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STRICT LOCAL MARTINGALES AND BUBBLES

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This paper deals with asset price bubbles modeled by strict local martingales. With any strict local martingale, one can associate a new measure, which is studied in detail in the first part of the paper. In the second part, we determine the "default term" apparent in risk-neutral option prices if the underlying stock exhibits a bubble modeled by a strict local martingale. Results for certain path dependent options and last passage time formulas are given.

1. Introduction. The goal of this paper is to determine the influence of asset price bubbles on the pricing of derivatives. Asset price bubbles have been studied extensively in the economic literature looking for explanations of why they arise, but have only recently gained attention in mathematical finance by Cox and Hobson [5], Pal and Protter [30], and Jarrow et al. [20–22]. When an asset price bubble exists, the market price of the asset is higher than its fundamental value. From a mathematical point of view, this is the case when the stock price process is modeled by a positive strict local martingale under the equivalent local martingale measure. Here, by a strict local martingale, we understand a local martingale, which is not a true martingale. Strict local martingales were first studied in the context of financial mathematics by Delbaen and Schachermayer [6]. Afterward, Elworthy et al. [10, 11] studied some of their properties including their tail behaviour. More recently, the interest in them grew again (cf., e.g., Mijatovic and Urusov [28]) because of their importance in the modelling of financial bubbles.

Obviously, there are options for which well-known results regarding their valuation in an arbitrage-free market hold true without modification, regardless of whether the underlying is a strict local martingale or a true martingale under the risk-neutral measure. One example is the put option with strike $K \geq 0$. If the underlying is modeled by a continuous local martingale X with $X_0 = 1$, it is shown by Madan et al. [25] that the risk-neutral value of the put option can be expressed in terms of the last passage time of the local martingale X at level K via

$$\mathbb{E}(K - X_T)^+ = \mathbb{E}\big((K - X_\infty)^+ \mathbb{1}_{\{\rho_K^X \leq T\}}\big) \qquad \text{with } \rho_K^X = \sup\{t \geq 0 | X_t = K\}.$$

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This formula does not require X to be a true martingale, but is also valid for strict local martingales. However, if we go from puts to calls, the strict locality of X is relevant. The general idea is to reduce the call case to the put case by a change of measure with Radon–Nikodym density process given by $(X_t)_{t\geq 0}$ as done in Madan et al. [25] in the case where X is a true martingale. However, if X is a strict local martingale, this does not define a measure any more. Instead, we first have to localize the strict local martingale and can thus only define measures on stopped sub-σ-algebras. Under certain conditions on the probability space, we can then extend the so-defined consistent family of measures to a measure defined on some larger σ-field. Under the new measure, the reciprocal of X turns into a true martingale. The conditions we impose are taken from Föllmer [15], who requires the filtration to be a standard system (cf. Definition 2.5). This way we get an extension of Theorem 4 in Delbaen and Schachermayer [6] to general probability spaces and càdlàg local martingales. We study the behavior of X and other local martingales under the new measure.

Using these technical results, we obtain decomposition formulas for some classes of European path-dependent options under the NFLVR condition. These formulas are extensions of Proposition 2 in Pal and Protter [30], which deals with nonpath-dependent options. We decompose the option value into a difference of two positive terms, of which the second one shows the influence of the stock price bubble.

Furthermore, we express the risk-neutral price of an exchange option in the presence of asset price bubbles as an expectation involving the last passage time at the strike level under the new measure. This result is similar to the formula for call options derived by Madan, Roynette and Yor [24] or Yen and Yor [37] for the case of reciprocal Bessel processes. We can further generalize their formula to the case where the candidate density process for the risk-neutral measure is only a strict local martingale. Then the NFLVR condition is not fulfilled and risk-neutral valuation fails, so that we have to work under the real-world measure. Since in this case the price of a zero coupon bond is decreasing in maturity even with an interest rate of zero, some people refer to this as a bond price bubble as opposed to the stock price bubbles discussed above; see, for example, Hulley [17]. In this general setup, we obtain expressions for the option value of European and American call options in terms of the last passage time and the explosion time of the deflated price process, which make some anomalies of the prices of call options in the presence of bubbles evident: European calls are not increasing in maturity any longer and the American call option premium is not equal to zero any more; see, for example, Cox and Hobson [5].

This paper is organized as follows: In the next section, we study strictly positive (strict) local martingales in more detail. On the one hand, we demonstrate ways of how one can obtain strict local martingales, while on the other hand we construct the above mentioned measure associated with a càdlàg strictly positive local martingale on a general filtered probability space with a standard system as filtration.

We give some examples of this construction in Section 3. In Section 4, we then apply our results to the study of asset price bubbles. After formally defining the financial market model, we obtain decomposition formulas for certain classes of European path-dependent options, which show the influence of stock price bubbles on the value of the options under the NFLVR condition. In Section 5, we further study the relationship between the original and the new measure constructed in Section 2.2, which we apply in Section 6 to obtain last passage time formulas for the European and American exchange option in the presence of asset price bubbles. Moreover, we show how this result can be applied to the real-world pricing of European and American call options. The last section contains some results about multivariate strict local martingales.

2. Càdlàg strictly positive strict local martingales. When dealing with continuous strictly positive strict local martingales, a very useful tool is the result from [6]; see also Proposition 6 in [30], which states that every such process defined as the coordinate process on the canonical space of trajectories can be obtained as the reciprocal of a "Doob h-transform"[2] with $h(x) = x$ of a continuous nonnegative true martingale. Conversely, any such transformation of a continuous nonnegative martingale, which hits zero with positive probability, yields a strict local martingale.

The goal of this section is to extend these results to càdlàg processes and general probability spaces satisfying some extra conditions, which were introduced in [31] and used in a similar context in [15]. While the construction of strict local martingales from true martingales follows from an application of the Lenglart–Girsanov theorem, the converse theorem relies as in [6] on the construction of the Föllmer exit measure of a strictly positive local martingale as done in [15] and [27].

2.1. *How to obtain strictly positive strict local martingales.* Examples of continuous strict local martingales have been known for a long time; the canonical example being the reciprocal of a Bessel process of dimension 3. This example can be generalized to a broader class of transient diffusions, which taken in natural scale turn out to be strict local martingales; see, for example, [10]. A natural way to construct strictly positive continuous strict local martingales is given in Theorem 1 of [6]. There it is shown that every uniformly integrable nonnegative martingale with positive probability to hit zero gives rise to a change of measure such that its reciprocal is a strict local martingale under the new measure. For the noncontinuous case and for not necessarily uniformly integrable martingales, we now give a simple extension of the just mentioned theorem from [6].

[2]Note that we abuse the word "Doob h-transform" in this context slightly, since Doob h-transforms are normally only defined in the theory of Markov processes.

THEOREM 2.1. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{Q})$ be the natural augmentation of some filtered probability space with $\mathcal{F} = \bigvee_{t\geq 0} \mathcal{F}_t$, that is, the filtration $(\mathcal{F}_t)$ is right-continuous and $\mathcal{F}_0$ contains all $\mathcal{F}_t$-negligible sets for all $t \geq 0$. Let Y be a nonnegative Q-martingale starting from $Y_0 = 1$. Set $\tau = \inf\{t \geq 0 : Y_t = 0\}$ and assume that $\mathsf{Q}(\tau < \infty) > 0$. Furthermore, suppose that Y does not jump to zero Q-almost surely. For all $t \geq 0$, define a probability measure P_t on $\mathcal{F}_t$ via $\mathsf{P}_t = Y_t.\mathsf{Q}|_{\mathcal{F}_t}$; in particular, $\mathsf{P}_t \ll \mathsf{Q}|_{\mathcal{F}_t}$. Assume that either Y is uniformly integrable under Q or that the nonaugmented probability space satisfies condition (P).[3] Then we can extend the consistent family $(\mathsf{P}_t)_{t\geq 0}$ to a measure P on the augmented space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$. Under the measure P the process Y does never reach zero and its reciprocal $1/Y$ is a strict local P-martingale.*

PROOF. Since the underlying probability space satisfies the natural assumptions, we may choose a càdlàg version of Y; see, for example, Propositions 3.1 and 3.3 in [29]. Especially, this means that τ is a well-defined stopping time. If Y is a uniformly integrable martingale, the measure P is defined on $\mathcal{F}$ by $d\mathsf{P} = Y_\infty\, d\mathsf{Q}$. In the other case, when the probability space fulfills condition (P), the existence of the measure P follows from Corollary 4.9 in [29]. Moreover, note that

$$\mathsf{P}(\tau < \infty) = \lim_{t\to\infty} \mathsf{P}(\tau \leq t) = \lim_{t\to\infty} \mathbb{E}^{\mathsf{Q}}(\mathbb{1}_{\{\tau\leq t\}} Y_t) = 0,$$

therefore, the process $1/Y$ is a P-almost surely well-defined semi-martingale. The result now follows from Corollary 3.10 in Chapter III of [19] applied to $M'_t := \frac{1}{Y_t}\mathbb{1}_{\{\tau>t\}}$, once we can show that $(M'_{t\wedge\tau_n} Y_{t\wedge\tau_n})$ with $\tau_n = \inf\{t \geq 0 : Y_t \leq \frac{1}{n}\}$ is a local Q-martingale for every $n \in \mathbb{N}$. But,

$$M'_{t\wedge\tau_n} Y_{t\wedge\tau_n} = \mathbb{1}_{\{\tau>t\wedge\tau_n\}} = 1 \qquad \mathsf{Q}\text{-a.s.},$$

because Y does not jump to zero Q-almost surely. This trivially proves the martingale property. Finally, the strictness of the local martingale $1/Y$ under P follows from

$$\mathbb{E}^{\mathsf{P}}\left(\frac{1}{Y_t}\right) = \mathsf{Q}(\tau > t) < 1$$

for t large enough, since by assumption $\mathsf{Q}(\tau < \infty) > 0$. $\square$

Starting with a Brownian motion stopped at zero under Q, it is easy to show that the associated strict local martingale under P is the reciprocal of the three-dimensional Bessel process, which is the canonical example of a strict local martingale (cf. Example 1 in [30]). Without stating the general result, the above construction is also applied in [4] to construct examples of strict local martingales

[3]Condition (P) first appeared in [31] and was later used in [29]. We recall its definition in the Appendix.

with jumps related to Dunkl Markov processes on the one hand (cf. Proposition 3 in [4]) and semi-stable Markov processes on the other hand (cf. Proposition 5 in [4]). Apart from the previous, there do not seem to be any well-known examples of strict local martingales with jumps. Note, however, that one can construct an example by taking any continuous strict local martingale and multiplying it with the stochastic exponential of an independent compound Poisson process or any other independent and strictly positive jump martingale.

In the following example, we construct a "nontrivial" positive strict local martingale with jumps by a shrinkage of filtration.

EXAMPLE 2.2. Consider the well-known reciprocal three-dimensional Bessel process Y as a function of a three-dimensional standard Brownian motion $B = (B^1, B^2, B^3)$ starting from $B_0 = (1, 0, 0)$, that is,

$$Y = \frac{1}{\sqrt{(B^1)^2 + (B^2)^2 + (B^3)^2}}.$$

We define the filtrations $(\mathcal{F}_t)_{t\geq 0}$ and $(\mathcal{G}_t)_{t\geq 0}$ through $\mathcal{F}_t = \sigma(B_s^1, B_s^2, B_s^3; s \leq t)$ and $\mathcal{G}_t = \sigma(B_s^1, B_s^2; s \leq t)$, as well as the filtration $(\mathcal{H}_t)_{t\geq 0}$ through

$$\mathcal{H}_t = \mathcal{F}_{\lfloor nt \rfloor/n} \vee \mathcal{G}_t = \sigma\left(B_s^1, B_s^2, s \leq t; B_u^3, u \leq \frac{\lfloor nt \rfloor}{n}\right)$$

for some $n \in \mathbb{N}$. It is shown in Theorem 15 of [16] that not only Y itself is a strict local $(\mathcal{F}_t)_{t\geq 0}$-martingale, but that also the optional projection of Y onto $(\mathcal{G}_t)_{t\geq 0}$ is a continuous local $(\mathcal{G}_t)_{t\geq 0}$-martingale. Since $\mathcal{G}_t \subset \mathcal{H}_t \subset \mathcal{F}_t$ for $t \geq 0$, it follows by Corollary 2 of [16] that then the optional projection of Y onto $(\mathcal{H}_t)_{t\geq 0}$, denoted by ${}^\circ Y$, is also a local martingale. However, since its expectation process is decreasing, ${}^\circ Y$ must be a strict local martingale that jumps at $t \in \frac{\mathbb{N}}{n}$. Indeed, since B^3 is a Brownian motion independent of B^1 and B^2, B_t^3 given $\mathcal{H}_t$ is normally distributed with mean $B^3_{\lfloor nt \rfloor/n}$ and variance $t - \frac{\lfloor nt \rfloor}{n}$. Therefore, ${}^\circ Y$ is given by the explicit formula ${}^\circ Y_t = u(B_t^1, B_t^2, B^3_{\lfloor nt \rfloor/n}, t)$, where

$$\begin{aligned} u(x, y, a, t) = &\int_{\mathbb{R}} (x^2 + y^2 + z^2)^{-1/2} \\ &\times \sqrt{\frac{1}{2\pi(t - \lfloor nt \rfloor/n)}} \exp\left(-\frac{1}{2(t - \lfloor nt \rfloor/n)}(z-a)^2\right) dz. \end{aligned}$$

REMARK 2.3. In the recent preprint [34], the method of filtration shrinkage is applied in greater generality to construct more sophisticated examples of strict local martingales with jumps.

EXAMPLE 2.4. As a further example, any nonnegative nonuniformly integrable $(\mathcal{F}_t)_{t\geq 0}$-martingale Z with $Z_0 = 1$ allows to construct a strictly positive strict local martingale Y relative to a new filtration $(\tilde{\mathcal{F}}_t)_{t\geq 0}$ through a deterministic change of time: simply set

$$Y_t = \begin{cases} \frac{1}{2}(1 + Z_{t/(1-t)}), & 0 \leq t < 1, \\ \frac{1}{2}\Big(1 + \lim_{t\to\infty} Z_t\Big), & 1 \leq t \end{cases}$$

and define $\tilde{\mathcal{F}}_t = \mathcal{F}_{t/(1-t)}$ for $t < 1$ and $\tilde{\mathcal{F}}_t = \mathcal{F}_\infty$ for $t \geq 1$. Since Z is *not* uniformly integrable, we have $\mathbb{E}Y_1 < Y_0 = Z_0 = 1$ almost surely. Note, however, that Y is a true martingale on the interval $[0, 1)$. Instead of setting Y constant for $t \geq 1$ one can also define Y to behave like any other strictly positive local martingale starting from $Y_1 := \frac{1}{2}(1 + \lim_{t\to\infty} Z_t)$ on $[1, \infty)$.

2.2. *From strictly positive strict local martingales to true martingales*. In the following, let $(\Omega, \mathcal{F}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \mathsf{P})$ be a filtered probability space. Furthermore, we denote by $(\mathcal{F}_t)_{t\geq 0}$ the right-continuous augmentation of $(\tilde{\mathcal{F}}_t)_{t\geq 0}$, that is, $\mathcal{F}_t := \tilde{\mathcal{F}}_{t+} = \bigcap_{s>t} \tilde{\mathcal{F}}_s$ for all $t \geq 0$. Note, however, that the filtration is *not* completed with the negligible sets of $\mathcal{F}$.

DEFINITION 2.5 (cf. [15]). Let T be a partially ordered nonvoid index set and let $(\tilde{\mathcal{F}}_t)_{t\in T}$ be a filtration on Ω. Then $(\tilde{\mathcal{F}}_t)_{t\in T}$ is called a standard system if:

- each measurable space $(\Omega, \tilde{\mathcal{F}}_t)$ is a standard Borel space, that is, $\tilde{\mathcal{F}}_t$ is σ-isomorphic to the σ-field of Borel sets on some complete separable metric space;
- for any increasing sequence $(t_i)_{i\in\mathbb{N}} \subset T$ and for any $A_1 \supset A_2 \supset \cdots \supset A_i \supset \cdots$, where A_i is an atom of $\tilde{\mathcal{F}}_{t_i}$, we have $\bigcap_i A_i \neq \varnothing$.

As noted in [29], the filtration $\tilde{\mathcal{F}}_t = \sigma(X_s, s \leq t)$, where $X_t(\omega) = \omega(t)$ is the coordinate process on the space $C(\mathbb{R}_+, \mathbb{R}_+)$ of nonexplosive nonnegative continuous functions, is not a standard system. However, it will be seen below that when dealing with strict local martingales it is natural to work on the space of all $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{\infty\}$-valued processes that are continuous up to some time $\alpha \in [0, \infty]$ and constant afterward. As noted in example (6.3) in [15], the filtration generated by the coordinate process on this space is indeed a standard system. More generally, we have the following lemma.

