measurable random vector. Replacing π by π_A in (1.2), after some simple algebra we obtain

$$\mathbb{E}\left[\frac{1 + \langle \pi, \chi \rangle}{1 + \langle \rho, \chi \rangle} \mathbb{I}_A\right] \leq \mathbb{P}[A].$$

As the latter holds for all $A \in \mathcal{F}'$, the proof of Lemma 1.4 is complete. □

We proceed by stating and proving another auxiliary result that will help to establish implication (i) ⇒ (ii) of Theorem 1.3. Lemma 1.5 can be seen as a "deflator version" of the discrete-time Dalang–Morton–Willinger version of the FTAP—see the original paper [12], as well as [28] for a rather elementary and short treatment. The idea is to essentially use myopic logarithmic expected utility maximization to define a deflator, using the result of Lemma 1.4 (Note that the idea of proving the FTAP using expected utility maximization is already present in [17].)

Define the set of dyadic rational numbers $\mathbb{D} := \{m/2^k \mid k \in \mathbb{N}, m \in \mathbb{N}\}$, which is dense in $\mathbb{R}_+$. Further, for $k \in \mathbb{N}$, define the set of trading times $\mathbb{T}^k := \{m/2^k \mid m \in \mathbb{N},\ 0 \leq m \leq k2^k\}$. Then, $\mathbb{T}^k \subset \mathbb{T}^{k'}$ for $k < k'$ and $\bigcup_{k \in \mathbb{N}} \mathbb{T}^k = \mathbb{D}$. In what follows, $\mathcal{X}_{\mathsf{S}}^k(1)$ denotes the subset of $\mathcal{X}_{\mathsf{S}}(1)$ consisting of wealth processes where trading only may happen at times in $\mathbb{T}^k$.

Lemma 1.5. *Under condition* $\mathrm{NA1}_{\mathsf{S}}$*, and for each* $k \in \mathbb{N}$*, there exists a wealth process* $\widetilde{X}^k \in \mathcal{X}_{\mathsf{S}}^k(1)$ *with* $\mathbb{P}[\widetilde{X}_t^k > 0] = 1$ *for all* $t \in \mathbb{T}^k$ *such that, by defining* $\widetilde{Y}^k := 1/\widetilde{X}^k$, $\mathbb{E}[\widetilde{Y}_t^k X_t \mid \mathcal{F}_s] \leq \widetilde{Y}_s^k X_s$ *holds for all* $X \in \mathcal{X}_{\mathsf{S}}^k(1)$*, where* $\mathbb{T}^k \ni s \leq t \in \mathbb{T}^k$.

Proof. The existence of such "numéraire portfolio" $\widetilde{X}^k$ essentially follows from [18, Theorem 4.12]. However, we give here a more elementary, self-contained argument, rather than using the latter heavy result. Throughout the proof we keep $k \in \mathbb{N}$ fixed, and we set $\mathbb{T}_{++}^k := \mathbb{T}^k \setminus \{0\}$.

First of all, it is straightforward to check that condition $\mathrm{NA1}_{\mathsf{S}}$ implies that each $X \in \mathcal{X}_{\mathsf{S}}$, and in particular also each S^i, $i \in \{1, \ldots, d\}$, cannot revive from bankruptcy. This implies that we can consider an alternative "multiplicative" characterization of wealth processes in

$\mathcal{X}_{\mathsf{S}}(1)$, as we now describe. Consider a process $\pi = (\pi_t)_{t \in \mathbb{T}^k_{++}}$ such that, for all $t \in \mathbb{T}^k_{++}, \pi_t \equiv (\pi^i_t)_{i \in \{1,\dots,d\}}$ is $\mathcal{F}_{t-1/2^k}$-measurable Δ^d-valued. Define $X^{(\pi)}_0 := 1$ and, for all $t \in \mathbb{T}^k_{++}$, $X^{(\pi)}_t := \prod_{\mathbb{T}^k_{++} \ni u \le t} \left(1 + \langle \pi_u, \Delta R^k_u \rangle\right)$, where, for $u \in \mathbb{T}^k_{++}$, $\Delta R^k_u = (\Delta R^{k,i}_u)_{i \in \{1,\dots,d\}}$ is such that $\Delta R^{k,i}_u = \left(S^i_u / S^i_{u-1/2^k} - 1\right) \mathbb{I}_{\{S^i_{u-1/2^k} > 0\}}$ for $i \in \{1, \dots, d\}$. Then, define a simple predictable d-dimensional process θ as follows: for $i \in \{1, \dots, d\}$ and $u \in]t - 1/2^k, t]$, where $t \in \mathbb{T}^k_{++}$, set $\theta^i_u = \left(\pi^i_t X^{(\pi)}_{t-1/2^k} / S^i_{t-1/2^k}\right) \mathbb{I}_{\{S^i_{t-1/2^k} > 0\}}$; otherwise, set $\theta = 0$. It is then straightforward to check that $X^{1,\theta}$, in the notation of (1.1), is an element of $\mathcal{X}^k_{\mathsf{S}}(1)$, as well as that $X^{1,\theta}_t = X^{(\pi)}_t$ holds for all $t \in \mathbb{T}^k$. We have then established that π generates a wealth process in $\mathcal{X}^k_{\mathsf{S}}(1)$. We claim that every wealth process of $\mathcal{X}^k_{\mathsf{S}}(1)$ can be generated this way. Indeed, starting with any predictable d-dimensional process θ such that $X^{1,\theta}$, in the notation of (1.1), is an element of $\mathcal{X}^k_{\mathsf{S}}(1)$, we define $\pi^i_t = \left(\theta^i_t S^i_{t-1/2^k} / X^{1,\theta}_{t-1/2^k}\right) \mathbb{I}_{\{X^{1,\theta}_{t-1/2^k} > 0\}}$ for $i \in \{1, \dots, d\}$ and $t \in \mathbb{T}^k_{++}$. Then, $\pi = (\pi_t)_{t \in \mathbb{T}^k_{++}}$ is Δ^d-valued, $\pi_t \equiv (\pi^i_t)_{i \in \{1,\dots,d\}}$ is $\mathcal{F}_{t-1/2^k}$-measurable for $t \in \mathbb{T}^k_{++}$, and π generates $X^{1,\theta}$ in the way described previously—in particular, $X^{1,\theta}_t = X^{(\pi)}_t$ holds for all $t \in \mathbb{T}^k$. (In establishing the claims above it is important that all wealth processes of $\mathcal{X}_{\mathsf{S}}$ cannot revive from bankruptcy.)

Continuing, Lemma 1.4 implies that for all $t \in \mathbb{T}^k$ there exists a Δ^d-valued and $\mathcal{F}_{t-1/2^k}$-measurable $\rho_t = (\rho^i_t)_{i \in \{1,\dots,d\}}$ such that, for all Δ^d-valued and $\mathcal{F}_{t-1/2^k}$-measurable and $\pi_t = (\pi^i_t)_{i \in \{1,\dots,d\}}$, we have

$$\mathbb{E}\left[\frac{1 + \langle \pi_t, \Delta R^k_t \rangle}{1 + \langle \rho_t, \Delta R^k_t \rangle} \,\middle|\, \mathcal{F}_{t-1/2^k} \right] \le 1.$$

Setting $\widetilde{X}^k$ to be the wealth process in $\mathcal{X}^k_{\mathsf{S}}(1)$ generated by ρ as described in the previous paragraph, the result of Lemma 1.5 is immediate. □

We proceed with the proof of implication (i) ⇒ (ii) of Theorem 1.3, using the notation from the statement of Lemma 1.5. For all $k \in \mathbb{N}$, $\widetilde{Y}^k$ satisfies $\widetilde{Y}^k_0 = 1$ and is a positive supermartingale when sampled from times in $\mathbb{T}^k$, since $1 \in \mathcal{X}^k_{\mathsf{S}}$. Therefore, for any $t \in \mathbb{D}$, the *convex hull* of the set $\{\widetilde{Y}^k_t \mid k \in \mathbb{N}\}$ is bounded in probability. We also claim that, under condition NA1$_{\mathsf{S}}$, for any $t \in \mathbb{R}_+$, the convex hull of the set $\{\widetilde{Y}^k_t \mid k \in \mathbb{N}\}$ is bounded away from zero in probability. Indeed, for any collection $(\alpha^k)_{k \in \mathbb{N}}$ such that $\alpha^k \ge 0$ for all $k \in \mathbb{N}$, having all but a finite number of α^k's non-zero and satisfying $\sum_{k=1}^\infty \alpha^k = 1$, we have

$$\frac{1}{\sum_{k=1}^\infty \alpha^k \widetilde{Y}^k} \le \sum_{k=1}^\infty \alpha^k \frac{1}{\widetilde{Y}^k} = \sum_{k=1}^\infty \alpha^k \widetilde{X}^k \in \mathcal{X}_{\mathsf{S}}(1).$$

Since, by Proposition 1.1, $\{X_t \mid X \in \mathcal{X}_{\mathsf{S}}(1)\}$ is bounded in probability for all $t \in \mathbb{R}_+$, the previous fact proves that the convex hull of the set $\{\widetilde{Y}^k_t \mid k \in \mathbb{N}\}$ is bounded away from zero in probability.

Now, using [13, Lemma A1.1], one can proceed as in the proof of [15, Lemma 5.2(a)] to infer the existence of a sequence $(\widehat{Y}^k)_{k \in \mathbb{N}}$ and some process $(\widehat{Y}_t)_{t \in \mathbb{D}}$ such that, for all $k \in \mathbb{N}$, $\widehat{Y}^k$ is a convex combination of $\widetilde{Y}^k, \widetilde{Y}^{k+1}, \dots$, and $\mathbb{P}[\lim_{k \to \infty} \widehat{Y}^k_t = \widehat{Y}_t,\ \forall t \in \mathbb{D}] = 1$. The discussion of the preceding paragraph ensures that $\mathbb{P}[0 < \widehat{Y}_t < \infty,\ \forall\, t \in \mathbb{D}] = 1$.