LEMMA 2.6. *Let $\Omega = D'(\mathbb{R}_+, \overline{\mathbb{R}}_+^n)$ be the space of functions from $\mathbb{R}_+$ into $\overline{\mathbb{R}}_+^n$ with componentwise right-continuous paths $(\omega_i(t))_{t\geq 0}$, $i = 1, \dots, n$, that have left limits on $(0, \alpha(\omega))$ for some $\alpha(\omega) \in [0, \infty]$ and remain constant on $[\alpha(\omega), \infty)$ at the value $\lim_{t\uparrow\alpha(\omega)} \omega_i(t)$ if this limit exists and at ∞ otherwise. We denote by $(X_t)_{t\geq 0}$ the coordinate process, that is, $X_t(\omega_1, \dots, \omega_n) = (\omega_1(t), \dots, \omega_n(t))$, and by $(\tilde{\mathcal{F}}_t)_{t\geq 0}$ the canonical filtration generated by the coordinate process, that is,*

$\tilde{\mathcal{F}}_t = \sigma(X_s; s \leq t)$. *Furthermore, set* $\mathcal{F} = \bigvee_{t \geq 0} \tilde{\mathcal{F}}_t$. *Then,* $(\tilde{\mathcal{F}}_t)_{t \geq 0}$ *is a standard system on the space* $(\Omega, \mathcal{F}, (\tilde{\mathcal{F}}_t)_{t \geq 0})$. *The same is true, if we replace* $D'(\mathbb{R}_+, \overline{\mathbb{R}}_+^n)$ *by its subspace* $C'(\mathbb{R}_+, \overline{\mathbb{R}}_+^n)$ *of functions which are componentwise continuous on some* $(0, \alpha(\omega))$ *and remain constant on* $[\alpha(\omega), \infty)$ *at the value* $\lim_{t \uparrow \alpha(\omega)} \omega_i(t)$ *if this limit exists and at* ∞ *otherwise.*

PROOF. We prove the claim for $\Omega = D'(\mathbb{R}_+, \overline{\mathbb{R}}_+^n)$. The case $\Omega = C'(\mathbb{R}_+, \overline{\mathbb{R}}_+^n)$ is done in a similar way. As in [9], we define a bijective mapping i from Ω to some subspace $A \subset (\overline{\mathbb{R}}_+^n)^{\mathbb{Q}}$ (where here $\mathbb{Q}$ denotes the set of all rational numbers), via $\omega \mapsto (X_r(\omega))_{r \in \mathbb{Q}}$. It is clear that i is bijective and we have $\mathcal{F} = i^{-1}(\mathcal{B}(A))$. Furthermore, a sequence $A_1 \supset A_2 \supset \cdots \supset A_i \supset \cdots$ of atoms of $\mathcal{F}_{t_i} = \sigma(X_s; s \leq t_i)$ defines a component-wise càdlàg function on the interval $[0, \lim t_i] \cap [0, \alpha(\omega))$, which is constant on $[0, \lim t_i] \cap [\alpha(\omega), \infty)$, for every increasing sequence $(t_i)_{i \in \mathbb{N}} \subset \mathbb{R}_+$. This function can easily be extended to an element of $D'(\mathbb{R}_+, \overline{\mathbb{R}}_+^n)$. $\square$

Recall that for any $(\mathcal{F}_t)_{t \geq 0}$-stopping time τ the sigma-algebra $\mathcal{F}_{\tau-}$ is defined as

$$\mathcal{F}_{\tau-} = \sigma\big(\tilde{\mathcal{F}}_0, \{\{\tau > t\} \cap \Gamma : \Gamma \in \mathcal{F}_t, t > 0\}\big).$$

LEMMA 2.7 (cf. [15], Remark 6.1). *Let* $(\tilde{\mathcal{F}}_t)_{t \geq 0}$ *be a standard system on* Ω. *Then for any increasing sequence* $(\tau_n)_{n \in \mathbb{N}}$ *of* $(\mathcal{F}_t)$*-stopping times the family* $(\mathcal{F}_{\tau_n -})_{n \in \mathbb{N}}$ *is also a standard system.*

NOTATION. When working on the subspace $(\Omega, \mathcal{F}_{\tau-})$ of $(\Omega, \mathcal{F})$, where τ is some $(\mathcal{F}_t)$-stopping time, we must restrict the filtration to $(\mathcal{F}_{t \wedge \tau -})_{t \geq 0}$, where with a slight abuse of notation we set $\mathcal{F}_{t \wedge \tau -} := \mathcal{F}_t \cap \mathcal{F}_{\tau-}$. In the following, we may also write $(\mathcal{F}_t)_{0 \leq t < \tau}$ for the filtration on $(\Omega, \mathcal{F}_{\tau-}, \mathsf{P})$.

Working with standard systems will allow us to derive for every strictly positive strict local P-martingale the existence of a measure Q on $(\Omega, \mathcal{F}_{\tau-}, (\mathcal{F}_t)_{0 \leq t < \tau})$, such that the reciprocal of the strict local P-martingale is a true Q-martingale. In Section 4, we will use this result to reduce calculations involving strict local martingales to the much easier case of true martingales.

From Theorem 4 in [6] and Proposition 6 in [30], we know that every continuous local martingale understood as the canonical process on $C(\mathbb{R}_+, \overline{\mathbb{R}}_+)$ gives rise to a new measure under which its reciprocal turns into a true martingale. In the context of arbitrage theory, similar results have recently been derived and applied by [14] and [36] for continuous processes in a Markovian setting. Theorem 2.12 below is an extension of these results to more general probability spaces and càdlàg processes. Its proof relies on the construction of the Föllmer measure (cf. [15] and [27]); nevertheless, we will give a detailed proof, since it is essential for the rest of the paper.

PROPOSITION 2.8. *Let $(\Omega, \mathcal{F}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \mathsf{P})$ be a filtered probability space and assume that $(\tilde{\mathcal{F}}_t)_{t\geq 0}$ is a standard system. Let X be a càdlàg local martingale on the space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$ with values in $(0,\infty)$ and $X_0 = 1$ P-almost surely. We define $\tau_n^X := \inf\{t \geq 0 : X_t > n\} \wedge n$ and $\tau^X = \lim_{n\to\infty} \tau_n^X$. Then there exists a unique probability measure Q on $(\Omega, \mathcal{F}_{\tau^X-}, (\mathcal{F}_{t\wedge\tau^X-})_{t\geq 0})$, such that $\frac{d\mathsf{P}}{d\mathsf{Q}}|_{\mathcal{F}_t\cap\mathcal{F}_{\tau^X-}} = \frac{1}{X_t}\mathbb{1}_{\{t<\tau^X\}}$ for all $t \geq 0$. Moreover, $1/X$ is a local Q-martingale on the interval $[0, \tau^X)$ which does not jump to zero Q-almost surely.*

PROOF. First, note that τ_n^X is an $(\mathcal{F}_t)_{t\geq 0}$-stopping time and the process $(X_{t\wedge\tau_n^X})_{t\geq 0}$ is a uniformly integrable $\{(\mathcal{F}_t)_{t\geq 0}, \mathsf{P}\}$-martingale for all $n \in \mathbb{N}$. Indeed, if (σ_m) is any localizing sequence for X such that $\mathbb{E}^{\mathsf{P}} X_{\sigma_m} = 1$ for all $m \in \mathbb{N}$, then

$$X_{\tau_n^X\wedge\sigma_m} \leq n \vee X_{\tau_n^X} \quad \text{and} \quad \mathbb{E}^{\mathsf{P}}(n \vee X_{\tau_n^X}) \leq n + \mathbb{E}^{\mathsf{P}} X_{\tau_n^X} \leq n+1$$

by the super-martingale property of X. By the dominated convergence theorem, we thus conclude that $\mathbb{E}^{\mathsf{P}} X_{\tau_n^X} = 1$, and thus (τ_n^X) is a localizing sequence as well.

Furthermore, $\mathsf{P}(\tau^X = \infty) = 1$, since a positive càdlàg local martingale does not explode almost surely. We define on $(\Omega, \mathcal{F}_{\tau_n^X})$ the probability measure $\tilde{\mathsf{Q}}_n$ via $\tilde{\mathsf{Q}}_n = X_{\tau_n^X} \cdot \mathsf{P}|_{\mathcal{F}_{\tau_n^X}}$ for all $n \in \mathbb{N}$. The family $(\tilde{\mathsf{Q}}_n)_{n\in\mathbb{N}}$ constitutes a consistent family of probability measures on $(\mathcal{F}_{\tau_n^X})_{n\geq 1}$: If $A \in \mathcal{F}_{\tau_n^X}$, then

$$\tilde{\mathsf{Q}}_{n+k}(A) = \mathbb{E}^{\mathsf{P}}(X_{\tau_{n+k}^X}\mathbb{1}_A) = \mathbb{E}^{\mathsf{P}}(X_{\tau_n^X}\mathbb{1}_A) = \tilde{\mathsf{Q}}_n(A),$$

that is, $\tilde{\mathsf{Q}}_{n+k}|_{\mathcal{F}_{\tau_n^X}} = \tilde{\mathsf{Q}}_n$ for all $n, k \in \mathbb{N}$. This induces a sequence of consistently defined measures $(\mathsf{Q}_n)_{n\in\mathbb{N}}$ on the sequence $(\mathcal{F}_{\tau_n^X-})_{n\in\mathbb{N}}$, which is a standard system by Lemma 2.7. Note that $\mathcal{F}_{\tau^X-} = \bigvee_{n\geq 1} \mathcal{F}_{\tau_n^X-}$, since $(\tau_n^X)_{n\geq 1}$ is increasing. We can thus apply Theorem 3.2 together with Theorem 4.1 in Chapter V of [31] (cf. also Theorem 6.2 in [15]), which yield the existence of a unique measure Q on $(\Omega, \mathcal{F}_{\tau^X-}, (\mathcal{F}_{t\wedge\tau^X-})_{t\geq 0})$ such that $\mathsf{Q}|_{\mathcal{F}_{\tau_n^X-}} = \mathsf{Q}_n = \tilde{\mathsf{Q}}_n|_{\mathcal{F}_{\tau_n^X-}}$. Moreover, since $\{\tau_n^X < \tau_m^X\} \in \mathcal{F}_{\tau_m^X-}$,

$$\begin{aligned}
&\mathsf{Q}(\tau_n^X < \tau^X)\\
&\quad= \lim_{m\to\infty} \mathsf{Q}(\tau_n^X < \tau_m^X) = \lim_{m\to\infty} \tilde{\mathsf{Q}}_m(\tau_n^X < \tau_m^X) = \lim_{m\to\infty} \mathbb{E}^{\mathsf{P}}(\mathbb{1}_{\{\tau_n^X<\tau_m^X\}} X_{\tau_m^X})\\
&\quad= \lim_{m\to\infty} \mathbb{E}^{\mathsf{P}}(\mathbb{1}_{\{\tau_n^X<\tau_m^X\}} X_{\tau_n^X}) = \mathbb{E}^{\mathsf{P}}(\mathbb{1}_{\{\tau_n^X<\tau^X\}} X_{\tau_n^X}) = \mathbb{E}^{\mathsf{P}}(X_{\tau_n^X}) = 1,
\end{aligned}$$

that is, $1/X$ does not jump to zero under Q. Therefore, if $\Lambda_n \in \mathcal{F}_{\tau_n^X}$, then

$$\begin{aligned}
\mathsf{Q}(\Lambda_n) &= \mathsf{Q}(\Lambda_n \cap \{\tau^X > \tau_n^X\}) = \lim_{m\to\infty} \mathsf{Q}(\Lambda_n \cap \{\tau_m^X > \tau_n^X\})\\
&= \lim_{m\to\infty} \mathbb{E}^{\mathsf{P}}(X_{\tau_m^X}\mathbb{1}_{\Lambda_n}\mathbb{1}_{\{\tau_m^X>\tau_n^X\}}) = \lim_{m\to\infty} \mathbb{E}^{\mathsf{P}}(X_{\tau_n^X}\mathbb{1}_{\Lambda_n}\mathbb{1}_{\{\tau_m^X>\tau_n^X\}})\\
&= \mathbb{E}^{\mathsf{P}}(X_{\tau_n^X}\mathbb{1}_{\Lambda_n}) = \tilde{\mathsf{Q}}_n(\Lambda_n).
\end{aligned}$$

Therefore, $\mathsf{Q}|_{\mathcal{F}_{\tau_n^X}} = \tilde{\mathsf{Q}}_n$ for all $n \in \mathbb{N}$.

Now let S be an $(\mathcal{F}_t)_{t\geq 0}$-stopping time. Note that $\{S < \tau_n^X\} \in \mathcal{F}_S$ and $\{S < \tau_n^X\} \in \mathcal{F}_{\tau_n^X}$. Thus,

$$\begin{aligned}\mathsf{Q}(S < \tau_n^X) &= \tilde{\mathsf{Q}}_n(S < \tau_n^X) = \mathbb{E}^{\mathsf{P}}(\mathbb{1}_{\{S<\tau_n^X\}} X_{\tau_n^X}) = \mathbb{E}^{\mathsf{P}}(\mathbb{1}_{\{S<\tau_n^X\}} \mathbb{E}^{\mathsf{P}}(X_{\tau_n^X} | \mathcal{F}_S)) \\ &= \mathbb{E}^{\mathsf{P}}(\mathbb{1}_{\{S<\tau_n^X\}} X_S).\end{aligned}$$

Since $\mathsf{P}(\tau_n^X < \tau^X = \infty) = 1$, taking the limit as $n \to \infty$ in the above equation yields

$$\mathsf{Q}(S < \tau^X) = \mathbb{E}^{\mathsf{P}}(\mathbb{1}_{\{S<\infty\}} X_S). \tag{1}$$

Applied to the stopping time $S_A := S\mathbb{1}_A + \infty\mathbb{1}_{A^c}$, where $A \in \mathcal{F}_S$, this gives

$$\mathsf{Q}(S < \tau^X, A) = \mathbb{E}^{\mathsf{P}}(\mathbb{1}_{A\cap\{S<\infty\}} X_S).$$

Especially, if S is finite P-almost surely, then $\mathsf{Q}(S < \tau^X, A) = \mathbb{E}^{\mathsf{P}}(X_S \mathbb{1}_A)$ for $A \in \mathcal{F}_S$. If $A \in \mathcal{F}_t \cap \mathcal{F}_{\tau^X-}$, then

$$\begin{aligned}\mathsf{P}(A) &= \lim_{n\to\infty} \mathsf{P}(A \cap \{t < \tau_n^X\}) = \lim_{n\to\infty} \mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_A \mathbb{1}_{\{t<\tau_n^X\}} \frac{1}{X_{\tau_n^X}}\right) \\ &= \lim_{n\to\infty} \mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_A \mathbb{1}_{\{t<\tau_n^X\}} \frac{1}{X_t}\right) = \mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_A \mathbb{1}_{\{t<\tau^X\}} \frac{1}{X_t}\right).\end{aligned}$$

Therefore, $\frac{d\mathsf{P}}{d\mathsf{Q}}|_{\mathcal{F}_t \cap \mathcal{F}_{\tau^X-}} = \frac{1}{X_t} \mathbb{1}_{\{t<\tau^X\}}$ for all $t \geq 0$.

Finally, note that because $(X_t^{\tau_n^X})_{t\geq 0}$ is a strictly positive uniformly integrable P-martingale for all $n \in \mathbb{N}$, $\mathsf{P}|_{\mathcal{F}_{\tau_n^X}} \sim \mathsf{Q}|_{\mathcal{F}_{\tau_n^X}}$ and

$$d\mathsf{P}|_{\mathcal{F}_{\tau_n^X}} = \frac{1}{X_{\tau_n^X}} d\mathsf{Q}|_{\mathcal{F}_{\tau_n^X}} \quad \Leftrightarrow \quad \left.\frac{d\mathsf{Q}}{d\mathsf{P}}\right|_{\mathcal{F}_{t\wedge\tau_n^X}} = X_{t\wedge\tau_n^X} \qquad \forall t \geq 0.$$

Thus,

$$\mathbb{E}^{\mathsf{Q}}\left(\frac{1}{X_{t\wedge\tau_n^X}} \Big| \mathcal{F}_s\right) = \mathbb{E}^{\mathsf{P}}\left(\frac{1}{X_{t\wedge\tau_n^X}} \cdot \frac{X_{t\wedge\tau_n^X}}{X_{s\wedge\tau_n^X}} \Big| \mathcal{F}_s\right) = \frac{1}{X_{s\wedge\tau_n^X}}$$

for $s \leq t$, that is, $\frac{1}{X}$ is a local Q-martingale on the interval $\bigcup_{n\in\mathbb{N}}[0, \tau_n^X] = [0, \tau^X)$. □

COROLLARY 2.9. *Under the assumptions of Proposition 2.8, X is a strict local P-martingale, if and only if $\mathsf{Q}(\tau^X < \infty) > 0$.*

PROOF. It follows directly from equation (1) that $\mathsf{Q}(t < \tau^X) = \mathbb{E}^{\mathsf{P}} X_t$, which is smaller than 1 for some t, iff X is a strict local martingale under P. □

REMARK 2.10. Corollary 2.9 makes clear why we cannot work with the natural augmentation of $(\tilde{\mathcal{F}}_t)_{t\geq 0}$. Indeed, we have $A_n := \{\tau^X \leq n\} \in \mathcal{F}_n \cap \mathcal{F}_{\tau^X-}$ and $\mathsf{P}(A_n) = 0$ for all $n \in \mathbb{N}$, while $\mathsf{Q}(A_n) > 0$ for some n if X is a strict local P-martingale. However, it is in general rather inconvenient to work without any augmentation, especially if one works with an uncountable number of stochastic processes. For this reason, a new kind of augmentation—called the (τ_n^X)-natural augmentation—is introduced in [23], which is suitable for the change of measure from P to Q undertaken here. Since for the financial applications in the second part of this paper the setup introduced above is already sufficient, we do not bother about this augmentation here and refer the interested reader to [23] for more technical details.

In the following, we extend the measure Q in an arbitrary way from $\mathcal{F}_{\tau^X-}$ to $\mathcal{F}_\infty = \bigvee_{t\geq 0} \tilde{\mathcal{F}}_t$. For notational convenience, we assume that $\mathcal{F} = \mathcal{F}_\infty$. In fact, it is always possible to extend a probability measure from $\mathcal{F}_{\tau^X-}$ to $\mathcal{F}$: since $(\Omega, \tilde{\mathcal{F}}_t)$ is a standard Borel space for every $t \geq 0$ and $(\Omega, \mathcal{F}_{\tau_n^X-})$ is a standard Borel space for all $n \in \mathbb{N}$ by Lemma 2.7, it follows from Theorem 4.1 in [31] that $(\Omega, \mathcal{F})$ and $(\Omega, \mathcal{F}_{\tau^X-})$ are also standard Borel spaces. Especially, they are countably generated which allows us to apply Theorem 3.1 of [12] that guarantees an extension of Q from $\mathcal{F}_{\tau^X-}$ to $\mathcal{F}$. Moreover, it does not matter for the results how we extend it, because all events that happen with positive probability under P take place before time τ^X under Q almost surely. However, if Y is any process on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$, then Y_t is only defined on $\{t < \tau^X\}$ under Q. Especially, if Y is a P-semi-martingale, then $Y^{\tau_n^X}$ is a Q-semi-martingale for each $n \in \mathbb{N}$ as follows from Girsanov's theorem, since $\mathsf{Q}|_{\mathcal{F}_{\tau_n^X}} \sim \mathsf{P}|_{\mathcal{F}_{\tau_n^X}}$. Therefore, Y is a Q-semi-martingale on the stochastic interval $\bigcup_{n\in\mathbb{N}}[0, \tau_n^X]$ or a "semi-martingale up to time τ^X" in the terminology of [18]. We note that in general it may not be possible to extend Y to the whole positive real line under Q in such a way that Y remains a semi-martingale. Indeed, according to Proposition 5.8 of [18] such an extension is possible if and only if Y_{τ^X-} exists in $\mathbb{R}_+$ Q-almost surely. We define the process $\tilde{Y}$ as

$$\tilde{Y}_t = \begin{cases} Y_t, & t < \tau^X, \\ \liminf\limits_{s\to\tau^X, s<\tau^X, s\in\mathbb{Q}} Y_s, & \tau^X \leq t < \infty. \end{cases} \tag{2}$$

Note that $\tilde{Y}_t = Y_t$ on $\{t < \tau^X\}$. The above definition specifies an extension of the process Y, which is a priori only defined up to time τ^X, to the whole positive real line. In the following, we will work with this extension.

LEMMA 2.11. *Under the assumptions of Proposition 2.8, we have* $\frac{1}{\tilde{X}_t} = \frac{1}{X_t}\mathbb{1}_{\{t<\tau^X\}}$. *Furthermore, the process* $(\frac{1}{\tilde{X}_t})_{t\geq 0}$ *is a true* Q*-martingale for any extension of* Q *from* $\mathcal{F}_{\tau^X-}$ *to* $\mathcal{F}$.

PROOF. First, note that Q-almost surely

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{X_{t\wedge\tau_n^X}} &= \limsup_{n\to\infty}\left(\frac{1}{X_t}\mathbb{1}_{\{t<\tau_n^X\}}+\frac{1}{X_{\tau_n^X}}\mathbb{1}_{\{t\geq\tau_n^X\}}\right)\\ &\leq \frac{1}{X_t}\mathbb{1}_{\{t<\tau^X\}}+\limsup_{n\to\infty}\frac{1}{n}\mathbb{1}_{\{t\geq\tau_n^X\}}=\frac{1}{X_t}\mathbb{1}_{\{t<\tau^X\}}\end{aligned}$$

and

$$\begin{aligned}\liminf_{n\to\infty}\frac{1}{X_{t\wedge\tau_n^X}} &= \liminf_{n\to\infty}\left(\frac{1}{X_t}\mathbb{1}_{\{t<\tau_n^X\}}+\frac{1}{X_{\tau_n^X}}\mathbb{1}_{\{t\geq\tau_n^X\}}\right)\\ &\geq \liminf_{n\to\infty}\frac{1}{X_t}\mathbb{1}_{\{t<\tau_n^X\}}=\frac{1}{X_t}\mathbb{1}_{\{t<\tau^X\}}.\end{aligned}$$

Thus, $\frac{1}{\tilde{X}_t}=\frac{1}{X_t}\mathbb{1}_{\{t<\tau^X\}}$. Furthermore,

$$\begin{aligned}0 &\leq \frac{1}{X_{\tau^X-}}\mathbb{1}_{\{\tau^X<\infty\}}=\lim_{k\to\infty}\frac{1}{X_{\tau^X-}}\mathbb{1}_{\{\tau^X<k\}}=\lim_{k\to\infty}\lim_{n\to\infty}\frac{1}{X_{\tau_n^X}}\mathbb{1}_{\{\tau^X<k\}}\\ &\leq \lim_{k\to\infty}\lim_{n\to\infty}\frac{1}{n}\mathbb{1}_{\{\tau^X<k\}}=0\end{aligned}$$

implies that $X_{\tau^X-}=\infty$ on $\{\tau^X<\infty\}$ Q-almost surely. From the proof of Proposition 2.8, we know that $\frac{1}{X^{\tau_n^X}}$ is a true Q-martingale for all $n\in\mathbb{N}$. By the definition of τ_n^X, we have for any integer $n\geq t$

$$X_{t\wedge\tau_n^X}=\tilde{X}_{t\wedge\tau_n^X}=\tilde{X}_{t\wedge\inf\{s\geq 0:\,\tilde{X}_s>n\}}\geq\tilde{X}_t\wedge 1\quad\Rightarrow\quad\frac{1}{X_{t\wedge\tau_n^X}}\leq\frac{1}{\tilde{X}_t\wedge 1}=1\vee\frac{1}{\tilde{X}_t}.$$

Because

$$\mathbb{E}^{\mathsf{Q}}\left(\frac{1}{\tilde{X}_t}\right)=\mathbb{E}^{\mathsf{Q}}\left(\liminf_{n\to\infty}\frac{1}{X_{t\wedge\tau_n^X}}\right)\leq\liminf_{n\to\infty}\mathbb{E}^{\mathsf{Q}}\left(\frac{1}{X_{t\wedge\tau_n^X}}\right)=1,$$

the dominated convergence theorem implies that for all $0\leq s\leq t$

$$\begin{aligned}\mathbb{E}^{\mathsf{Q}}\left(\frac{1}{\tilde{X}_t}\Big|\mathcal{F}_s\right) &= \mathbb{E}^{\mathsf{Q}}\left(\lim_{n\to\infty}\frac{1}{X_{t\wedge\tau_n^X}}\Big|\mathcal{F}_s\right)=\lim_{n\to\infty}\mathbb{E}^{\mathsf{Q}}\left(\frac{1}{X_{t\wedge\tau_n^X}}\Big|\mathcal{F}_s\right)\\ &= \lim_{n\to\infty}\frac{1}{X_{s\wedge\tau_n^X}}=\frac{1}{\tilde{X}_s}.\end{aligned}$$

□

To simplify notation, we identity in the following the process X with $\tilde{X}$. We summarize our results so far in the following theorem.

THEOREM 2.12. *Let $(\Omega,\mathcal{F},(\tilde{\mathcal{F}}_t)_{t\geq 0},\mathsf{P})$ be a filtered probability space and assume that $(\tilde{\mathcal{F}}_t)_{t\geq 0}$ is a standard system. Let X be a càdlàg local martingale on*

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$ *with values in* $(0,\infty)$ *and* $X_0 = 1$ P-*almost surely. We define* $\tau_n^X := \inf\{t \geq 0 : X_t > n\} \wedge n$ *and* $\tau^X = \lim_{n\to\infty} \tau_n^X$. *Then there exists a probability measure* Q *on* $(\Omega, \mathcal{F}_\infty)$ *such that* $1/X$ *is a* Q-*martingale, which does not jump to zero* Q-*almost surely, and such that* $\mathsf{Q}(A, \tau^X > t) = \mathbb{E}^{\mathsf{P}}(X_t \mathbb{1}_A)$ *for all* $t \geq 0$ *and* $A \in \mathcal{F}_t$. *In particular,* $\mathsf{P}|_{\mathcal{F}_t} \ll \mathsf{Q}|_{\mathcal{F}_t}$ *for all* $t \geq 0$.

Note that in the case where X is a strict local P-martingale Theorem 2.12 is a precise converse to Theorem 2.1, if one identifies X of Theorem 2.12 with $1/Y$ of Theorem 2.1.

3. Examples.

In this section, we shed new light on some known examples of strict local martingales by applying the theory from the last section for illustration.

3.1. *Continuous local martingales.* For the following examples, we work on the path space $C'(\mathbb{R}_+, \overline{\mathbb{R}}_+)$ with W denoting the coordinate process. Here, $(\mathcal{F}_t)_{t\geq 0}$ is the right-continuous augmentation of the canonical filtration generated by the coordinate process and P is Wiener measure.

3.1.1. *Exponential local martingales.* Suppose that X has dynamics

$$dX_t = X_t b(Y_t)\,dW_t, \qquad X_0 = 1,$$

where Y is assumed to be a (possibly explosive) diffusion with

$$dY_t = \mu(Y_t)\,dt + \sigma(Y_t)\,dW_t, \qquad Y_0 = y \in \mathbb{R}.$$

Here, $b(\cdot)$, $\mu(\cdot)$ and $\sigma(\cdot)$ are chosen such that both SDEs allow for strong solutions and guarantee X to be strictly positive. Exponential local martingales of this type are further studied in [28]. Under Q the dynamics of $\frac{1}{X}$ up to time τ^X are

$$d\left(\frac{1}{X_t}\right) = -\frac{b(Y_t)}{X_t}\,dW_t^{\mathsf{Q}}$$

for a Q-Brownian motion W^{Q} defined up to time τ^X, and the Q-dynamics of Y_t up to time τ^X are

$$dY_t = \left[\mu(Y_t) + \sigma(Y_t)b(Y_t)\right]dt + \sigma(Y_t)\,dW_t^{\mathsf{Q}}.$$

Notably, the criterion whether X is a strict local or a true P-martingale from [28], Theorem 2.1, is deterministic and only involves the functions b, σ and μ via the scale function of the original diffusion Y under P and an auxiliary diffusion $\tilde{Y}$, whose dynamics are identical with the Q-dynamics of Y stated above.

3.1.2. *Diffusions in natural scale.* We now take X to be a local P-martingale of the form

$$dX_t = \sigma(X_t)\,dW_t, \qquad X_0 = 1,$$

assuming that $\sigma(x)$ is locally bounded and bounded away from zero for $x > 0$ and $\sigma(0) = 0$. Using the results from [8], we know that X is strictly positive, whenever

$$\int_0^1 \frac{x}{\sigma^2(x)}\,dx = \infty,$$

which we shall assume in the following. Furthermore, X is a strict local martingale, if and only if

$$\int_1^\infty \frac{x}{\sigma^2(x)}\,dx < \infty.$$

We know that $\frac{1}{X}$ is a Q-martingale, where $\frac{d\mathsf{P}}{d\mathsf{Q}}|_{\mathcal{F}_t} = \frac{1}{X_t}$, with decomposition

$$d\left(\frac{1}{X_t}\right) = -\frac{\sigma(X_t)}{X_t^2}\,dW_t^{\mathsf{Q}} = \overline{\sigma}\left(\frac{1}{X_t}\right)dW_t^{\mathsf{Q}}$$

for a Q-Brownian motion W^{Q} defined up to time τ^X and $\overline{\sigma}(y) := -y^2 \cdot \sigma(\frac{1}{y})$. Note that

$$\int_1^\infty \frac{y}{\overline{\sigma}^2(y)}\,dy = \int_0^1 \frac{x}{\sigma^2(x)}\,dx = \infty,$$

which confirms that $\frac{1}{X}$ is a true Q-martingale. We see that, if X is a strict local martingale under P, then

$$\int_0^1 \frac{y}{\overline{\sigma}^2(y)}\,dy = \int_1^\infty \frac{x}{\sigma^2(x)}\,dx < \infty,$$

that is, $\frac{1}{X}$ hits zero in finite time Q-almost surely.

3.2. *Jump example.* [4] Let $\Omega = D'(\mathbb{R}_+, \overline{\mathbb{R}})$ with $(\xi_t)_{t\geq 0}$ denoting the coordinate process and $(\mathcal{F}_t)_{t\geq 0}$ being the right-continuous augmentation of the canonical filtration generated by the coordinate process. Assume that under P, $(\xi_t)_{t\geq 0}$ is a one-dimensional Lévy process with $\xi_0 = 0$, $\mathbb{E}^{\mathsf{P}} \exp(b\xi_t) = \exp(t\rho(b)) < \infty$ for all $t \geq 0$ and characteristic exponent

$$\Psi(\lambda) = ia\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{\mathbb{R}} \bigl(1 - e^{i\lambda x} + i\lambda x \mathbb{1}_{\{|x|<1\}}\bigr)\pi(dx),$$

[4] This example is taken from [4]. However, we corrected a small mistake concerning the time-scaling.

where $a \in \mathbb{R}$, $\sigma^2 \geq 0$ and π is a positive measure on $\mathbb{R} \setminus \{0\}$ such that $\int (1 \wedge |x|^2)\pi(dx) < \infty$. Define

$$X_t = Y_t^b \exp\left(-\rho(b) \int_0^t \frac{ds}{Y_s}\right),$$

where $(Y_t)_{t \geq 0}$ is a semi-stable Markov process, that is, $(\frac{1}{c} Y_{ct}^{(x)})_{t \geq 0} \stackrel{(d)}{=} (Y_t^{(xc^{-1})})_{t \geq 0}$ for all $c > 0$, implicitly defined via

$$\exp(\xi_t) = Y_{\int_0^t \exp(\xi_s)\, ds}.$$

Following [4], $(X_t)_{t \geq 0}$ is a positive strict local martingale if a and b satisfy

$$-a + \int_{|x|>1} x\pi(dx) \geq 0,$$

$$-a + b\sigma^2 - \int_{|x|<1} x(1 - e^{bx})\pi(dx) + \int_{|x|>1} xe^{bx}\pi(dx) < 0.$$

Furthermore, under the new measure Q the process

$$\frac{1}{X_t} = Y_t^{-b} \exp\left(\rho(b) \int_0^t \frac{ds}{Y_s}\right)$$

is a true martingale, where now $(\xi_t)_{t \geq 0}$ has characteristic exponent $\tilde{\Psi}$ with

$$\tilde{\Psi}(u) = \Psi(u - ib) - \Psi(-ib).$$

4. Application to financial bubbles I: Decomposition formulas. In this section, we apply our results to option pricing in the presence of strict local martingales. For this, we assume that the following standing assumption (S) holds throughout the entire section:

(S) *X is assumed to be a càdlàg strictly positive local martingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathsf{P})$, whose filtration is the right-continuous augmentation of a standard system and $\mathcal{F} = \bigvee_{t \geq 0} \mathcal{F}_t$. We assume that $X_0 = 1$ and set $\tau_n^X = \inf\{t \geq 0 | X_t > n\} \wedge n$ for all $n \in \mathbb{N}$ and $\tau^X = \lim_{n \to \infty} \tau_n^X$. Furthermore, we denote by Q any extension to $(\Omega, \mathcal{F})$ of the measure associated with X, defined in Theorem 2.12.*

We consider a financial market model which satisfies the NFLVR property as defined in [7]. We denote by P an equivalent local martingale measure (ELMM). Assuming that the interest rate equals zero, we interpret X as the (discounted) stock price process, which is a local martingale under P. In this context, the question of whether X is a strict local or a true P-martingale determines whether there exists a stock price bubble. If X is a strict local P-martingale, the fundamental value of the asset (given by the conditional expectation) deviates from its actual market price X. Several authors (cf., e.g., [5, 21, 22, 30]) have interpreted this as the existence of a stock price bubble, which we formally define as follows.

DEFINITION 4.1. With the previous notation, the *asset price bubble* for the stock price process X between time $t \geq 0$ and time $T \geq t$ is equal to the $\mathcal{F}_t$-measurable random variable

$$\gamma_X(t, T) := X_t - \mathbb{E}^{\mathsf{P}}(X_T | \mathcal{F}_t).$$

REMARK 4.2. For $t = 0$, we recover the 'default' function $\gamma_X(0, T) = X_0 - \mathbb{E}^{\mathsf{P}} X_T$ of the local martingale X, which was introduced in [11]. Here, the term 'default' refers to the locality property of X and measures its failure of being a martingale. In [10, 11], the authors derive several expressions for the default function in terms of the first hitting time, the local time and the last passage time of the local martingale.

REMARK 4.3. Note that the above definition of a bubble depends on the measure P, which may be viewed as the subjective valuation measure of a certain economic agent. From the agent's point of view, the asset price contains a bubble. Only in a complete market, that is, if and only if P is the unique ELMM, the notion of a bubble becomes universal without any element of subjectivity.

In Proposition 7 of [30], the price of a nonpath-dependent option written on a stock, whose price process is a (strict) local martingale, is decomposed into a "normal" ("nonbubble") term and a default term. In the following, we give an extension of this theorem to a certain class of path-dependent options. For this, let us introduce the following notation for all $k \in \mathbb{N}$:

$$\mathbb{R}^k_+ = \{x \in \mathbb{R}^k : x_l \geq 0, l = 1, \dots, k\}, \qquad \mathbb{R}^k_{++} = \{x \in \mathbb{R}^k : x_l > 0, l = 1, \dots, k\}.$$

THEOREM 4.4. *Let $0 \leq t_1 < t_2 < \cdots < t_n < \infty$ and consider a Borel-measurable nonnegative function $h : \mathbb{R}^n_{++} \to \mathbb{R}_+$. Define the function $g(x) := x_n \cdot h(\frac{1}{x_1}, \dots, \frac{1}{x_n})$ for all $x = (x_1, \dots, x_n) \in \mathbb{R}^n_{++}$. Then*

$$\mathbb{E}^{\mathsf{P}} h(X_{t_1}, \dots, X_{t_n}) = \mathbb{E}^{\mathsf{Q}}\left(g\left(\frac{1}{X_{t_1}}, \dots, \frac{1}{X_{t_n}}\right)\mathbb{1}_{\{\tau^X > t_n\}}\right).$$

Now suppose that the following limits exist in $\mathbb{R}_+$ for $y_i \in \mathbb{R}_{++}$, $i = 1, \dots, n-1$:

$$\lim_{|z| \to 0} g(y_1, \dots, y_k; z_1, \dots, z_{n-k}) =: \eta_k(y_1, \dots, y_k), \qquad k = 1, \dots, n-1,$$

$$\lim_{|z| \to 0} g(z_1, \dots, z_n) =: \eta_0.$$

Define $\overline{g} : A \to \mathbb{R}_+$ as the extension of g from $\mathbb{R}^n_{++}$ to $A \subset \mathbb{R}^n_+$, where A is defined as $A := \{x \in \mathbb{R}^n_+ : \text{if } x_k = 0 \text{ for some } k = 1, \dots, n, \text{ then } x_l = 0\ \forall l \geq k\}$. Then

$$\mathbb{E}^{\mathsf{P}} h(X_{t_1}, \dots, X_{t_n}) = \mathbb{E}^{\mathsf{Q}} \overline{g}\left(\frac{1}{X_{t_1}}, \dots, \frac{1}{X_{t_n}}\right) - \sum_{k=0}^{n-1} \mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_{\{t_k < \tau^X \leq t_{k+1}\}} \cdot \eta_k(X^k)\right), \tag{3}$$

where we set $t_0 = 0$ and $X^k = (\frac{1}{X_{t_1}}, \dots, \frac{1}{X_{t_k}})$ for $k = 1, \dots, n-1$, $X^0 \equiv 0$.