Let $\mathbb{D} \ni s \leq t \in \mathbb{D}$. Then, $s \in \mathbb{T}^k$ and $t \in \mathbb{T}^k$ for all large enough $k \in \mathbb{N}$. According to the conditional version of Fatou's Lemma, for all $X \in \bigcup_{k=1}^{\infty} \mathcal{X}_{\mathsf{S}}^k$ we have that

$$\mathbb{E}[\widehat{Y}_t X_t \mid \mathcal{F}_s] \leq \liminf_{k\to\infty} \mathbb{E}[\widehat{Y}_t^k X_t \mid \mathcal{F}_s] \leq \liminf_{k\to\infty} \widehat{Y}_s^k X_s = \widehat{Y}_s X_s. \tag{1.3}$$

It follows that $(\widehat{Y}_t X_t)_{t\in\mathbb{D}}$ is a supermartingale for all $X \in \bigcup_{k=1}^{\infty} \mathcal{X}_{\mathsf{S}}^k$. (Observe here that we sample the process $\widehat{Y} X$ only at times contained in $\mathbb{D}$.) In particular, $(\widehat{Y}_t)_{t\in\mathbb{D}}$ is a supermartingale.

For any $t \in \mathbb{R}_+$ define $Y_t := \lim_{s\downarrow\downarrow t, s\in\mathbb{D}} \widehat{Y}_s$—the limit is taken in the $\mathbb{P}$-a.s. sense, and exists in view of the supermartingale property of $(\widehat{Y}_t)_{t\in\mathbb{D}}$. It is straightforward that Y is a càdlàg process; it is also adapted because $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ is right-continuous. (The argument to obtain adapted and right-continuous versions of martingale-type processes is classical—see, for example, [19, Section 1.3.A, Proposition 3.14].) Now, for $t \in \mathbb{R}_+$, let $T \in \mathbb{D}$ be such that $T > t$; a combination of the right-continuity of both Y and the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$, the supermartingale property of $(\widehat{Y}_t)_{t\in\mathbb{D}}$, and Lévy's martingale convergence Theorem (see [30, Theorem 14.2]), give $\mathbb{E}[\widehat{Y}_T \mid \mathcal{F}_t] \leq Y_t$. Since $\mathbb{P}[\widehat{Y}_T > 0] = 1$, we obtain $\mathbb{P}[Y_t > 0] = 1$. Right-continuity of the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$, coupled with (1.3), imply that $\mathbb{E}[Y_t X_t \mid \mathcal{F}_s] \leq Y_s X_s$ for all $\mathbb{R}_+ \ni s \leq t \in \mathbb{R}_+$ and $X \in \bigcup_{k=1}^{\infty} \mathcal{X}_{\mathsf{S}}^k$. In particular, Y is a càdlàg nonnegative supermartingale; since $\mathbb{P}[Y_t > 0] = 1$ holds for all $t \in \mathbb{R}_+$, we conclude that $\mathbb{P}[Y_t > 0, \forall t \in \mathbb{R}_+] = 1$.

Of course, $1 \in \mathcal{X}_{\mathsf{S}}^k$ and $S^i \in \mathcal{X}_{\mathsf{S}}^k$ hold for all $k \in \mathbb{N}$ and $i \in \{1, \ldots, d\}$. It follows that Y is a supermartingale, as well as that YS^i is a supermartingale for all $i \in \{1, \ldots, d\}$. In particular, Y and $YS = (YS^i)_{i\in\{1,\ldots,d\}}$ are semimartingales. Consider any $X^{x,\theta}$ in the notation of (1.1). Using the integration-by-parts formula (both for $YX^{x,\theta}$ and for YS), we obtain

$$YX^{x,\theta} = x + \int_0^{\cdot} \left(X_{t-}^{x,\theta} - \langle \theta_t, S_{t-} \rangle \right) \mathrm{d}Y_t + \int_0^{\cdot} \langle \theta_t, \mathrm{d}(Y_t S_t) \rangle .$$

If $X^{x,\theta} \in \mathcal{X}_{\mathsf{S}}(x)$, we have $X_{-}^{x,\theta} - \langle \theta, S_- \rangle \geq 0$, as well as $\theta^i \geq 0$ for $i \in \{1, \ldots, d\}$. Then, the supermartingale property of Y and YS^i, $i \in \{1, \ldots, d\}$, gives that $YX^{x,\theta}$ is a supermartingale. Therefore, $Y \in \mathcal{Y}_{\mathsf{S}}$, i.e., $\mathcal{Y}_{\mathsf{S}} \neq \emptyset$. □

Proof ((ii) ⇒ (i))**.** Let $Y \in \mathcal{Y}_{\mathsf{S}}$, and fix $T \in \mathbb{R}_+$. Then, $\sup_{X\in\mathcal{X}_{\mathsf{S}}(1)} \mathbb{E}[Y_T X_T] \leq 1$. In particular, the set $\{Y_T X_T \mid X \in \mathcal{X}_{\mathsf{S}}(1)\}$ is bounded in probability. Since $\mathbb{P}[Y_T > 0] = 1$, the set $\{X_T \mid X \in \mathcal{X}_{\mathsf{S}}(1)\}$ is bounded in probability as well. An invocation of Proposition 1.1 finishes the argument. □

Proof ((ii) ⇒ (iii))**.** Let $Y \in \mathcal{Y}_{\mathsf{S}}$. Since $S^i \in \mathcal{X}_{\mathsf{S}}, YS^i$ is a supermartingale, thus a semimartingale, for all $i \in \{1, \ldots, d\}$. Also, the fact that $Y > 0$ and Itô's formula give that $1/Y$ is a semimartingale. Therefore, $S^i = (1/Y)(YS^i)$ is a semimartingale for all $i \in \{1, \ldots, d\}$. Furthermore, since YS^i is a nonnegative supermartingale, we have $Y_t S_t^i = 0$ for all $t \geq \zeta^{S^i}$ on $\{\zeta^{S^i} < \infty\}$, for $i \in \{1, \ldots, d\}$. Now, using $Y > 0$ again, we obtain that $S_t^i = 0$ holds for all $t \geq \zeta^{S^i}$ on $\{\zeta^{S^i} < \infty\}$. In other words, each S^i, $i \in \{1, \ldots, d\}$, cannot revive after bankruptcy. □

Proof ((iv) ⇒ (i))**.** Since S is a semimartingale, we can consider continuous-time trading. For $x \in \mathbb{R}_+$, let $\mathcal{X}(x)$ be the set of all wealth processes $X^{x,\theta} := x + \int_0^{\cdot} \langle \theta_t, \mathrm{d}S_t \rangle$, where θ is d-dimensional, predictable and S-integrable, "$\int_0^{\cdot} \langle \theta_t, \mathrm{d}S_t \rangle$" denotes a vector stochastic integral, $X^{x,\theta} \geq 0$ and $0 \leq \langle \theta, S_- \rangle \leq X_-^{x,\theta}$. (Observe that the qualifying subscript "S" denoting simple trading has been dropped in the definition of $\mathcal{X}_{\mathsf{S}}(x)$, since we are considering continuous-time trading.) Of course, $\mathcal{X}_{\mathsf{S}}(x) \subseteq \mathcal{X}(x)$. We shall show in the next paragraph that $\{X_T \mid X \in \mathcal{X}(1)\}$ is bounded in probability for all $T \in \mathbb{R}_+$, therefore establishing condition $\mathrm{NA1}_{\mathsf{S}}$, in view of Proposition 1.1.

For all $i \in \{1, \dots, d\}$, write $S^i = S^i_0 \mathcal{E}(R^i)$, where R^i is a semimartingale with $R^i_0 = 0$. Let $R := (R^i)_{i=1,\dots,d}$. It is straightforward to see that $\mathcal{X}(1)$ coincides with the class of all processes of the form $\mathcal{E}\left(\int_0^\cdot \langle \pi_t, \mathrm{d}R_t\rangle\right)$, where π is predictable and take values in the d-dimensional simplex $\Delta^d := \left\{z = (z^i)_{i=1,\dots,d} \in \mathbb{R}^d \mid z^i \geq 0 \text{ for } i = 1, \dots, d, \text{ and } \sum_{i=1}^d z^i \leq 1\right\}$. Since, for all $T \in \mathbb{R}_+$,

$$\log\left(\mathcal{E}\left(\int_0^T \langle \pi_t, \mathrm{d}R_t\rangle\right)\right) \leq \int_0^T \langle \pi_t, \mathrm{d}R_t\rangle$$

holds for all Δ^d-valued and predictable π, it suffices to show the boundedness in probability of the class of all $\int_0^T \langle \pi_t, \mathrm{d}R_t\rangle$, where π ranges in all Δ^d-valued and predictable processes. Write $R = B + M$, where B is a process of finite variation and M is a local martingale with $|\Delta M^i| \leq 1$, $i \in \{1, \dots, d\}$. Then, $\int_0^T |\langle \pi_t, \mathrm{d}B_t\rangle| \leq \sum_{i=1}^d \int_0^T |\mathrm{d}B^i_t| < \infty$. This establishes the boundedness in probability of the class of all $\int_0^T \langle \pi_t, \mathrm{d}B_t\rangle$, where π ranges in all Δ^d-valued and predictable processes. We have to show that the same holds for the class of all $\int_0^T \langle \pi_t, \mathrm{d}M_t\rangle$, where π is Δ^d-valued and predictable. For $k \in \mathbb{N}$, let $\tau^k := \inf\{t \in \mathbb{R}_+ \mid \sum_{i=1}^d [M^i, M^i]_t \geq k\} \wedge T$, Note that $[M^i, M^i]_{\tau^k} = [M^i, M^i]_{\tau^k -} + |\Delta M^i_{\tau^k}|^2 \leq k + 1$ holds for all $i \in \{1, \dots, d\}$. Therefore, using the notation $\|\eta\|_{\mathbb{L}^2} := \sqrt{\mathbb{E}[|\eta|^2]}$ for a random variable η, we obtain