In particular, if $\eta_k(\cdot) \equiv c_k$, $k = 1, \dots, n-1$, *are constant, then*

$$\mathbb{E}^{\mathsf{P}} h(X_{t_1}, \dots, X_{t_n}) = \mathbb{E}^{\mathsf{Q}} \overline{g}\left(\frac{1}{X_{t_1}}, \dots, \frac{1}{X_{t_n}}\right) - \sum_{k=0}^{n-1} c_k \cdot \mathsf{Q}(t_k < \tau^X \le t_{k+1}). \tag{4}$$

PROOF. First, note that

$$\mathbb{1}_{\{\tau^X > t_n\}} = \mathbb{1}_{\{\tau^X > t_1\}} \mathbb{1}_{\{\tau^X > t_2\}} \cdots \mathbb{1}_{\{\tau^X > t_{n-1}\}} \mathbb{1}_{\{\tau^X > t_n\}}.$$

Using the change of measure $d\mathsf{P}|_{\mathcal{F}_{t_n}} = \frac{1}{X_{t_n}}\, d\mathsf{Q}|_{\mathcal{F}_{t_n}}$ on $\{\tau^X > t_n\}$, we deduce

$$\begin{aligned}
&\mathbb{E}^{\mathsf{P}} h(X) \\
&\quad = \mathbb{E}^{\mathsf{Q}}\left(g\left(\frac{1}{X}\right)\mathbb{1}_{\{\tau^X > t_n\}}\right) \\
&\quad = \mathbb{E}^{\mathsf{Q}}\left(g\left(\frac{1}{X}\right)\mathbb{1}_{\{\tau^X > t_1\}} \cdots \mathbb{1}_{\{\tau^X > t_n\}}\right) \\
&\quad = \mathbb{E}^{\mathsf{Q}}\Bigg(\mathbb{1}_{\{\tau^X > t_1\}} \mathbb{E}^{\mathsf{Q}}\Bigg(\mathbb{1}_{\{\tau^X > t_2\}} \cdots \\
&\qquad\quad \times \mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_{\{\tau^X > t_{n-1}\}} \mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_{\{\tau^X > t_n\}} g\left(\frac{1}{X}\right)\Big|\mathcal{F}_{t_{n-1}}\right)\Big|\mathcal{F}_{t_{n-2}}\right) \cdots \Big|\mathcal{F}_{t_2}\Bigg)\Big|\mathcal{F}_{t_1}\Bigg)\Bigg).
\end{aligned}$$

Because on $\{\tau^X > t_{n-1}\}$, we have

$$\begin{aligned}
&\mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_{\{\tau^X > t_n\}} g\left(\frac{1}{X}\right)\Big|\mathcal{F}_{t_{n-1}}\right) \\
&\qquad = \mathbb{E}^{\mathsf{Q}}\left(\overline{g}\left(\frac{1}{X}\right)\Big|\mathcal{F}_{t_{n-1}}\right) - \mathbb{E}^{\mathsf{Q}}(\mathbb{1}_{\{t_{n-1} < \tau^X \le t_n\}} \eta_{n-1}(X^{n-1}) | \mathcal{F}_{t_{n-1}}),
\end{aligned}$$

it follows that

$$\begin{aligned}
\mathbb{E}^{\mathsf{P}} h(X) = {} & \mathbb{E}^{\mathsf{Q}}\Bigg(\mathbb{1}_{\{\tau^X > t_1\}} \mathbb{E}^{\mathsf{Q}}\Bigg(\mathbb{1}_{\{\tau^X > t_2\}} \cdots \\
& \times \mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_{\{\tau^X > t_{n-2}\}} \mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_{\{\tau^X > t_{n-1}\}} \overline{g}\left(\frac{1}{X}\right)\Big|\mathcal{F}_{t_{n-2}}\right) \cdots \Big|\mathcal{F}_{t_1}\right)\Bigg) \\
& - \mathbb{E}^{\mathsf{Q}}(\mathbb{1}_{\{\tau^X > t_1\}} \mathbb{1}_{\{\tau^X > t_2\}} \cdots \mathbb{1}_{\{\tau^X > t_{n-1}\}} \mathbb{1}_{\{t_{n-1} < \tau^X \le t_n\}} \eta_{n-1}(X^{n-1}))\Bigg).
\end{aligned}$$

Similarly, on $\{\tau^X > t_{n-2}\}$ we have

$$\begin{aligned}
&\mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_{\{\tau^X > t_{n-1}\}} \overline{g}\left(\frac{1}{X}\right)\Big|\mathcal{F}_{t_{n-2}}\right) \\
&\qquad = \mathbb{E}^{\mathsf{Q}}\left(\overline{g}\left(\frac{1}{X}\right)\Big|\mathcal{F}_{t_{n-2}}\right) - \mathbb{E}^{\mathsf{Q}}(\mathbb{1}_{\{t_{n-2} < \tau^X \le t_{n-1}\}} \eta_{n-2}(X^{n-2}) | \mathcal{F}_{t_{n-2}}),
\end{aligned}$$

and we deduce that

$$\mathbb{E}^{\mathsf{P}} h(X) = \mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_{\{\tau^X > t_1\}} \mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_{\{\tau^X > t_2\}} \cdots \mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_{\{\tau^X > t_{n-2}\}} \overline{g}\left(\frac{1}{X}\right) \middle| \mathcal{F}_{t_{n-3}}\right) \cdots \middle| \mathcal{F}_{t_1}\right)\right)$$
$$- \mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_{\{t_{n-2} < \tau^X \le t_{n-1}\}} \eta_{n-2}(X^{n-2})\right) - \mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_{\{t_{n-1} < \tau^X \le t_n\}} \eta_{n-1}(X^{n-1})\right).$$

Iterating this procedure results in

$$\begin{aligned}\mathbb{E}^{\mathsf{P}} h(X) &= \mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_{\{\tau^X > t_1\}} \overline{g}\left(\frac{1}{X}\right)\right) - \sum_{k=1}^{n-1} \mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_{\{t_k < \tau^X \le t_{k+1}\}} \eta_k(X^k)\right) \\ &= \mathbb{E}^{\mathsf{Q}} \overline{g}\left(\frac{1}{X}\right) - \mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_{\{\tau^X \le t_1\}} \eta_0\right) - \sum_{k=1}^{n-1} \mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_{\{t_k < \tau^X \le t_{k+1}\}} \eta_k(X^k)\right). \qquad \square\end{aligned}$$

REMARK 4.5. The sum following the minus sign in the above decompositions (3) and (4) will be called the *default term*. This is motivated by the following observation:

$$\begin{aligned}\gamma_X(t,T) &= X_t - \mathbb{E}^{\mathsf{P}}(X_T | \mathcal{F}_t) = X_t - X_t \cdot \mathsf{Q}(\tau^X > T | \mathcal{F}_t) \\ &= X_t \cdot \mathsf{Q}(\tau^X \le T | \mathcal{F}_t) \qquad \mathsf{P}\text{-a.s.}\end{aligned} \tag{5}$$

Here, the second equality in (5) is justified by the following calculation, valid for any $\mathcal{F}_t$-measurable set A:

$$\begin{aligned}\mathbb{E}^{\mathsf{P}}(\mathbb{1}_A X_T) &= \mathsf{Q}(A, \tau^X > T) = \mathsf{Q}(A, \tau^X > t, \tau^X > T) \\ &= \mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_{\{A, \tau^X > t\}} \mathsf{Q}(\tau^X > T | \mathcal{F}_t)\right) \\ &= \mathbb{E}^{\mathsf{P}}\left(\mathbb{1}_A X_t \cdot \mathsf{Q}(\tau^X > T | \mathcal{F}_t)\right) \qquad \mathsf{P}\text{-a.s.}\end{aligned}$$

Taking expectations with respect to P in (5) yields

$$\begin{aligned}\mathbb{E}^{\mathsf{P}} \gamma_X(t,T) &= \mathbb{E}^{\mathsf{P}}\left(X_t \cdot \mathsf{Q}(\tau^X \le T | \mathcal{F}_t)\right) = \mathbb{E}^{\mathsf{Q}}\left(\mathbb{1}_{\{\tau^X > t\}} \mathsf{Q}(\tau^X \le T | \mathcal{F}_t)\right) \\ &= \mathsf{Q}(t < \tau^X \le T).\end{aligned}$$

Thus, the default term is directly related to the expected bubble of the underlying. It measures how much the failure of the martingale property by X affects the option price. If X is a true martingale, it will equal zero.

The convergence conditions that must be fulfilled in Theorem 4.4 may seem to be rather strict. However, below we give a few examples of options which satisfy those conditions.

EXAMPLE 4.6. Let us consider a modified call option with maturity T and strike K, where the holder has the option to reset the strike value to the current stock price at certain points in time $t_1 < t_2 < \cdots < t_n < T$, that is, the payoff profile of the option is given by

$$H(X) = \bigl(X_T - \min(K, X_{t_1}, X_{t_2}, \dots, X_{t_n})\bigr)^+.$$

With the notation in Theorem 4.4 and setting $t_{n+1} = T$, it follows that

$$\eta_0 = \eta_1 = \cdots = \eta_n = 1$$

and the option value can be decomposed as

$$\begin{aligned}\mathbb{E}^{\mathsf{P}} h(X) &= \mathbb{E}^{\mathsf{Q}}\biggl(1 - \frac{1}{X_T} \cdot \min(K, X_{t_1}, \dots, X_{t_n})\biggr)^+ - \sum_{k=0}^{n} \mathsf{Q}\bigl(t_k < \tau^X \le t_{k+1}\bigr) \\ &= \mathbb{E}^{\mathsf{Q}}\biggl(1 - \frac{1}{X_T} \cdot \min(K, X_{t_1}, \dots, X_{t_n})\biggr)^+ - \gamma_X(0, T).\end{aligned}$$

Therefore, this modified call option has the same default as the normal call option (cf. equation (14) in [30]).

EXAMPLE 4.7. Let us consider a call option on the ratio of the stock price at times T and $S \le T$ with strike $K \in \mathbb{R}_+$, that is,

$$h(X) = \biggl(\frac{X_T}{X_S} - K\biggr)^+$$

for $S < T \in \mathbb{R}_+$. In this case,

$$\eta_0 = 0, \qquad \eta_1(y) = y$$

and the decomposition of the option value is given by

$$\mathbb{E}^{\mathsf{P}} h(X) = \mathbb{E}^{\mathsf{Q}}\biggl(\frac{1}{X_S} - \frac{K}{X_T}\biggr)^+ - \mathbb{E}^{\mathsf{Q}}\biggl(\mathbb{1}_{\{S < \tau^X \le T\}} \frac{1}{X_S}\biggr).$$

EXAMPLE 4.8. A *chooser option* with maturity T and strike K entitles the holder to decide at time $S < T$, whether the option is a call or a put. He will choose the call, if its value is as least as high as the value of the put option with strike K and maturity T at time S. However, in the presence of asset price bubbles, that is, when the underlying is a strict local martingale, put-call-parity does not hold, but instead we have

$$\mathbb{E}^{\mathsf{P}}\bigl((X_T - K)^+ | \mathcal{F}_S\bigr) - \mathbb{E}^{\mathsf{P}}\bigl((K - X_T)^+ | \mathcal{F}_S\bigr) = \mathbb{E}^{\mathsf{P}}(X_T | \mathcal{F}_S) - K.$$

Therefore, the payoff of the chooser option equals

$$h(X_S, X_T) = (X_T - K)^+ \mathbb{1}_{\{\mathbb{E}^{\mathsf{P}}(X_T | \mathcal{F}_S) \ge K\}} + (K - X_T)^+ \mathbb{1}_{\{\mathbb{E}^{\mathsf{P}}(X_T | \mathcal{F}_S) < K\}}.$$

Let us assume that X is Markovian. Then we can express $\mathbb{E}^{\mathsf{P}}(X_T|\mathcal{F}_S)$ as a function of X_S, say $\mathbb{E}^{\mathsf{P}}(X_T|\mathcal{F}_S) = m(X_S)$, and the limits defined in Theorem 4.4 exist, if m is monotone for large values, and equal

$$\eta_1(y) = \mathbb{1}_{\{m(1/y)\geq K\}}, \qquad \eta_0 = \lim_{x\to\infty} \mathbb{1}_{\{m(x)\geq K\}}.$$

Thus, the value of the chooser option can be decomposed as

$$\begin{aligned}\mathbb{E}^{\mathsf{P}} h(X_S, X_T) = \mathbb{E}^{\mathsf{Q}}\left(\frac{h(X_S, X_T)}{X_T}\right) - \mathsf{Q}\big(m(X_S) \geq K, S < \tau^X \leq T\big) \\ - \lim_{x\to\infty} \mathbb{1}_{\{m(x)\geq K\}} \mathsf{Q}\big(\tau^X \leq S\big).\end{aligned}$$

If X is the reciprocal of a BES(3)-process under P, it is calculated in Section 2.2.2 in [5] that

$$m(X_S) = \mathbb{E}^{\mathsf{P}}(X_T|X_S) = X_S\left(1 - 2\Phi\left(-\frac{1}{X_S\sqrt{T-S}}\right)\right).$$

Therefore,

$$\begin{aligned}\lim_{x\to\infty} m(x) = \lim_{x\to\infty} \mathbb{E}^{\mathsf{P}}(X_T|X_S = x) = \lim_{x\to\infty} 2\varphi\left(-\frac{1}{x\sqrt{T-S}}\right)\frac{1}{\sqrt{T-S}} \\ = \frac{\sqrt{2}}{\sqrt{\pi(T-S)}}\end{aligned}$$

and

$$\eta_1(y) = \mathbb{1}_{\{1/y(1-2\Phi(-y/\sqrt{T-S}))\geq K\}}, \qquad \eta_0 = \mathbb{1}_{\{\sqrt{2}/\sqrt{\pi(T-S)}>K\}}.$$

REMARK 4.9. Here, we take the approach of valuating options by risk-neutral expectations. While there may be other approaches, risk-neutral expectations do not create arbitrage in the market, even though the stock itself is not priced that way. Indeed, P remains an ELMM in the enlarged market also after adding any asset $V_t = \mathbb{E}^{\mathsf{P}}[H|\mathcal{F}_t]$, $t \leq T$, for some integrable $H \in \mathcal{F}_T$. Interestingly, by choosing $H = X_T$ we may have $V_0 < X_0$ (in the case where X is a strict local martingale). But it is impossible to short X and take a long position on V all the way up to T because of credit constraints, therefore, NFLVR is not violated.

In the following, we give another extension of Proposition 7 in [30] to Barrier options, that is, we allow the options to be knocked-in or knocked-out by passing some pre-specified level.

THEOREM 4.10. *Consider any nonnegative Borel-measurable function* $h:\mathbb{R}_{++} \to \mathbb{R}_+$ *and define* $g(x) = x \cdot h(\frac{1}{x})$ *for* $x > 0$. *Suppose that* $\lim_{x\to 0} g(x) =: \eta < \infty$ *exists and denote by* $\overline{g}:\mathbb{R}_+ \to \mathbb{R}_+$ *the extension of* g *with* $\overline{g}(0) = \eta$. *Define* $\hat{m}_T^X := \min_{t\leq T} X_t$, $m_T^X := \max_{t\leq T} X_t$ *as well as* $T_a^X := \inf\{t \geq 0 : X_t \leq a\}$ *for*

$a \in \mathbb{R}_+$. Then for any bounded stopping time T and for any real numbers $D \leq 1$ and $F \geq 1$:

$$\text{(DI)}\quad \mathbb{E}^{\mathsf{P}}\big(h(X_T)\mathbb{1}_{\{\hat{m}_T^X \leq D\}}\big) = \mathbb{E}^{\mathsf{Q}}\bigg(g\bigg(\frac{1}{X_T}\bigg)\mathbb{1}_{\{\hat{m}_T^X \leq D\}}\bigg) - \eta \cdot \mathsf{Q}\big(T_D^X < \tau^X \leq T\big),$$

$$\text{(DO)}\quad \mathbb{E}^{\mathsf{P}}\big(h(X_T)\mathbb{1}_{\{\hat{m}_T^X \geq D\}}\big) = \mathbb{E}^{\mathsf{Q}}\bigg(g\bigg(\frac{1}{X_T}\bigg)\mathbb{1}_{\{\hat{m}_T^X \geq D\}}\bigg) - \eta \cdot \mathsf{Q}\big(T_D^X = \infty, \tau^X \leq T\big),$$

$$\text{(UI)}\quad \mathbb{E}^{\mathsf{P}}\big(h(X_T)\mathbb{1}_{\{m_T^X \geq F\}}\big) = \mathbb{E}^{\mathsf{Q}}\bigg(g\bigg(\frac{1}{X_T}\bigg)\mathbb{1}_{\{m_T^X \geq F\}}\bigg) - \eta \cdot \mathsf{Q}\big(\tau^X \leq T\big),$$

$$\text{(UO)}\quad \mathbb{E}^{\mathsf{P}}\big(h(X_T)\mathbb{1}_{\{m_T^X \leq F\}}\big) = \mathbb{E}^{\mathsf{Q}}\bigg(g\bigg(\frac{1}{X_T}\bigg)\mathbb{1}_{\{m_T^X \leq F\}}\bigg).$$

Before proving the theorem, we remark that the result is intuitively reasonable because the default only plays a role if the option is active. Especially note that the default term for Up-and-Out options (UO) is equal to zero, since in this case we can replace X by the uniformly integrable martingale $X^{\tilde{\tau}_{F+1}^X}$ in the definition of the option's payoff function, where $\tilde{\tau}_a^X := \inf\{t \geq 0 : X_t > a\}$ for any $a \geq 1$.

PROOF OF THEOREM 4.10. Keeping in mind that $D \leq 1$ and $F \geq 1$, it follows from the absolute continuity relationship between P and Q that

$$\begin{aligned}
\mathbb{E}^{\mathsf{P}}\big(h(X_T)\mathbb{1}_{\{\hat{m}_T^X \leq D\}}\big) &= \mathbb{E}^{\mathsf{Q}}\bigg(g\bigg(\frac{1}{X_T}\bigg)\mathbb{1}_{\{\tau^X > T, \hat{m}_T^X \leq D\}}\bigg) = \mathbb{E}^{\mathsf{Q}}\bigg(g\bigg(\frac{1}{X_T}\bigg)\mathbb{1}_{\{\tau^X > T \geq T_D^X\}}\bigg) \\
&= \mathbb{E}^{\mathsf{Q}}\bigg(g\bigg(\frac{1}{X_T}\bigg)\mathbb{1}_{\{\hat{m}_T^X \leq D\}}\bigg) - \eta \cdot \mathsf{Q}\big(T_D^X \leq T, \tau^X \leq T\big) \\
&= \mathbb{E}^{\mathsf{Q}}\bigg(g\bigg(\frac{1}{X_T}\bigg)\mathbb{1}_{\{\hat{m}_T^X \leq D\}}\bigg) - \eta \cdot \mathsf{Q}\big(T_D^X < \tau^X \leq T\big).
\end{aligned}$$

This proves the formula for the Down-and-In barrier option (DI). The other three formulas can be proven in a similar way by noting that

$$\begin{aligned}
\mathsf{Q}\big(\tau^X \leq T < T_D^X\big) &= \mathsf{Q}\big(\tau^X \leq T, T_D^X = \infty\big), \\
\mathsf{Q}\big(\tilde{\tau}_F^X \leq T, \tau^X \leq T\big) &= \mathsf{Q}\big(\tau^X \leq T\big), \\
\mathsf{Q}\big(\tau^X \leq T < \tilde{\tau}_F^X\big) &= 0.
\end{aligned}$$

□

REMARK 4.11. Above we used the risk-neutral pricing approach to calculate the value of some options written on a stock which may have an asset price bubble, as suggested by the first fundamental theorem of asset pricing. The derived decompositions show that there is an important difference in the option value depending on whether the underlying is a strict local or a true martingale under the risk-neutral measure, which is reflected in the default term. Even though we do not create arbitrage opportunities when pricing options by their fundamental values calculated above, several authors have suggested to "correct" the option price to account for the strictness of the local martingale (cf., e.g., [2, 20–22, 26]). In [2], the price of a contingent claim is defined as the minimal super-replicating cost under both measures P and Q corresponding to two different currencies, where the process X is interpreted as the exchange rate between them. While the authors of [20–22] work under the additional No Dominance assumption, which is strictly stronger than NFLVR, and allow for bubbles in the option prices within this framework, in [26] the following pricing formulas for European and American call options written on (continuous) X with strike K and maturity T are suggested:

$$C_E^{\text{strict}}(K,T) := \lim_{n\to\infty} \mathbb{E}^{\mathsf{P}}(X_{T\wedge\sigma_n} - K)^+,$$
$$C_A^{\text{strict}}(K,T) := \sup_{\sigma\in\mathcal{T}_{0,T}} \lim_{n\to\infty} \mathbb{E}^{\mathsf{P}}(X_{\sigma\wedge\sigma_n} - K)^+$$

for some localizing sequence $(\sigma_n)_{n\in\mathbb{N}}$ of the (strict) local martingale X. It is proven in [26] that these definitions are independent of the chosen localizing sequence and that $C_E^{\text{strict}} = C_A^{\text{strict}}$. However, a generalization of this definition to any other option $h(\cdot)$ on X with maturity T is problematic: the independence of the chosen localizing sequence $(\sigma_n)_{n\in\mathbb{N}}$ is not true in general, so one may have to choose $\sigma_n = \tau_n^X$ as defined above. Moreover, in general $\lim_{n\to\infty} \mathbb{E}^{\mathsf{P}} h(X_T^{\sigma_n})$ may not be well defined and equal to $\mathbb{E}^{\mathsf{P}} h(X_T)$, even when X is a true martingale, as the following example shows.

EXAMPLE 4.12. Suppose that $(\log(X_t) + t/2)_{t\geq 0}$ is a Brownian motion, that is, X is a geometric Brownian motion, and consider the claim $h(X_T)$ with continuous payoff function

$$h(x) = \sum_{n\in\mathbb{N}} \mathbb{1}_{\{n-a_n\leq x\leq n+a_n\}} f_n\left(n - \frac{n|x-n|}{a_n}\right) \qquad \text{with } f_n(z) = \frac{1}{\mathsf{P}(\tau_n^X \leq 1)} \cdot \frac{z}{n},$$

where each $a_n \in (0,1)$ is chosen small enough such that

$$2n^2 \cdot \mathsf{P}(n - a_n \leq X_1 \leq n + a_n) \leq \mathsf{P}(\tau_n^X \leq 1).$$

Let us set $T = 1$ and $\sigma_n = \tau_n^X$ for all $n \in \mathbb{N}$. In this case,

$$\mathbb{E}^{\mathsf{P}} h(X_{1\wedge\tau_n^X}) \geq \mathsf{P}(\tau_n^X \leq 1) f_n(n) = 1, \qquad n \in \mathbb{N},$$

but

$$\mathbb{E}^{\mathsf{P}} h(X_1) \leq \sum_{n\in\mathbb{N}} \mathsf{P}(n - a_n \leq X_1 \leq n + a_n) f_n(n)$$
$$\leq \sum_{n\in\mathbb{N}} \frac{\mathsf{P}(\tau_n^X \leq 1)}{2n^2} \cdot f_n(n) = \frac{\pi^2}{12} < 1.$$

Since in this example there are no asset price bubbles, it does not seem correct to trade the option for a price which differs from its fundamental value. Therefore, in the case where we have a decomposition of the fundamental option value as above or more generally as proven in Theorem 4.4, this suggests that the most sensible approach to correct the option value for bubbles in the underlying is to set the default term equal to zero. Equivalently, we can also set τ^X equal to infinity under the measure Q. This even gives a way of correcting the option value for stock price bubbles in the general case, where a decomposition formula may not be available, leaving open the question of why this should give an arbitrage-free pricing rule. By doing so, we would basically treat the price process as if it were a true martingale. However, we want to emphasize that it is not necessary to correct the price at all, since the fundamental value gives an arbitrage-free price as explained in Remark 4.9.

5. Relationship between P and Q.

In the following, we study the relationship between the original measure P and the measure Q in more detail. We suppose that assumption (S) is valid throughout the entire section.

LEMMA 5.1. *Set $X = \tilde{X}$, that is, $X_t = \infty$ on $\{t \geq \tau^X\}$. Then*, $\mathsf{Q}(X_\infty = \infty) = 1 \Leftrightarrow \mathsf{P}(X_\infty = 0) = 1$.

PROOF. Since X is a P-super-martingale and $\frac{1}{X}$ a Q-martingale, both converge and, therefore, X_∞ is almost surely well defined under both measures.

$\underline{\Leftarrow}$: Assume that $\mathsf{P}(X_\infty = 0) = 1$. Because $1/X$ is a Q-martingale, we have by Fatou's lemma for all $u > 0$,

$$\begin{aligned}
\mathbb{E}^{\mathsf{Q}}\left(\frac{1}{X_\infty}\mathbb{1}_{\{\tau^X > t, X_t > u\}}\right) &\leq \liminf_{n\to\infty} \mathbb{E}^{\mathsf{Q}}\left(\frac{1}{X_{t+n}}\mathbb{1}_{\{\tau^X > t, X_t > u\}}\right) \\
&= \mathbb{E}^{\mathsf{Q}}\left(\frac{1}{X_t}\mathbb{1}_{\{\tau^X > t, X_t > u\}}\right) \\
&= \mathsf{P}(X_t > u).
\end{aligned}$$

By dominated convergence for $t \to \infty$,

$$\mathbb{E}^{\mathsf{Q}}\left(\frac{1}{X_\infty}\mathbb{1}_{\{\tau^X = \infty, X_\infty > u\}}\right) \leq \mathsf{P}(X_\infty \geq u) = 0 \qquad \forall u > 0.$$

This implies that

$$\mathbb{E}^{\mathsf{Q}}\left(\frac{1}{X_\infty}\mathbb{1}_{\{\tau^X = \infty, X_\infty > 0\}}\right) = 0.$$

Since $\frac{1}{X}$ is a Q-martingale,

$$\mathbb{E}^{\mathsf{Q}}\left(\frac{1}{X_\infty}\right) \leq \mathbb{E}^{\mathsf{Q}}\left(\frac{1}{X_t}\right) = 1.$$

Thus, $\mathsf{Q}(X_\infty = 0) = 0$ and

$$\mathbb{E}^{\mathsf{Q}}\left(\frac{1}{X_\infty}\mathbb{1}_{\{\tau^X=\infty\}}\right) = 0 \quad \Leftrightarrow \quad \frac{1}{X_\infty}\mathbb{1}_{\{\tau^X=\infty\}} = 0 \qquad \mathsf{Q}\text{-a.s.}$$

Since $\frac{1}{X_\infty}\mathbb{1}_{\{\tau^X<\infty\}} = 0$, it follows that $\frac{1}{X_\infty} = 0$ Q-almost surely.

$\underline{\Rightarrow}$: Assume that $\mathsf{Q}(X_\infty = \infty) = 1$. Because X is a P-super-martingale, we have

$$\mathbb{E}^{\mathsf{P}} X_\infty \le \mathbb{E}^{\mathsf{P}} X_t \le 1$$

and

$$\mathbb{E}^{\mathsf{P}}(X_\infty \mathbb{1}_{\{X_t<k\}}) \le \mathbb{E}^{\mathsf{P}}(X_t \mathbb{1}_{\{X_t<k\}}) = \mathsf{Q}(t < \tau^X, X_t < k) = \mathsf{Q}(X_t < k) \qquad \forall k \ge 0.$$

For $t \to \infty$ by dominated convergence then

$$\mathbb{E}^{\mathsf{P}}(X_\infty \mathbb{1}_{\{X_\infty<k\}}) \le \mathsf{Q}(X_\infty < k) = 0 \qquad \forall k \ge 0.$$

This implies that $X_\infty \mathbb{1}_{\{X_\infty<k\}} = 0$ P-a.s. for all $k \ge 0$. Therefore, $\mathsf{P}(X_\infty \in \{0, \infty\}) = 1$. Since $\mathbb{E}^{\mathsf{P}}(X_\infty) \le 1$, it follows that $\mathsf{P}(X_\infty = \infty) = 0$, and thus $X_\infty = 0$ P-almost surely. $\square$

Until here, we have only considered the behaviour of the local P-martingale X under Q. But how do other processes change their behaviour, when passing from P to Q? This question is of particular interest, since we want to apply our results to the pricing of options written on more than one underlying stock. Let us assume that besides X there exists another process Y on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge0}, \mathsf{P})$. For all $n \in \mathbb{N}$ we set $\tau_n^Y = \inf\{t \ge 0 : Y_t > n\} \wedge n$ and $\tau^Y = \lim_{n\to\infty} \tau_n^Y$. Note that in what follows we identify Y with the process $\tilde{Y}$ defined above.

LEMMA 5.2. *Let Y be a nonnegative càdlàg local P-martingale. Then* $\mathsf{Q}(\tau^X \le \tau^Y) = 1$.

PROOF.

$$\mathsf{Q}(\tau^Y < \tau^X) = \lim_{n\to\infty} \mathsf{Q}(\tau^Y < \tau_n^X) = \lim_{n\to\infty} \mathbb{E}^{\mathsf{P}}(X_{\tau_n^X} \mathbb{1}_{\{\tau^Y<\tau_n^X\}}) = 0. \qquad \square$$

Moreover, we introduce condition (T): $\mathsf{Q}(\tau^X = \tau^Y < \infty) = 0$.

Clearly, (T) is always fulfilled if X is a true martingale. Moreover, condition (T) also holds, if X and Y are independent under P. Indeed, in this case for every $n \in \mathbb{N}$

$$\begin{aligned}
&\mathsf{Q}(\tau^Y = \tau^X < n) \\
&\quad = \lim_{m\to\infty} \mathsf{Q}(\tau_m^Y < \tau^X < n) = \lim_{m\to\infty}\lim_{k\to\infty} \mathsf{Q}(\tau_m^Y < \tau_k^X < n) \\
&\quad = \lim_{m\to\infty}\lim_{k\to\infty} \mathbb{E}^{\mathsf{P}}(X_{\tau_k^X} \mathbb{1}_{\{\tau_m^Y<\tau_k^X<n\}}) \le \lim_{m\to\infty}\lim_{k\to\infty} \mathbb{E}^{\mathsf{P}}(X_{\tau_k^X} \mathbb{1}_{\{\tau_m^Y<n\}}) \\
&\quad = \lim_{m\to\infty}\lim_{k\to\infty} \mathbb{E}^{\mathsf{P}} X_{\tau_k^X} \cdot \mathsf{P}(\tau_m^Y < n) = \lim_{m\to\infty} \mathsf{P}(\tau_m^Y < n) = 0.
\end{aligned}$$

However, in general it is hard to check condition (T), since it requires some knowledge of the joint distribution of τ_n^X and τ_m^Y for n, m large.

If X and Y are assumed to be càdlàg processes under P, they are also almost surely càdlàg under Q before time τ^X because P and Q are equivalent on every $\mathcal{F}_{\tau_n^X}$. Furthermore, since $\frac{1}{X}$ is a Q-martingale, it does not explode and, therefore, $X_{t-} \neq 0$ and $X_t \neq 0$ Q-almost surely for all $t \geq 0$. Thus, the process $Z := \frac{Y}{X}$ does also have almost surely càdlàg paths before time τ^X. Since from time τ^X on everything is constant, the only crucial question is whether $Z = \frac{Y}{X}$ has a left-limit at τ^X.

LEMMA 5.3. *Let Y be a nonnegative local P-martingale. Then $Z_t := (\frac{Y_t}{X_t})_{0 \leq t < \tau^X}$ is a local martingale on $(\Omega, \mathcal{F}_{\tau^X -}, (\mathcal{F}_{t \wedge \tau^X -})_{t \geq 0}, \mathsf{Q})$. Furthermore, setting $Z_t := \tilde{Z}_t$ and $X_t = \infty$ on $\{t \geq \tau^X\}$ is the unique way to define Z and X after time τ^X such that $\frac{1}{X}$ and Z remain nonnegative càdlàg local martingales on $[0, \infty)$ for all possible extensions of the measure Q from $\mathcal{F}_{\tau^X -}$ to $\mathcal{F} = \bigvee_{t \geq 0} \mathcal{F}_t$.*