$$\left\| \int_0^{\tau^k} \langle \pi_t, \mathrm{d}M_t\rangle \right\|_{\mathbb{L}^2} \leq \sum_{i=1}^d \left\| \int_0^{\tau^k} \pi^i_t \,\mathrm{d}M^i_t \right\|_{\mathbb{L}^2} \leq \sum_{i=1}^d \left\| \sqrt{[M^i, M^i]_{\tau^k}} \right\|_{\mathbb{L}^2} \leq d\sqrt{k+1}.$$

Fix $\epsilon > 0$. Let $k = k(\epsilon)$ be such that $\mathbb{P}[\tau^k < T] < \epsilon/2$, and also let $\ell := d\sqrt{2(k+1)/\epsilon}$. Then,

$$\begin{aligned}
\mathbb{P}\left[\int_0^T \langle \pi_t, \mathrm{d}M_t\rangle > \ell\right] &\leq \mathbb{P}\left[\tau^k < T\right] + \mathbb{P}\left[\int_0^{\tau^k} \langle \pi_t, \mathrm{d}M_t\rangle > \ell\right] \\
&\leq \frac{\epsilon}{2} + \left| \frac{\left\| \int_0^{\tau^k} \langle \pi_t, \mathrm{d}M_t\rangle \right\|_{\mathbb{L}^2}}{\ell} \right|^2 \leq \epsilon.
\end{aligned}$$

The last estimate is uniform over all Δ^d-valued and predictable π. We have, therefore, established the boundedness in probability of the class of all $\int_0^T \langle \pi_t, \mathrm{d}M_t\rangle$, where π ranges in all Δ^d-valued and predictable processes. This completes the proof. □

1.6. *Proof of Theorem 1.3, statement (2)*

In view of statement (1) of Theorem 1.3, we only need to show the validity of (iii) ⇔ (iv) under the extra assumption of statement (2). This equivalence is really [10, Proposition 2.2], but we present the few details for completeness.

For the implication (iii) ⇒ (iv), simply define $R^i := \int_0^\cdot (1/S^i_{t-})\,\mathrm{d}S^i_t$ for $i \in \{1, \dots, d\}$, The latter process is a well-defined semimartingale because, for each $i \in \{1, \dots, d\}$, S^i is a semimartingale, S^i_- is locally bounded away from zero on the stochastic interval $[\![0, \zeta^{S^i}]\!]$, and $S = 0$ on $[\![\zeta^{S^i}, \infty[\![$.

Now, for (iv) $\Rightarrow$ (iii), it is clear that S is a semimartingale. Furthermore, for all $i \in \{1, \ldots, d\}$, S^i cannot revive from bankruptcy; this follows because stochastic exponentials stay at zero once they hit zero. □

2. On and beyond the main result

2.1. Comparison with existing literature

Theorem 7.2 of the seminal paper [13] establishes the semimartingale property of S under condition NFLVR for simple admissible strategies, coupled with a local boundedness assumption on S (always together with the càdlàg property and adaptedness). The assumptions of Theorem 1.3 are different than the ones in [13]. Condition $\mathrm{NA1_S}$ (valid for simple, no-short-sales trading) is weaker than NFLVR for simple admissible strategies. Furthermore, local boundedness from above is not required in our context, but we do require that each S^i, $i \in \{1, \ldots, d\}$, is nonnegative. In fact, as we shall argue in Section 2.3, nonnegativity of each S^i, $i \in \{1, \ldots, d\}$, can be weakened by local boundedness from below, indeed making Theorem 1.3 a generalization of [13, Theorem 7.2]. Note that if the components of S are unbounded both above and below, not even condition NFLVR (stated for simple strategies, of course) is enough to ensure the semimartingale property of S; see [13, Example 7.5].

Interestingly, and in contrast to [13], the proof of Theorem 1.3 provided here does *not* use the deep Bichteler–Dellacherie theorem on the characterization of semimartingales as "good integrators" (see [7,27], where one *starts* by defining semimartingales as good integrators and obtains the classical definition as a byproduct). Actually, and in view of Proposition 1.1, statement (2) of Theorem 1.3 can be seen as a "multiplicative" counterpart of the Bichteler–Dellacherie theorem. Its proof exploits two simple facts: (a) positive supermartingales are semimartingales, which follows directly from the Doob–Meyer decomposition theorem; and (b) reciprocals of strictly positive supermartingales are semimartingales, which is a consequence of Itô's formula. Crucial in the proof is also the concept of the numéraire portfolio.

Remark 2.1. After the present paper was written, the very interesting preprint [4] appeared, in which the authors establish a result that polishes [13, Theorem 7.2]. In the latter paper, the Bichteler–Dellacherie theorem is not assumed, but rather obtained by utilizing a connection to a no-arbitrage notion which is also a weakening of the NFLVR condition, but in a different direction than the one used here.

2.2. On the actual strength of condition $\mathrm{NA1_S}$

As Theorem 1.3 shows, S is a semimartingale under condition $\mathrm{NA1_S}$. In that case, we can consider the class of nonnegative wealth processes corresponding to *continuous-time* no-short-sales trading, containing all $X^{x,\theta} \equiv x + \int_0^\cdot \langle \theta_t, \mathrm{d}S_t \rangle$ with $x \in \mathbb{R}_+$, θ being predictable, and such that $X^{x,\theta}_- \leq \langle \theta, S_- \rangle$ and $\theta^i \geq 0$ for $i \in \{1, \ldots, d\}$ hold. The exact same argument as in the last paragraph of the proof of (i) $\Rightarrow$ (ii) in Section 1.5 shows that YX is a nonnegative supermartingale for all wealth processes X resulting from continuous-time no-short-sales trading. This implies (see the proof of (ii) $\Rightarrow$ (i) in Section 1.5 and the proof of Proposition 1.1) that there are no arbitrages of the first kind in the class of wealth processes resulting from continuous-time no-short-sales trading. To recapitulate, condition $\mathrm{NA1_S}$ actually implies *both* that S is a semimartingale *and* that there are no arbitrages of the first kind in the class of wealth processes resulting from continuous-time no-short-sales trading.

2.3. The semimartingale property of S when each $S^i, i \in \{1, \ldots, d\}$, is locally bounded from below

As mentioned previously, implication (i) ⇒ (iii) actually holds even when each $S^i, i \in \{1, \ldots, d\}$, is locally bounded from below, which we shall establish now. We still, of course, assume that each $S^i, i \in \{1, \ldots, d\}$, is adapted and càdlàg. Since "no-short-sales" strategies have ambiguous meaning when asset prices can become negative, we need to make some changes in the class of admissible wealth processes. For $x \in \mathbb{R}_+$, let $\mathcal{X}'_{\mathsf{s}}(x)$ denote the class of all wealth processes $X^{x,\theta}$ using simple trading as in (1.1) that satisfy $X^{x,\theta} \geq 0$. Further, set $\mathcal{X}'_{\mathsf{s}} = \bigcup_{x \in \mathbb{R}_+} \mathcal{X}'_{\mathsf{s}}(x)$. Define condition $\mathrm{NA1}'_{\mathsf{s}}$ for the class $\mathcal{X}'_{\mathsf{s}}$ in the obvious manner, replacing "$\mathcal{X}_{\mathsf{s}}$" with "$\mathcal{X}'_{\mathsf{s}}$" throughout in Section 1.2. Assume then that condition $\mathrm{NA1}'_{\mathsf{s}}$ holds. To show that S is a semimartingale, it is enough to show that $(S_{\tau^k \wedge t})_{t \in \mathbb{R}_+}$ is a semimartingale for each $k \in \mathbb{N}$, where $(\tau^k)_{k \in \mathbb{N}}$ is a localizing sequence such that $S^i \geq -k$ on $[\![0, \tau^k]\!]$ for all $i \in \{1, \ldots, d\}$ and $k \in \mathbb{N}$. In other words, we might as well assume that $S^i \geq -k$ for all $i \in \{1, \ldots, d\}$. Define $\tilde{S}^i := k + S^i$; then, $\tilde{S}^i$ is nonnegative for all $i \in \{1, \ldots, d\}$. Let $\tilde{S} = (\tilde{S}^i)_{i \in \{1, \ldots, d\}}$. If $\tilde{\mathcal{X}}_{\mathsf{s}}$ is (in self-explanatory notation) the collection of all wealth processes resulting from simple, no-short-sales strategies investing in $\tilde{S}$, it is straightforward that $\tilde{\mathcal{X}}_{\mathsf{s}} \subseteq \mathcal{X}'_{\mathsf{s}}$. Therefore, $\mathrm{NA1}_{\mathsf{s}}$ holds for simple, no-short-sales strategies investing in $\tilde{S}$; using implication (i) ⇒ (iii) in statement (1) of Theorem 1.3, we obtain the semimartingale property of $\tilde{S}$. The latter is of course equivalent to S being a semimartingale.