PROOF. First, we show that $Z = \frac{Y}{X}$ is a local Q-martingale on $\bigcup_{n \in \mathbb{N}} [0, \tau_n^X]$ with localizing sequence $(\tau_n^Y \wedge \tau_n^X)_{n \in \mathbb{N}}$. Indeed, we have for all $t \geq 0$ and $n \in \mathbb{N}$,

$$\begin{aligned}
\mathbb{E}^{\mathsf{Q}}(Z_{\tau_n^Y \wedge \tau_n^X} | \mathcal{F}_t) &= \mathbb{E}^{\mathsf{Q}}\left(\frac{Y_{\tau_n^Y \wedge \tau_n^X}}{X_{\tau_n^Y \wedge \tau_n^X}} \middle| \mathcal{F}_t \right) = \mathbb{E}^{\mathsf{P}}\left(\frac{Y_{\tau_n^Y \wedge \tau_n^X}}{X_{t \wedge \tau_n^Y \wedge \tau_n^X}} \middle| \mathcal{F}_t \right) = \frac{Y_{t \wedge \tau_n^Y \wedge \tau_n^X}}{X_{t \wedge \tau_n^Y \wedge \tau_n^X}} \\
&= Z_{t \wedge \tau_n^Y \wedge \tau_n^X}
\end{aligned}$$

and by Lemma 5.2 we know that $\tau_n^X \wedge \tau_n^Y \to \tau^X$ Q-almost surely. Since Z is a nonnegative local super-martingale up to time τ^X, we can apply Fatou's lemma twice with $s \leq t$:

$$\begin{aligned}
\tilde{Z}_s &= \liminf_{u \to \tau^X, u < \tau^X, u \in \mathbb{Q}} Z_{s \wedge u} = \liminf_{u \to \tau^X, u < \tau^X, u \in \mathbb{Q}} \lim_{n \to \infty} Z_{s \wedge u \wedge \tau_n^X \wedge \tau_n^Y} \\
&\geq \liminf_{u \to \tau^X, u < \tau^X, u \in \mathbb{Q}} \lim_{n \to \infty} \mathbb{E}^{\mathsf{Q}}(Z_{t \wedge u \wedge \tau_n^X \wedge \tau_n^Y} | \mathcal{F}_s) \geq \liminf_{u \to \tau^X, u < \tau^X, u \in \mathbb{Q}} \mathbb{E}^{\mathsf{Q}}(Z_{t \wedge u} | \mathcal{F}_s) \\
&\geq \mathbb{E}^{\mathsf{Q}}\Big(\liminf_{u \to \tau^X, u < \tau^X, u \in \mathbb{Q}} Z_{t \wedge u} | \mathcal{F}_s \Big) = \mathbb{E}^{\mathsf{Q}}(\tilde{Z}_t | \mathcal{F}_s),
\end{aligned}$$

where the second inequality is due to the fact that $\mathbb{E}^{\mathsf{Q}}(Z_{t \wedge u \wedge \tau_n^X \wedge \tau_n^Y} | \mathcal{F}_s) \geq \mathbb{E}^{\mathsf{Q}}(Z_{t \wedge u} | \mathcal{F}_s)$ by the super-martingale property. By the convergence theorem for positive super-martingales, we conclude that $\tilde{Z}_{\tau^X -} = Z_{\tau^X -}$ exists Q-almost surely in $\mathbb{R}_+$. To see that $\tilde{Z}$ is indeed a local martingale and not only a super-martingale, we show that $\tilde{Z}^{\tau_n^Z}$ is a uniformly integrable martingale for all $n \in \mathbb{N}$, where $\tau_n^Z = \inf\{t \geq 0 | Z_t > n\} \wedge n$. Since $\tilde{Z}$ is a nonnegative super-martingale, it is sufficient to prove that the expectation of $\tilde{Z}^{\tau_n^Z}$ is constant:

$$\begin{aligned}\mathbb{E}^{\mathsf{Q}}\tilde{Z}_{\tau_n^Z} &= \mathbb{E}^{\mathsf{Q}}(\tilde{Z}_{\tau_n^Z}\mathbb{1}_{\{\tau_n^Z<\tau^X\}} + \tilde{Z}_{\tau_n^Z}\mathbb{1}_{\{\tau_n^Z\geq\tau^X\}})\\ &= \lim_{m\to\infty}\mathbb{E}^{\mathsf{Q}}(Z_{\tau_n^Z}\mathbb{1}_{\{\tau_n^Z<\tau_m^X\wedge\tau_m^Y\}}) + \mathbb{E}^{\mathsf{Q}}(\tilde{Z}_{\tau^X-}\mathbb{1}_{\{\tau_n^Z\geq\tau^X\}})\\ &= \lim_{m\to\infty}\mathbb{E}^{\mathsf{Q}}(Z_{\tau_m^X\wedge\tau_m^Y}\mathbb{1}_{\{\tau_n^Z<\tau_m^X\wedge\tau_m^Y\}}) + \mathbb{E}^{\mathsf{Q}}\Big(\lim_{m\to\infty} Z_{\tau_m^X\wedge\tau_m^Y}\mathbb{1}_{\{\tau_n^Z\geq\tau^X\}}\Big)\\ &= \lim_{m\to\infty}\mathbb{E}^{\mathsf{Q}}Z_{\tau_m^X\wedge\tau_m^Y} - \lim_{m\to\infty}\mathbb{E}^{\mathsf{Q}}(Z_{\tau_m^X\wedge\tau_m^Y}\mathbb{1}_{\{\tau^X>\tau_n^Z\geq\tau_m^X\wedge\tau_m^Y\}}) = Z_0.\end{aligned}$$

To prove the uniqueness of the extension of Z for all possible extensions of Q to $\mathcal{F}$, define for all $n\in\mathbb{N}$, $\tau_n^Z = \inf\{t\geq 0: \overline{Z}_t > n\}$, where $\overline{Z}$ is an arbitrary càdlàg extension of $(Z_t)_{t<\tau^X}$. Then $(\tau_n^Z)_{n\in\mathbb{N}}$ is a localizing sequence for $\overline{Z}$ for all possible extensions of Q. Fix one of these extensions and call it Q^0. We have

$$\mathbb{E}^{\mathsf{Q}^0}\big(\overline{Z}_t^{\tau_n^Z}|\mathcal{F}_s\big) = \overline{Z}_s^{\tau_n^Z} \qquad \forall n\in\mathbb{N}.$$

Now for fix $n\in\mathbb{N}$ define the new measure Q^n on $\mathcal{F}$ via

$$\frac{d\mathsf{Q}^n}{d\mathsf{Q}^0} = \frac{\overline{Z}_{\tau_n^Z}}{\overline{Z}_{\tau^X-}^{\tau_n^Z}}.$$

Note that Q^n is also an extension of Q from $\mathcal{F}_{\tau^X-}$ to $\mathcal{F}$. Furthermore, for all $\varepsilon\geq 0$,

$$\begin{aligned}\overline{Z}_{\tau^X-}^{\tau_n^Z} &= \mathbb{E}^{\mathsf{Q}^n}\big(\overline{Z}_{\tau^X+\varepsilon}^{\tau_n^Z}|\mathcal{F}_{\tau^X-}\big) = \mathbb{E}^{\mathsf{Q}^0}\bigg(\frac{\overline{Z}_{\tau_n^Z}}{\overline{Z}_{\tau^X-}^{\tau_n^Z}}\cdot\overline{Z}_{\tau^X+\varepsilon}^{\tau_n^Z}\Big|\mathcal{F}_{\tau^X-}\bigg)\\ &= \mathbb{E}^{\mathsf{Q}^0}\bigg(\frac{(\overline{Z}_{\tau^X+\varepsilon}^{\tau_n^Z})^2}{\overline{Z}_{\tau^X-}^{\tau_n^Z}}\Big|\mathcal{F}_{\tau^X-}\bigg),\end{aligned}$$

because $\overline{Z}^{\tau_n^Z}$ must also be a uniformly integrable martingale under Q^n. Therefore, $\overline{Z}^{\tau_n^Z}$ and $(\overline{Z}^{\tau_n^Z})^2$ are both Q^0-martingales after time τ^X-, which implies that $\overline{Z}_{\varepsilon+\tau^X} = \overline{Z}_{\tau^X-}$ for all $\varepsilon\geq 0$. Thus, $\overline{Z}\equiv\tilde{Z}$ is uniquely determined. $\square$

As usual to simplify notation, we will identify Z with the process $\tilde{Z}$ in the following.

REMARK 5.4.

- Note that if condition (T) is satisfied, then $Z_{\tau^X} = Z_{\tau^X-} = 0$ on $\{\tau^X<\infty\}$ Q-almost surely.
- Even though we proved that Z_{τ^X-} exists Q-a.s. and also X_{τ^X-} is well defined, this does not allow us to infer any conclusions about the set $\{Y_{\tau^X-}$ exists in $\mathbb{R}_+\}$ in general.

- For our purposes it is sufficient that local Q-martingales are càdlàg almost everywhere, since we are only interested in pricing and do not deal with an uncountable number of processes. One should, however, have in mind that in order to have *everywhere* regular paths some kind of augmentation is needed (cf. [23]).

REMARK 5.5. If $\Omega = C'(\mathbb{R}_+, \overline{\mathbb{R}}_+^2)$ is the path space introduced in Lemma 2.6, (X, Y) is the coordinate process, and $(\tilde{\mathcal{F}}_t)_{t\geq 0}$ is the canonical filtration generated by (X, Y), then under the assumptions of Lemma 5.3 we can extend Q to $\mathcal{F} = \bigvee_{t\geq 0} \mathcal{F}_t$ such that

$$\mathsf{Q}\big(\omega_1(t) = \infty, \omega_2(t) = \omega_2(\tau^X -)\ \forall t \geq \tau^X\big) = 1.$$

LEMMA 5.6. *Let Y be a nonnegative local P-martingale and set $Z := \frac{Y}{X}$.*

(1) *If X is a P-martingale, then Z is a strict local Q-martingale if and only if Y is a strict local P-martingale.*
(2) *Assume that X is a strict local P-martingale. Then*:

   (a) *If Y is a P-martingale, then Z is a Q-martingale and $Z_{\tau^X} = 0$ on $\{\tau^X < \infty\}$.*
   (b) *If Z is a strict local Q-martingale or Z is a Q-martingale with $\mathsf{Q}(\tau^X < \infty, Z_{\tau^X} > 0) > 0$, then Y is a strict local P-martingale.*
   (c) *If Z is a Q-martingale and if condition* (T) *holds, then Y is a P-martingale.*
   (d) *If Y is a strict local P-martingale and if condition* (T) *holds, then Z is a strict local Q-martingale.*

PROOF.

(1) This is obvious, because Q and P are locally equivalent, if X is a true P-martingale.
(2) First note that

$$\begin{aligned}
\mathbb{E}^{\mathsf{P}} Y_0 = \mathbb{E}^{\mathsf{Q}} Z_0 \geq \mathbb{E}^{\mathsf{Q}} Z_t &= \mathbb{E}^{\mathsf{Q}}(Z_t \mathbb{1}_{\{t<\tau^X\}}) + \mathbb{E}^{\mathsf{Q}}(Z_t \mathbb{1}_{\{t\geq\tau^X\}}) \\
&= \mathbb{E}^{\mathsf{Q}}\left(\frac{Y_t}{X_t}\mathbb{1}_{\{t<\tau^X\}}\right) + \mathbb{E}^{\mathsf{Q}}(Z_{\tau^X} \mathbb{1}_{\{t\geq\tau^X\}}) \\
&= \mathbb{E}^{\mathsf{P}} Y_t + \mathbb{E}^{\mathsf{Q}}(Z_{\tau^X} \mathbb{1}_{\{t\geq\tau^X\}}) \geq \mathbb{E}^{\mathsf{P}} Y_t.
\end{aligned}$$

   (a) Since Y is a positive local P-martingale, we have

   Y is a true P-martingale

$$\begin{aligned}
&\Leftrightarrow \quad \mathbb{E}^{\mathsf{P}} Y_t = \mathbb{E}^{\mathsf{P}} Y_0 \qquad \text{for all } t \geq 0, \\
&\Leftrightarrow \quad \mathbb{E}^{\mathsf{Q}} Z_t = \mathbb{E}^{\mathsf{Q}} Z_0 \qquad \text{for all } t \geq 0,\ Z_{\tau^X}\mathbb{1}_{\{\tau^X<\infty\}} = 0\ \mathsf{Q}\text{-a.s.}
\end{aligned}$$

(b) Follows from (a).

(c) If (T) holds, $Z_{\tau^X} = 0$ on $\{\tau^X < \infty\}$ Q-almost surely; cf. Remark 5.5. Therefore, since Z is a Q-martingale, the above inequality turns into an equality and Y is a true P-martingale.

(d) Follows from (c). □

EXAMPLE 5.7 (Continuation of Example 3.1.2). For the following example, we work on the path space $C'(\mathbb{R}_+, \overline{\mathbb{R}}_+^2)$ with (X, Y) denoting the coordinate process and $(\mathcal{F}_t)_{t\geq 0}$ being the right-continuous augmentation of the canonical filtration generated by the coordinate process. Remember from Example 3.1.2 that for $\sigma(x)$ locally bounded and bounded away from zero for $x > 0$, $\sigma(0) = 0$, the local P-martingale

$$dX_t = \sigma(X_t)\,dW_t, \qquad X_0 = 1,$$

is strictly positive whenever

$$\int_0^1 \frac{x}{\sigma^2(x)}\,dx = \infty,$$

and under Q with $\frac{d\mathsf{P}}{d\mathsf{Q}}|_{\mathcal{F}_t} = \frac{1}{X_t}$ the reciprocal process is a true martingale with decomposition

$$d\left(\frac{1}{X_t}\right) = -\frac{\sigma(X_t)}{X_t^2}\,dW_t^{\mathsf{Q}} = \overline{\sigma}\left(\frac{1}{X_t}\right)dW_t^{\mathsf{Q}}$$

for the Q-Brownian motion $W_t^{\mathsf{Q}} = W_t - \int_0^t \frac{\sigma(X_s)}{X_s}\,ds$ defined on the set $\{t < \tau^X\}$ and $\overline{\sigma}(y) := -y^2 \cdot \sigma(\frac{1}{y})$.

Now let us assume that Y is also a local martingale under P with dynamics

$$dY_t = \gamma(Y_t)\,dB_t,$$

where γ fulfills the same assumptions as σ and B is another P-Brownian motion such that $\langle B, W\rangle_t = \rho t$. Then $\frac{Y}{X}$ is a Q-local martingale with decomposition

$$d\left(\frac{Y_t}{X_t}\right) = \frac{\gamma(Y_t)}{X_t}\,dB_t^{\mathsf{Q}} + Y_t\overline{\sigma}\left(\frac{1}{X_t}\right)dW_t^{\mathsf{Q}},$$

where B^{Q} is a Q-BM defined up to time τ^X such that $\langle B^{\mathsf{Q}}, W^{\mathsf{Q}}\rangle_t = \rho t$ on $\{t < \tau^X\}$.

6. Application to financial bubbles II: Last passage time formulas. In Section 4, we have seen how one can determine the influence bubbles have on option pricing formulas through a decomposition of the option value into a "normal" term and a default term (cf. Theorems 4.4 and 4.10). However, this approach only works well for options written on one underlying. It is rather difficult to give a universal way of how to determine the influence of asset price bubbles on the valuation of more complicated options and we will not do this here in all generality. Instead, we will do the analysis for a special example, the so called exchange option, which allows us to connect results about last passage times with the change of measure that was defined in Section 2.2.

Again we suppose that assumption (S) holds throughout the entire section. In addition, we assume that there exists another strictly positive process Y on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$, which is also a local P-martingale. Furthermore, in the following we will assume that X and Y are *continuous*. As in Section 5, we define $Z := \frac{Y}{X}$, which is a local Q-martingale.

6.1. *Exchange option.* With the interpretation of X and Y as two stock price processes and assuming an interest rate of $r = 0$, we can define the price of a European exchange option with strike $K \in \mathbb{R}_+$ (also known as the ratio of notionals) and maturity $T \in \mathbb{R}_+$ as

$$E(K, T) := \mathbb{E}^{\mathsf{P}}(X_T - KY_T)^+.$$

The corresponding price of the American option is given by

$$A(K, T) := \sup_{\sigma \in \mathcal{T}_{0,T}} \mathbb{E}^{\mathsf{P}}(X_\sigma - KY_\sigma)^+,$$

where $\mathcal{T}_{0,T}$ is the set of all stopping times σ, which take values in $[0, T]$. Let us define the last passage time $\rho_K := \sup\{t \geq 0 | Z_t = \frac{1}{K}\}$, where as usual the supremum of the empty set is equal to zero. In the next theorem, the prices of the European and American exchange option are expressed in terms of the last passage time ρ_K in the spirit of [33].

THEOREM 6.1. *For all $K, T \geq 0$, the prices of the European and American exchange option are given by*

$$E(K, T) = \mathbb{E}^{\mathsf{Q}}\big((1 - KZ_{\tau^X})^+ \mathbb{1}_{\{\rho_K \leq T < \tau^X\}}\big),$$
$$A(K, T) = \mathbb{E}^{\mathsf{Q}}\big((1 - KZ_{\tau^X})^+ \mathbb{1}_{\{\rho_K \leq T\}}\big).$$

PROOF. Assume $\sigma \in \mathcal{T}_{0,T}$. As seen above, $Z = \frac{Y}{X}$ is a nonnegative local Q-martingale, thus a supermartingale, which converges almost surely to $Z_\infty = Z_{\tau^X}$. From Corollary 3.4 in [3], respectively Theorem 2.5 in [33] we have the identity

$$\left(\frac{1}{K} - Z_\sigma\right)^+ = \mathbb{E}^{\mathsf{Q}}\left(\left(\frac{1}{K} - Z_{\tau^X}\right)^+ \mathbb{1}_{\{\rho_K \leq \sigma\}} \Big| \mathcal{F}_\sigma\right). \tag{6}$$

Multiplying the above equation with the $\mathcal{F}_\sigma$-measurable random variable $K\mathbb{1}_{\{\tau^X > \sigma\}}$ and taking expectations under Q yields

$$\mathbb{E}^{\mathsf{Q}}\big((1 - KZ_\sigma)^+ \mathbb{1}_{\{\tau^X > \sigma\}}\big) = \mathbb{E}^{\mathsf{Q}}\big((1 - KZ_{\tau^X})^+ \mathbb{1}_{\{\rho_K \leq \sigma < \tau^X\}}\big).$$

Changing the measure via $d\mathsf{P}|_{\mathcal{F}_\sigma} = \frac{1}{X_\sigma}\, d\mathsf{Q}|_{\mathcal{F}_\sigma}$, we obtain

$$\begin{aligned}\mathbb{E}^{\mathsf{P}}(X_\sigma - KY_\sigma)^+ &= \mathbb{E}^{\mathsf{P}}\big(\mathbb{1}_{\{\tau^X > \sigma\}} X_\sigma (1 - KZ_\sigma)^+\big) \\ &= \mathbb{E}^{\mathsf{Q}}\big((1 - KZ_{\tau^X})^+ \mathbb{1}_{\{\rho_K \le \sigma < \tau^X\}}\big),\end{aligned} \tag{7}$$

since $\mathbb{1}_{\{\tau^X > \sigma\}} = 1$ P-almost surely. Taking $\sigma = T$ the formula for the European option is proven. For the American option value we note that in the proof of Theorem 1.4 in [1] it is shown that

$$A(K,T) = \lim_{n\to\infty} \mathbb{E}^{\mathsf{P}}\left(Y_{\tau_n^X \wedge T}\left(\frac{1}{Z_{\tau_n^X \wedge T}} - K\right)^+\right) = \lim_{n\to\infty} \mathbb{E}^{\mathsf{P}}(X_{\tau_n^X \wedge T} - KY_{\tau_n^X \wedge T})^+.$$

Setting $\sigma = \tau_n^X \wedge T$ in equality (7), it follows that

$$\begin{aligned}&A(K,T) \\ &\quad= \lim_{n\to\infty} \mathbb{E}^{\mathsf{P}}(X_{\tau_n^X \wedge T} - KY_{\tau_n^X \wedge T})^+ = \lim_{n\to\infty} \mathbb{E}^{\mathsf{Q}}\big((1 - KZ_{\tau^X})^+ \mathbb{1}_{\{\rho_K \le \tau_n^X \wedge T < \tau^X\}}\big) \\ &\quad= \lim_{n\to\infty} \mathbb{E}^{\mathsf{Q}}\big((1 - KZ_{\tau^X})^+ \mathbb{1}_{\{\rho_K \le \tau_n^X \wedge T\}}\big) = \mathbb{E}^{\mathsf{Q}}\big((1 - KZ_{\tau^X})^+ \mathbb{1}_{\{\rho_K \le \tau^X \wedge T\}}\big) \\ &\quad= \mathbb{E}^{\mathsf{Q}}\big((1 - KZ_{\tau^X})^+ \mathbb{1}_{\{\rho_K \le T\}}\big),\end{aligned}$$

where the last equality follows from the fact that $Z_{\tau^X} = \frac{1}{K}$ on $\{\rho_K > \tau^X\} = \{\rho_K = \infty\}$. $\square$

REMARK 6.2. Assume that $\mathsf{Q}(\tau^X < \infty) = 1$, that is, $\mathbb{E}^{\mathsf{P}} X_t \overset{t\to\infty}{\longrightarrow} 0$. If we take $Y \equiv 1$ in the above theorem, we get the formula for the standard European call option expressed as a function of the last passage time of X as it can be found in [37] for the special case of Bessel processes or in [24]:

$$E(K,T) = \mathsf{Q}\big(\rho_K \le T < \tau^X\big). \tag{8}$$

More generally, for arbitrary Y formula (8) is still true, if (T) holds and $\mathsf{Q}(\tau^X < \infty) = 1$.

REMARK 6.3. We can also express the price of a barrier exchange option in terms of the last passage time of Z at level $\frac{1}{K}$ as done in Theorem 6.1 for exchange options without barriers. For example, in the case of the Down-and-In exchange option we simply have to multiply equation (6) with the $\mathcal{F}_\sigma$-measurable random variable $\mathbb{1}_{\{\hat{m}_\sigma^X \le D\}}$.

We now analyze a few special cases of Theorem 6.1 in more detail:

(1) X is a true P-martingale.

If X is a true P-martingale, the price process for X exhibits no asset price bubble. Then, regardless of whether the stock price process Y has an asset price bubble or not, we know that Q is locally equivalent to P and $\mathsf{Q}(\tau^X = \infty) = 1$. Therefore,

$$E(K,T) = A(K,T) = \mathbb{E}^{\mathsf{Q}}\big((1 - KZ_\infty)^+ \mathbb{1}_{\{\rho_K \le T\}}\big)$$

and the European and American exchange option values are equal. For $Y \equiv 1$, this formula is well known (cf. [33]).

(2) Y is a true P-martingale.

We recall from Lemma 5.6 that in this case $Z_{\tau^X} = 0$ on $\{\tau^X < \infty\}$ Q-almost surely. Denoting $\tau_0^Z = \inf\{t \ge 0 | Z_t = 0\}$ this translates into $\mathsf{Q}(\tau^X = \tau_0^Z) = 1$, since

$$\mathsf{Q}\big(\tau_0^Z < \tau^X\big) = \lim_{n\to\infty} \mathsf{Q}\big(\tau_0^Z < \tau_n^X\big) = \lim_{n\to\infty} \mathbb{E}^{\mathsf{P}}(X_{\tau_n^X} \mathbb{1}_{\{\tau_0^Z < \tau_n^X\}}) = 0.$$

Therefore,

$$\begin{aligned} E(K,T) &= \mathsf{Q}\big(\rho_K \le T < \tau_0^Z\big), \\ A(K,T) &= \mathsf{Q}\big(\rho_K \le T \wedge \tau^X\big) = \mathsf{Q}\big(\rho_K \le T \wedge \tau_0^Z\big) = \mathsf{Q}(\rho_K \le T), \end{aligned}$$

where the last equality follows from the fact that the last passage time of the level $\frac{1}{K}$ by Z cannot be greater than its first hitting time of 0. Note that in this case the above formula for $E(K,T)$ is similar to the one for the European call option given in [24], Proposition 7; see also [37] for the case of the reciprocal Bessel process of dimension greater than two.

Especially, the American option premium is equal to

$$\begin{aligned} &A(K,T) - E(K,T) \\ &\quad = \mathsf{Q}(\rho_K \le T) - \mathsf{Q}\big(\rho_K \le T < \tau_0^Z\big) = \mathsf{Q}\big(\rho_K \le T, \tau_0^Z \le T\big) \\ &\quad = \mathsf{Q}\big(\tau_0^Z \le T\big) = \mathsf{Q}\big(\tau^X \le T\big) = \gamma_X(0,T), \end{aligned}$$

which is just the default of the local P-martingale X or, in other words, the bubble of the stock X between 0 and T.

(3) X and Y are both strict local P-martingales: An example.

Let X and Y be the reciprocals of two independent BES(3)-processes under P and assume that $X_0 = x \in \mathbb{R}_+$, while $Y_0 = 1$. (Note that this normalization is different from the previous one. However, since the density of X respectively Y is explicitly known in this case, we can do calculations directly under P. This allows us to point out some anomalies of the option value in the presence of strict local martingales.)

We apply the formula for the European call option value written on the reciprocal BES(3)-process from Example 3.6 in [5] and integrate over Y:

$$\begin{aligned} E(K,T) = \int_0^\infty x\Big[\Phi\Big(\frac{x-zK}{xzK\sqrt{T}}\Big) - \Phi\Big(-\frac{1}{x\sqrt{T}}\Big) \\ + \Phi\Big(\frac{1}{x\sqrt{T}}\Big) - \Phi\Big(\frac{zK+x}{xzK\sqrt{T}}\Big)\Big]\mathsf{P}(Y_T \in dz) \\ - K\int_0^\infty z\Big\{\Phi\Big(\frac{zK+x}{xzK\sqrt{T}}\Big) - \Phi\Big(\frac{zK-x}{xzK\sqrt{T}}\Big) \\ + x\sqrt{T}\Big[\varphi\Big(\frac{zK+x}{xzK\sqrt{T}}\Big) - \varphi\Big(\frac{x-zK}{xzK\sqrt{T}}\Big)\Big]\Big\}\mathsf{P}(Y_T \in dz), \end{aligned}$$

where

$$\mathsf{P}(Y_T \in dz) = \frac{1}{z^3}\frac{dz}{\sqrt{2\pi T}}\Big(\exp\Big(-\frac{(1/z-1)^2}{2T}\Big) - \exp\Big(-\frac{(1/z+1)^2}{2T}\Big)\Big).$$

Since $\mathbb{E}^{\mathsf{P}} X_T \stackrel{x\to\infty}{\longrightarrow} \frac{2}{\sqrt{2\pi T}}$ as shown in [17], the option value converges to a finite positive value as the initial stock price $X_0 = x$ goes to infinity. Therefore, the convexity of the payoff function does not carry over to the option value. This anomaly for stock price bubbles has been noticed before by, for example, [5, 17]. We refer for the economic intuition of this phenomenon to [17], where a detailed analysis of stock and bond price bubbles modelled by the reciprocal BES(3)-process is done.

Furthermore, recall that by Jensen's inequality the European exchange option value is increasing in maturity if X and Y are true martingales. However, in our example the option value is *not* increasing in maturity anymore: Indeed, because of $E(K,T) \le \mathbb{E}^{\mathsf{P}} X_T \stackrel{T\to\infty}{\longrightarrow} 0$ the option value converges to zero as $T \to \infty$. Taking $Y \equiv 1$, this behaviour has been noticed before by, for example, [5, 17, 26, 30] and is also directly evident from the representation of $E(K,T)$ in Theorem 6.1.

6.2. *Real-world pricing*. Here, we want to give another interpretation of Theorem 6.1. Note that from a mathematical point of view we have only assumed that X and Y are strictly positive local P-martingales for the result. Above we have interpreted P as the risk-neutral probability and X, Y as two stock price processes. Now note that we have the identity $(X - KY)^+ = Y(\frac{1}{Z} - K)^+$. This motivates the following alternative financial setting: we take P to be the historical probability and assume that also $\mathsf{P}(Y_0 = 1) = 1$. Normalizing the interest rate to be equal to zero, the process $S := \frac{1}{Z}$ denotes the (discounted) stock price process, while Y is a candidate for the density of an equivalent local martingale measure (ELMM). Since Y and $X = YS$ are both strictly positive local P-martingales, they are P-super-martingales and cannot reach infinity under P. Thus, $S = \frac{1}{Z}$ is also strictly positive under P and does not attain infinity under P either.

As before, X and Y are both allowed to be either strict local or true P-martingales. While the question of whether $X = YS$ is a true martingale or not is related to the existence of a stock price bubble as discussed earlier, the question of whether Y is a strict local martingale or not is connected to the absence of arbitrage. If Y is a uniformly integrable P-martingale, an ELMM for Z exists and the market satisfies NFLVR. However, as shown in [13] and explained in [1], even if Y is only a strict local martingale, a super-hedging strategy for any contingent claim written on S exists. Therefore, the "normal" call option pricing formulas

$$E(K,T) = \mathbb{E}^{\mathsf{P}}\big(Y_T(S_T - K)^+\big), \qquad A(K,T) = \sup_{\sigma \in \mathcal{T}_{0,T}} \mathbb{E}^{\mathsf{P}}\big(Y_\sigma(S_\sigma - K)^+\big)$$

are still reasonable when Y is only a strict local martingale. This pricing method is also known as "real-world pricing," since we cannot work under an ELMM directly, but must define the option value under the real-world measure (cf. [32]). Note that if Y is a true martingale, we can define an ELMM P^* for S on $\mathcal{F}_T$ via $\mathsf{P}^*|_{\mathcal{F}_T} = Y_T \cdot \mathsf{P}|_{\mathcal{F}_T}$ and the market satisfies the NFLVR property until time $T \in \mathbb{R}_+$. In this case, we obtain the usual pricing formulas

$$E(K,T) = \mathbb{E}^{\mathsf{P}^*}(S_T - K)^+ \quad \text{respectively} \quad A(K,T) = \sup_{\sigma \in \mathcal{T}_{0,T}} \mathbb{E}^{\mathsf{P}^*}(S_\sigma - K)^+.$$

Following [17], we can interpret the situation when Y is only a strict local martingale as the existence of a bond price bubble as opposed to the stock price bubble discussed above. This is motivated by the fact that the real-world price of a zero-coupon bond is strictly less than the (discounted) pay-off of one, if Y is a strict local martingale. Of course, it is possible to make a risk-free profit in this case via an admissible trading strategy. From Theorem 6.1, we have the following corollary.

COROLLARY 6.4. *For all $K, T \geq 0$, the values of the European and American call option under real-world pricing are given by*

$$E(K,T) = \mathbb{E}^{\mathsf{Q}}\left(\left(1 - \frac{K}{S_{\tau^X}}\right)^+ \mathbb{1}_{\{\rho_K^S \leq T < \tau^X\}}\right),$$

$$A(K,T) = \mathbb{E}^{\mathsf{Q}}\left(\left(1 - \frac{K}{S_{\tau^X}}\right)^+ \mathbb{1}_{\{\rho_K^S \leq T\}}\right)$$

with $\rho_K^S = \sup\{t \geq 0 | S_t = K\}$.

From the above formulas for the European and American call options, it can easily be seen that their values are generally different, unless $X = YS$ is a true P-martingale (in this case $\tau^X = \infty$ Q-a.s.). Therefore, Merton's no early exercise theorem does not hold anymore (cf. also [1, 5, 21, 22]).

Furthermore, note that we have the following formula for any bounded stopping time T:

$$E(K,T) = \mathbb{E}^{\mathsf{P}}(X_T - KY_T)^+ = \mathbb{E}^{\mathsf{Q}}(1 - KZ_T)^+ - \mathbb{E}^{\mathsf{Q}}\big(\mathbb{1}_{\{\tau^X \leq T\}}(1 - KZ_T)^+\big),$$

where the second term equals $\mathsf{Q}(\tau^X \le T)$, if (T) holds. For $Y \equiv 1$, this decomposition of the European call value is shown in [30].

Now we show that also the asymptotic behaviour of the European and American call option is unusual, when we allow X and / or Y to be strict local P-martingales. From the definition of the European call option value, we easily see that

$$\lim_{K\to 0} E(K,T) = \mathbb{E}^{\mathsf{P}}(Y_T S_T) = \mathbb{E}^{\mathsf{P}} X_T = \mathsf{Q}(\tau^X > T), \qquad \lim_{K\to\infty} E(K,T) = 0.$$

Moreover, using the last passage time formula for the American call derived above, it follows that

$$\lim_{K\to 0} A(K,T) = \lim_{K\to 0} \mathsf{Q}(\rho_K^S \le T) = 1,$$

since Z does not explode Q-a.s., and hence S is strictly positive under Q. Similarly, denoting $\rho_{1/K}^Z = \sup\{t \ge 0 | Z_t = \frac{1}{K}\}$, we get

$$\begin{aligned}\lim_{K\to\infty} A(K,T) &= \lim_{K\to\infty} \mathsf{Q}(\rho_K^S \le T, S_{\tau^X} = \infty) = \lim_{K\to\infty} \mathsf{Q}(\rho_{1/K}^Z \le T, Z_{\tau^X} = 0)\\ &= \mathsf{Q}(Z_{\tau^X} = Z_T = 0) = \mathsf{Q}(T \ge \tau^X, Z_{\tau^X} = 0),\end{aligned}$$

which may be strictly positive and equals $\mathsf{Q}(T \ge \tau^X) = \gamma_X(0,T)$ under (T). For the asymptotics in T, we have

$$\lim_{T\to\infty} E(K,T) = \mathbb{E}^{\mathsf{Q}}\left(\left(1 - \frac{K}{S_{\tau^X}}\right)^+ \mathbb{1}_{\{\tau^X = \infty\}}\right),$$

$$\lim_{T\to\infty} A(K,T) = \mathbb{E}^{\mathsf{Q}}\left(1 - \frac{K}{S_{\tau^X}}\right)^+,$$

and from the definition of the call option it is also clear that

$$\lim_{T\to 0} E(K,T) = \lim_{T\to 0} A(K,T) = (1-K)^+.$$

6.2.1. *American option premium under real-world pricing.* We keep the notation and interpretation introduced at the beginning of Section 6.2. However, we do not assume that Z and/or X are continuous anymore.

LEMMA 6.5. *Let $h : \mathbb{R}_{++} \to \mathbb{R}_+$ be a Borel-measurable function s.t. $\lim_{x\to\infty} \frac{h(x)}{x} =: \eta$ exists in $\mathbb{R}_+$. Define $g : \mathbb{R}_+ \to \mathbb{R}_+$ via $g(x) = x \cdot h(\frac{1}{x})$ for $x > 0$ and $g(0) = \eta$. We denote by $E(h,T) = \mathbb{E}^{\mathsf{P}}(Y_T h(S_T))$ the value of the European option with maturity T and payoff function h and by $A(h,T)$ the value of the corresponding American option. Then*

$$E(h,T) = \mathbb{E}^{\mathsf{Q}} g(Z_T) - \mathbb{E}^{\mathsf{Q}}(\mathbb{1}_{\{\tau^X \le T\}} g(Z_{\tau^X})).$$

Furthermore, if in addition h is convex with $h(0) = 0$, $h(x) \le x$ for all $x \in \mathbb{R}_+$ and $\eta = 1$, then

$$A(h,T) = \mathbb{E}^{\mathsf{Q}} g(Z_T).$$

PROOF. For the European option value, we have

$$\begin{aligned}E(h,T) &= \mathbb{E}^{\mathsf{P}}\big(Y_T h(S_T)\big) = \mathbb{E}^{\mathsf{Q}}\big(g(Z_T)\mathbb{1}_{\{\tau^X > T\}}\big)\\ &= \mathbb{E}^{\mathsf{Q}} g(Z_T) - \mathbb{E}^{\mathsf{Q}}\big(\mathbb{1}_{\{\tau^X \le T\}} g(Z_{\tau^X})\big).\end{aligned}$$

And for the American option value we get

$$\begin{aligned}A(h,T) &= \lim_{n\to\infty} \mathbb{E}^{\mathsf{P}}\big(Y_{T\wedge\tau_n^X} h(S_{T\wedge\tau_n^X})\big) = \lim_{n\to\infty} \mathbb{E}^{\mathsf{Q}}\left(Z_{T\wedge\tau_n^X} h\left(\frac{1}{Z_{T\wedge\tau_n^X}}\right)\right)\\ &= \lim_{n\to\infty} \mathbb{E}^{\mathsf{Q}} g(Z_{T\wedge\tau_n^X}) = \mathbb{E}^{\mathsf{Q}} g(Z_{T\wedge\tau^X}) = \mathbb{E}^{\mathsf{Q}} g(Z_T),\end{aligned}$$

where the first equality is proven in [1] under the above stated assumptions on h and the fourth equality follows by dominated convergence since $g \le 1$ is a bounded and continuous function. $\square$

Under the assumptions of Lemma 6.5, the American option premium is thus equal to

$$A(h,T) - E(h,T) = \mathbb{E}^{\mathsf{Q}}\big(\mathbb{1}_{\{\tau^X \le T\}} g(Z_{\tau^X})\big).$$

Note that Lemma 6.5 is a generalization of Theorem A1 in [5]. Indeed, if Y is a uniformly integrable P-martingale (i.e., NFLVR is satisfied), $Z_{\tau^X} = 0$ on $\{\tau^X < \infty\}$ by part 2(a) of Lemma 5.6. Thus,

$$A(h,T) = E(h,T) + g(0)\cdot \mathsf{Q}\big(\tau^X \le T\big) = E(h,T) + \gamma_X(0,T).$$

7. Multivariate strictly positive (strict) local martingales. So far the measure Q defined in Theorem 2.12 above is only associated with the local P-martingale X in the sense that $X_{\tau_n^X}.\mathsf{P}|_{\mathcal{F}_{\tau_n^X}} = \mathsf{Q}|_{\mathcal{F}_{\tau_n^X}}$ for all $n \in \mathbb{N}$ and that $\frac{1}{X}$ is a true martingale under Q. One may now naturally wonder whether, given two (or more) positive local P-martingales X and Y, there exists a measure Q, under which $\frac{1}{X}$ and $\frac{1}{Y}$ are both local (or even true) martingales. Obviously, this is the case, if X and Y are independent under P. In this section, we will consider the case where X and Y are continuous local P-martingales, but not necessarily independent.