One might wonder why we do not simply ask from the outset that each $S^i, i \in \{1, \ldots, d\}$, is locally bounded from below, since it certainly contains the case where each $S^i, i \in \{1, \ldots, d\}$, is nonnegative. The reason is that by restricting trading to using only no-short-sales strategies (which we can do when each $S^i, i \in \{1, \ldots, d\}$, is nonnegative) enables us to be as general as possible in extracting the semimartingale property of S from the $\mathrm{NA1}_{\mathsf{s}}$ condition. Consider, for example, the discounted asset-price process given by $S = a\mathbb{I}_{[\![0,1[\![} + b\mathbb{I}_{[\![1,\infty[\![}$, where $b > a > 0$. This is a *really* elementary example of a nonnegative semimartingale. Now, if we allow for any form of simple trading, as long as it keeps the wealth processes nonnegative, it is clear that condition $\mathrm{NA1}'_{\mathsf{s}}$ will fail (since it is known that at time $t = 1$ there will be a jump of size $(b - a) > 0$ in the discounted asset-price process). On the other hand, if we only allow for no-short-sales strategies, $\mathrm{NA1}_{\mathsf{s}}$ will hold—this is easy to see directly using Proposition 1.1, since $X_T \leq (b - a)/a$ for all $T \geq 1$ and $X \in \mathcal{X}_{\mathsf{s}}(1)$. Therefore, we can conclude that S is a semimartingale using implication (i) ⇒ (iii) in statement (1) of Theorem 1.3. (Of course, one might argue that there is no need to invoke Theorem 1.3 for the simple example here. The point is that allowing for all nonnegative wealth processes results in a rather weak sufficient criterion for the semimartingale property of S.) Note also, in passing, that the above simple example gives an elementary case where $\mathcal{Y}_{\mathsf{s}} \neq \emptyset$ (for example, in view of Theorem 1.3), but where an equivalent supermartingale measure cannot exist, as the process S is nondecreasing and not identically constant.

2.4. The semimartingale property of S via bounded indirect utility

There has been previous work in the literature obtaining the semimartingale property of S using the finiteness of the value function of a utility maximization problem via use of only simple strategies—see, for instance, [1,6,22]. In all cases, there has been an assumption of local boundedness (or even continuity) on S. We shall offer a result in the same spirit, dropping the local boundedness requirement. We shall assume *either* that discounted asset-price processes are nonnegative and *only* no-short-sales simple strategies are considered (which allows for a sharp result), *or* that discounted asset-price processes are locally bounded from below. In the latter case, Proposition 2.2 that follows is a direct generalization of the corresponding result in [1], where the authors consider locally bounded (both above and below) discounted asset-price processes. In the statement of Proposition 2.2 below, we use the notation $\mathcal{X}'_{\mathsf{s}}(x)$ introduced previously in Section 2.3.

Proposition 2.2. *Let $S = (S^i)_{i=1,\dots,d}$ be such that S^i is an adapted and càdlàg process for $i \in \{1, \dots, d\}$. Also, let $U : \mathbb{R}_+ \mapsto \mathbb{R} \cup \{-\infty\}$ be a nondecreasing function with $U > -\infty$ on $]0, \infty]$ and $U(\infty) = \infty$. Fix some $x > 0$. Finally, let T be a finite stopping time. Assume that either:*

- *each S^i, $i \in \{1, \dots, d\}$, is nonnegative and $\sup_{X \in \mathcal{X}_{\mathsf{s}}(x)} \mathbb{E}[U(X_T)] < \infty$, or*
- *each S^i, $i \in \{1, \dots, d\}$, is locally bounded from below and $\sup_{X \in \mathcal{X}'_{\mathsf{s}}(x)} \mathbb{E}[U(X_T)] < \infty$.*

Then, the process $(S_{T \wedge t})_{t \in \mathbb{R}_+}$ is a semimartingale.

Proof. Assume first that each S^i, $i \in \{1, \dots, d\}$, is nonnegative and that $\sup_{X \in \mathcal{X}_{\mathsf{s}}(x)} \mathbb{E}[U(X_T)] < \infty$. Since we only care about the semimartingale property of $(S_{T \wedge t})_{t \in \mathbb{R}_+}$, assume without loss of generality that $S_t = S_{T \wedge t}$ for all $t \in \mathbb{R}_+$. Suppose that condition $\mathrm{NA1}_{\mathsf{s}}$ fails. According to Proposition 1.1 and Remark 1.2, there exists a sequence $(\widetilde{X}^n)_{n \in \mathbb{N}}$ of elements in $\mathcal{X}_{\mathsf{s}}(x)$ and $p > 0$ such that $\mathbb{P}[\widetilde{X}^n_T > 2n] \geq p$ for all $n \in \mathbb{N}$. For all $n \in \mathbb{N}$, let $X^n := (x + \widetilde{X}^n)/2 \in \mathcal{X}_{\mathsf{s}}(x)$. Then, $\sup_{X \in \mathcal{X}_{\mathsf{s}}(x)} \mathbb{E}[U(X_T)] \geq \liminf_{n \to \infty} \mathbb{E}[U(X^n_T)] \geq (1 - p)U(x/2) + p \liminf_{n \to \infty} U(n) = \infty$. This is a contradiction to $\sup_{X \in \mathcal{X}_{\mathsf{s}}(x)} \mathbb{E}[U(X_T)] < \infty$. We conclude that $(S_{T \wedge t})_{t \in \mathbb{R}_+}$ is a semimartingale using implication (i) $\Rightarrow$ (iii) in statement (1) of Theorem 1.3.