THEOREM 7.1. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathsf{P})$ be a filtered probability space, where $(\mathcal{F}_t)_{t\ge 0}$ is the right-continuous augmentation of a standard system. Assume that X and Y are two strictly positive continuous local P-martingales with $d\langle X\rangle_t = f_t\,dt$, $d\langle Y\rangle_t = g_t\,dt$ and $d\langle X, Y\rangle_t = h_t\,dt$. Suppose that for all $t > 0$, the stochastic integral*

$$M_t = \int_0^t \frac{(f_s Y_s - h_s X_s) g_s}{Y_s X_s (f_s g_s - h_s^2)}\,dX_s + \int_0^t \frac{(g_s X_s - h_s Y_s) f_s}{Y_s X_s (f_s g_s - h_s^2)}\,dY_s$$

is well-defined. Denote by τ the explosion time of $\mathcal{E}(M)$. Then there exists a measure Q *on $\mathcal{F}_\infty$, under which $\frac{1}{\tilde{X}}$ and $\frac{1}{\tilde{Y}}$ defined via*

$$\tilde{X}_t = X_t \mathbb{1}_{\{t<\tau\}} + \liminf_{s\to\tau, s<\tau, s\in\mathbb{Q}} X_s \mathbb{1}_{\{\tau\le t<\infty\}},$$

$$\tilde{Y}_t = Y_t \mathbb{1}_{\{t<\tau\}} + \liminf_{s\to\tau, s<\tau, s\in\mathbb{Q}} Y_s \mathbb{1}_{\{\tau\le t<\infty\}}$$

are both continuous nonnegative local Q*-martingales and $d\mathsf{P}|_{\mathcal{F}_t} = \frac{1}{\mathcal{E}(M)_t} \times \mathbb{1}_{\{t<\tau\}} d\mathsf{Q}|_{\mathcal{F}_t}$ for all $t \ge 0$.*

PROOF. The stochastic exponential $\mathcal{E}(M)$ is a continuous local P-martingale with localizing sequence

$$\tau_n := \inf\{t \ge 0 : \mathcal{E}(M)_t > n\} \wedge n.$$

We define a consistent family of probability measures Q_n on $\mathcal{F}_{\tau_n}$ by

$$\frac{d\mathsf{Q}_n}{d\mathsf{P}}\bigg|_{\mathcal{F}_{\tau_n}} = \mathcal{E}(M)_{\tau_n}, \qquad n \in \mathbb{N}.$$

Using the same trick as in the proof of Theorem 2.12, we restrict each measure Q_n to $\mathcal{F}_{\tau_n-}$. Since $(\mathcal{F}_{\tau_n-})_{n\in\mathbb{N}}$ is a standard system by Lemma 2.7, there exists a unique measure Q on $\mathcal{F}_{\tau-}$, such that $\mathsf{Q}|_{\mathcal{F}_{\tau_n}} = \mathsf{Q}_n$ for all $n \in \mathbb{N}$. For any stopping time S and $A \in \mathcal{F}_S$, we get

$$\mathsf{Q}(S < \tau_n, A) = \mathbb{E}^{\mathsf{P}}\big(\mathcal{E}(M)_{S\wedge\tau_n}\mathbb{1}_{\{S<\tau_n, A\}}\big) = \mathbb{E}^{\mathsf{P}}\big(\mathcal{E}(M)_S\mathbb{1}_{\{S<\tau_n, A\}}\big).$$

Taking $n \to \infty$ results in

$$\mathsf{Q}(S < \tau, A) = \mathbb{E}^{\mathsf{P}}\big(\mathcal{E}(M)_S\mathbb{1}_{\{S<\infty, A\}}\big).$$

It follows that P is locally absolutely continuous with respect to Q before τ. We choose an arbitrary extension of Q from $\mathcal{F}_{\tau-}$ to $\mathcal{F}_\infty$ as discussed on page 1836. Next, according to Girsanov's theorem applied on $\mathcal{F}_{\tau_n}$,

$$\begin{aligned}
N_{t\wedge\tau_n} &:= X_t^{\tau_n} - \langle M^{\tau_n}, X^{\tau_n}\rangle_t \\
&= X_t^{\tau_n} - \int_0^{t\wedge\tau_n} \frac{(f_s Y_s - h_s X_s)g_s}{Y_s X_s(f_s g_s - h_s^2)}\, d\langle X\rangle_s \\
&\quad - \int_0^{t\wedge\tau_n} \frac{(g_s X_s - h_s Y_s) f_s}{Y_s X_s(f_s g_s - h_s^2)}\, d\langle X, Y\rangle_s \\
&= X_t^{\tau_n} - \int_0^{t\wedge\tau_n} \frac{(f_s Y_s - h_s X_s)g_s f_s + (g_s X_s - h_s Y_s) f_s h_s}{Y_s X_s(f_s g_s - h_s^2)}\, ds \\
&= X_t^{\tau_n} - \int_0^{t\wedge\tau_n} \frac{f_s}{X_s}\, ds
\end{aligned}$$

is a local Q-martingale. We apply Itô's formula:

$$\begin{aligned}\frac{1}{X_{t\wedge\tau_n}} &= \frac{1}{X_0} - \int_0^{t\wedge\tau_n} \frac{dX_s}{X_s^2} + \int_0^{t\wedge\tau_n} \frac{d\langle X\rangle_s}{X_s^3} \\ &= \frac{1}{X_0} - \int_0^{t\wedge\tau_n} \frac{dN_s}{X_s^2} - \int_0^{t\wedge\tau_n} \frac{f_s}{X_s^3}\,ds + \int_0^{t\wedge\tau_n} \frac{f_s}{X_s^3}\,ds = \frac{1}{X_0} - \int_0^{t\wedge\tau_n} \frac{dN_s}{X_s^2}.\end{aligned}$$

Thus, $\frac{1}{X^{\tau_n}}$ is a local Q-martingale for all $n \in \mathbb{N}$. Since $\frac{1}{X}$ is continuous, $(\tau_m^{1/X})_{m\in\mathbb{N}}$ is a localizing sequence for $\frac{1}{X^{\tau_n}}$ on $(\Omega, \mathcal{F}_{\tau_n}, \mathsf{Q})$ for all $n \in \mathbb{N}$, where

$$\tau_m^{1/X} := \inf\left\{t \geq 0 : \frac{1}{X_t} > m\right\} \wedge m, \qquad \tau^{1/X} := \lim_{m\to\infty} \tau_m^{1/X}.$$

Moreover, we have

$$\mathsf{Q}(\tau^{1/X} < \tau) = \lim_{n\to\infty} \mathsf{Q}(\tau^{1/X} < \tau_n) = \lim_{n\to\infty} \mathbb{E}^{\mathsf{P}}\left(\mathcal{E}(M)_{\tau_n} \mathbb{1}_{\{\tau^{1/X} < \tau_n\}}\right) = 0,$$

because X is strictly positive under P. Since a process which is locally a local martingale is a local martingale itself, we conclude that $\frac{1}{X}$ is a positive local Q-martingale up to time τ with localizing sequence $(\tau_n \wedge \tau_n^{1/X})_{n\in\mathbb{N}}$. Especially, $\lim_{n\to\infty} X_{\tau_n} = \lim_{n\to\infty} X_{\tau_n \wedge \tau_n^{1/X}}$ exists Q-almost surely. Thus, $\frac{1}{\tilde{X}}$ is a continuous positive Q-super-martingale and $\tau_n^{1/X} \to \infty$ Q-almost surely. Therefore,

$$1 \geq \mathbb{E}^{\mathsf{Q}}\left(\frac{1}{\tilde{X}_{\tau_n^{1/X}}}\right) = \lim_{m\to\infty} \mathbb{E}^{\mathsf{Q}}\left(\frac{1}{\tilde{X}_{\tau_n^{1/X} \wedge \tau_m}}\right) \geq \lim_{m\to\infty} \mathbb{E}^{\mathsf{Q}}\left(\frac{1}{\tilde{X}_{\tau_m^{1/X} \wedge \tau_m}}\right) = 1,$$

where the two inequalities follow by the super-martingale property. Hence, $\frac{1}{\tilde{X}}$ is a local Q-martingale.

For $\frac{1}{\tilde{Y}}$, the claim follows by analogous calculations. $\square$

But are $\frac{1}{\tilde{X}}$ and $\frac{1}{\tilde{Y}}$ in the setting of Theorem 7.1 actually true Q-martingales or just local Q-martingales? In general, there does not seem to be an easy answer to this question. However, if X (resp. Y) is a homogeneous diffusion, one can show the following extension of the above theorem.

LEMMA 7.2. *In the setting of Theorem 7.1 assume that X follows the P-dynamics*

$$dX_t = \sigma(X_t)\,dB_t$$

for some P-Brownian motion B, where $\sigma(\cdot)$ is locally bounded and bounded away from zero on $(0,\infty)$ and $\sigma(0) = 0$. Then $\frac{1}{\tilde{X}}$ is a Q-martingale, where the measure Q is constructed in Theorem 7.1.

PROOF. Note that, with the notation used in the proof of Theorem 7.1, up to time τ the process N follows the dynamics

$$dN_t = \sigma(X_t)\, dB_t^{\mathsf{Q}},$$

where

$$B_t^{\mathsf{Q}} := B_t - \int_0^t \frac{\sigma(X_s)}{X_s}\, ds$$

is a Q-Brownian motion on $[0, \tau)$ by Lévy's theorem. Hence, the Q-dynamics of $\frac{1}{X}$ up to time τ are given by

$$d\left(\frac{1}{X_t}\right) = -\frac{\sigma(X_t)}{X_t^2}\, dB_t^{\mathsf{Q}} =: \overline{\sigma}\left(\frac{1}{X_t}\right) dB_t^{\mathsf{Q}} \tag{9}$$

and we are in a situation similar to Example 3.1.2. Especially, $\frac{1}{X}$ is a stopped homogeneous diffusion under Q. Recall that since X is strictly positive under P, we must have

$$\int_0^1 \frac{x}{\sigma^2(x)}\, dx = \infty.$$

But any diffusion on an auxiliary probability space with the dynamics described in (9) satisfies

$$\int_1^\infty \frac{x}{\overline{\sigma}^2(x)}\, dx = \int_0^1 \frac{y}{\sigma^2(y)}\, dy = \infty$$

and is hence a true martingale by the criterion of [8], cf. also Example 3.1.2. Naturally, any stopped diffusion with the same dynamics is a martingale as well. Since the fact whether $\frac{1}{X}$ is a true martingale or not only depends on its distributional properties, we may therefore conclude that $\frac{1}{X}$ is indeed a Q-martingale. $\square$

REMARK 7.3. Theorem 7.1 deals with two strictly positive local P-martingales. It is, however, obvious that one can get a similar result for $n \geq 2$ strictly positive local P-martingales. Also note that the construction in Theorem 7.1 is only possible if the local quadratic covariation matrix of the local P-martingales is sufficiently nondegenerate. Moreover, it is interesting that the statement of Lemma 7.2 contains no further restrictions on the stochastic behaviour of Y.

We briefly want to describe a different approach focusing on "conformal local martingales" in $\mathbb{R}^d$, $d > 2$, which is dealt with in [30].

DEFINITION 7.4. A continuous local martingale X, taking values in $\mathbb{R}^d$, is called a conformal local martingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$, if $\langle X^i, X^j\rangle = \langle X^1\rangle \mathbb{1}_{\{i=j\}}$ P-almost surely for all $1 \leq i, j \leq d$.

In [30], the authors make the restriction that the conformal local martingale does not enter some compact neighborhood of the origin in $\mathbb{R}^d$. Using simple localization arguments as in Theorem 2.12 above, one can get rid off this assumption which seems somehow inappropriate when dealing with stock price processes. This yields the following extended version of Lemma 12 in [30]. We denote by $|\cdot|$ the Euclidean norm in $\mathbb{R}^d$.

THEOREM 7.5. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$ be a filtered probability space such that $(\mathcal{F}_t)_{t\geq 0}$ is the right-continuous augmentation of a standard system. For $d > 2$, let $X = (X^1, \dots, X^d)$ be a conformal local P-martingale. Suppose that $X_0 = x_0$ with $|x_0| = 1$ and define $\tau := \inf\{t \geq 0 | |X_t| = 0\}$. Then there exists a measure Q on $\mathcal{F}_\infty$, such that $\mathsf{Q}|_{\mathcal{F}_t} \gg \mathsf{P}|_{\mathcal{F}_t}$ for all $t \geq 0$ and such that*

$$Y_t := \begin{cases} \dfrac{X_t}{|X_t|^2}, & t < \tau, \\ \liminf\limits_{s\to\tau, s<\tau, s\in\mathbb{Q}} \dfrac{X_s}{|X_s|^2}, & t \geq \tau \end{cases}$$

is a conformal uniformly-integrable Q-martingale.

PROOF. Note that $\mathsf{P}(\tau < \infty) = 0$ by Knight's theorem because a standard d-dimensional Brownian motion does not return to the origin almost surely for $d > 2$. We define the stopping times $\tau_n := \inf\{t \geq 0 : |X_t| \leq \frac{1}{n}\}$. As in Lemma 11 in [30], it follows that $(|X_{t\wedge\tau_n}|^{2-d})_{t\geq 0}$ is a uniformly integrable P-martingale for all $n \in \mathbb{N}$, because $|\cdot|^{2-d}$ is harmonic. We define a consistent family of probability measures Q_n on $\mathcal{F}_{\tau_n}$ by

$$\left.\frac{d\mathsf{Q}_n}{d\mathsf{P}}\right|_{\mathcal{F}_{\tau_n}} = |X_{\tau_n}|^{2-d}, \qquad n \in \mathbb{N}.$$

Using the same trick as in the proof of Theorem 2.12, we restrict each measure Q_n to $\mathcal{F}_{\tau_n -}$. Since $(\mathcal{F}_{\tau_n -})_{n\in\mathbb{N}}$ is a standard system, there exists a unique measure Q on $\mathcal{F}_{\tau-}$, such that $\mathsf{Q}|_{\mathcal{F}_{\tau_n}} = \mathsf{Q}_n$ for all $n \in \mathbb{N}$. For any stopping time S, we thus get

$$\mathsf{Q}(S < \tau_n) = \mathbb{E}^{\mathsf{P}}\big(|X_{\tau_n}|^{2-d}\mathbb{1}_{\{S<\tau_n\}}\big) = \mathbb{E}^{\mathsf{P}}\big(|X_S|^{2-d}\mathbb{1}_{\{S<\tau_n\}}\big).$$

Choosing $S = t < \infty$, $A \in \mathcal{F}_t$ and taking $n \to \infty$ results in

$$\mathsf{Q}\big(A \cap \{t < \tau\}\big) = \mathbb{E}^{\mathsf{P}}\big(|X_t|^{2-d}\mathbb{1}_A\big).$$

Therefore, P is locally absolutely continuous to Q before τ. As explained on page 1836 there exists an extension of Q from $\mathcal{F}_{\tau-}$ to $\mathcal{F}_\infty$, which we also denote by Q.

From Lemma 12 in [30], we know that $\frac{X_{t\wedge\tau_n}}{|X_{t\wedge\tau_n}|^2}$ is a conformal Q_n-martingale. Furthermore,

$$\Big(\mathbb{E}^{\mathsf{Q}} \sup_{t<\tau} |Y_t^i|\Big)^2 \leq \mathbb{E}^{\mathsf{Q}} \sup_{t<\tau} |Y_t^i|^2 \leq 1, \qquad 1 \leq i \leq d.$$

Thus, Y is a continuous uniformly integrable Q-martingale by Exercise 1.48 in Chapter IV of [35]. Clearly, Y is also conformal. □

APPENDIX: CONDITION (P)

In Theorem 2.1, we mentioned condition (P), which was introduced in Definition 4.1 in [29] following [31] as follows.

DEFINITION A.6. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$ be a filtered measurable space, such that $\mathcal{F}$ is the σ-algebra generated by $(\mathcal{F}_t)_{t\geq 0} : \mathcal{F} = \bigvee_{t\geq 0} \mathcal{F}_t$. We shall say that the property (P) holds if and only if $(\mathcal{F}_t)_{t\geq 0}$ enjoys the following conditions:

- For all $t \geq 0$, $\mathcal{F}_t$ is generated by a countable number of sets.
- For all $t \geq 0$, there exists a Polish space Ω_t, and a surjective map π_t from Ω to Ω_t, such that $\mathcal{F}_t$ is the σ-algebra of the inverse images by π_t of Borel sets in Ω_t, and such that for all $B \in \mathcal{F}_t$, $\omega \in \Omega$, $\pi_t(\omega) \in \pi_t(B)$ implies $\omega \in B$.
- If $(\omega_n)_{n\geq 0}$ is a sequence of elements of Ω such that for all $N \geq 0$,

$$\bigcap_{n\geq 0}^{N} A_n(\omega_n) \neq \varnothing,$$

  where $A_n(\omega_n)$ is the intersection of the sets in $\mathcal{F}_n$ containing ω_n, then

$$\bigcap_{n\geq 0}^{\infty} A_n(\omega_n) \neq \varnothing.$$

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