Under the assumption that each S^i, $i \in \{1, \dots, d\}$ is locally bounded from below and that $\sup_{X \in \mathcal{X}'_{\mathsf{s}}(x)} \mathbb{E}[U(X_T)] < \infty$, the proof is exactly the same as the one in the preceding paragraph, provided that one replaces "$\mathcal{X}_{\mathsf{s}}$" with "$\mathcal{X}'_{\mathsf{s}}$" throughout, and uses the fact that condition $\mathrm{NA1}'_{\mathsf{s}}$ for the class $\mathcal{X}'_{\mathsf{s}}$ implies the semimartingale property for S, as was discussed in Section 2.3. □

2.5. *On the implication* (iii) ⇒ (i) *in Theorem 1.3*

If we do not require the additional assumption on S in statement (2) of Theorem 1.3, implication (iii) $\Rightarrow$ (i) might fail. We present below a counterexample where this happens.

On $(\Omega, \mathcal{F}, \mathbb{P})$, let W be a standard, one-dimensional Brownian motion (with respect to its own natural filtration—we have not defined $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ yet). Define the process ξ via $\xi_t := \exp(-t/4 + W_t)$ for $t \in \mathbb{R}_+$. Since $\lim_{t \to \infty} W_t/t = 0$, $\mathbb{P}$-a.s., it is straightforward to check that $\xi_\infty := \lim_{t \to \infty} \xi_t = 0$, and actually that $\int_0^\infty \xi_t \, \mathrm{d}t < \infty$, both holding $\mathbb{P}$-a.s. Write $\xi = A + M$ for the Doob–Meyer decomposition of the continuous submartingale ξ under its natural filtration, where $A = (1/4) \int_0^\cdot \xi_t \, \mathrm{d}t$ and $M = \int_0^\cdot \xi_t \, \mathrm{d}W_t$. Due to $\int_0^\infty \xi_t \, \mathrm{d}t < \infty$, we have $A_\infty < \infty$ and $[M, M]_\infty = \int_0^\infty |\xi_t|^2 \, \mathrm{d}t < \infty$, where $[M, M]$ is the quadratic variation process of M. In the terminology of [9], ξ is a semimartingale up to infinity. If we define S via $S_t = \xi_{t/(1-t)}$ for $t \in [0, 1[$ and $S_t = 0$ for $t \in [1, \infty[$, then S is a nonnegative semimartingale. Define $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ to be the augmentation of the natural filtration of S. Observe that $\zeta^S = 1$ and $S_{\zeta^S -} = 0$; the condition of statement (2) of Theorem 1.3 is not satisfied. In order to establish that $\mathrm{NA1}_{\mathsf{s}}$ fails, and in view of Proposition 1.1, it is sufficient to show that $\{X_1 \mid X \in \mathcal{X}_{\mathsf{s}}(1)\}$ is not bounded in probability. Using continuous-time trading, define a wealth process $\widehat{X}$ for $t \in [0, 1[$, via $\widehat{X}_0 = 1$ and the dynamics $\mathrm{d}\widehat{X}_t/\widehat{X}_t = (1/4)(\mathrm{d}S_t/S_t)$ for $t \in [0, 1[$. Then, $\widehat{X}_t = \exp\big((1/16)(t/(1-t)) + (1/4)W_{t/(1-t)}\big)$for $t \in [0, 1[$, which implies that $\mathbb{P}[\lim_{t\uparrow\uparrow 1} \widehat{X}_t = \infty] = 1$, where "$t \uparrow\uparrow 1$" means that t *strictly* increases to 1. Here, the percentage of investment is $1/4 \in [0, 1]$, i.e., $\widehat{X}$ is the result of a no-short-sales strategy. One can then find an approximating sequence $(X^k)_{k\in\mathbb{N}}$ such that $X^k \in \mathcal{X}_{\mathsf{S}}(1)$ for all $k \in \mathbb{N}$, as well as $\mathbb{P}[|X^k_1 - \widehat{X}_{1-1/k}| < 1] > 1 - 1/k$. (Approximation results of this sort are discussed in greater generality in [29].) Then, $(X^k_1)_{k\in\mathbb{N}}$ is not bounded in probability; therefore, $\mathrm{NA1}_{\mathsf{S}}$ fails. Of course, in this example we also have (iii) ⇒ (iv) of Theorem 1.3 failing.

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