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BOUNDARY-CROSSING PROBABILITIES OF SOME RANDOM FIELDS
RELATED TO LIKELIHOOD RATIO TESTS FOR EPIDEMIC ALTERNATIVES

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Abstract
We consider the likelihood ratio tests to detect an epidemic alternative in the
following two cases of normal observations: (1) the alternative specifies a square
wave drift in the mean value of an i.i.d. sequence; (2) the alternative permits a square
wave drift in the intercept of a simple linear regression model. To develop the
approximations for the significance levels leads us to consider boundary-crossing
problems of some two-dimensional discrete-time Gaussian fields. By the method
which was proposed originally by Woodroofe (1976) and adapted to study maxima
of some random fields by Siegmund (1988), some large deviations for the conditional
non-linear boundary-crossing probabilities are developed. Some results of Monte
Carlo experiments confirm the accuracy of these approximations.

LARGE DEVIATION; GAUSSIAN FIELDS
AMS 1991 SUBJECT CLASSIFICATION: PRIMARY 60F10
SECONDARY 60G15; 62F99

1. Introduction

In 1976 Woodroofe proposed a method of developing large deviation approximations
for boundary-crossing probabilities of one-dimensional discrete-time random processes,
which is to split the probability into a sum by means of the first crossing time, in which
every summand is the probability that the first crossing occurs at a fixed time. Siegmund
(1988) showed that this method could be modified to study maxima of some random
fields with two-dimensional time. In this paper, we try to adapt this method to develop
the large deviations for some complicated conditional boundary-crossing probabilities
of Gaussian fields, which are related to the likelihood ratio tests for some epidemic
alternatives in an independent identically distributed normal random variable se-
quence, and also in a simple normal linear regression model.

The problem of testing a change-point or an epidemic alternative has been widely
discussed in different formulations with a variety of applications. One of the most
distinguished topics is the likelihood ratio test (or the slightly generalized likelihood

Received 7 June 1991; revision received 4 November 1991.
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Research supported partially by the Alexander von Humboldt-Stiftung, the National Science Foundation
of China, and the Deutsche Forschungsgemeinschaft.
ratio tests in case that the model contains some nuisance parameters, cf. James et al.
(1987)). To evaluate its significance level leads to some boundary-crossing probabilities,
which can be precisely calculated on rare occasions. For most of the applications, some
asymptotic approximation, for example, the large deviation, is unavoidable. Let
\( Y_1, \ldots, Y_n \) be independent random variables, \( F \) and \( G \) be two different
distribution functions. The change-point testing problem is to test the null hypothesis
\[
H: \quad Y_1, \ldots, Y_n \text{ are identically distributed with } F;
\]
against the alternative
\[
K_1: \quad \text{there exist } 1 \leq n < m \text{ for which } Y_1, \ldots, Y_n \text{ are identically distributed}
\text{ with } F \text{ while } Y_{n+1}, \ldots, Y_m \text{ are identically distributed with } G.
\]
The testing of an epidemic alternative is to test the null hypothesis \( H \) versus the
alternative
\[
K_2: \quad \text{there exist } 1 \leq i < j \leq m \text{ for which } Y_1, \ldots, Y_i, Y_{j+1}, \ldots, Y_m \text{ are identically}
\text{ distributed with } F \text{ while } Y_{i+1}, \ldots, Y_j \text{ are identically distributed with } G.
\]
The hypothesis \( K_2 \) has been called an epidemic alternative because an epidemic state \( G \)
runs from time \( i + 1 \) through \( j \) after which the normal state \( F \) is restored (cf. Levin and
Kline (1985)). When \( F \) and \( G \) are completely specified, the problem is relatively easy
since the related boundary-crossing problems are of constant boundaries (cf. Siegmund
(1986), (1988), Hogan and Siegmund (1986), James et al. (1987)). However the case in
which the distribution functions are of known form but contain some unknown
parameters is more interesting and important in practice. In such a case, the significance
level of the likelihood ratio test presents a non-linear boundary-crossing probability. A
most widely utilized assumption for such a case is that \( F = N(\mu, \sigma^2) \) and \( G =
N(\mu + \delta, \sigma^2) \) with unknown \( \mu \) and \( \delta \). Siegmund (1986) developed the large deviation
approximation for the significance level for the likelihood ratio test for a change point
when \( \sigma^2 \) is known. James et al. (1988) got the parallel result with unknown \( \sigma^2 \). Kim and
Siegmund (1989) extended the above results to a simple normal linear regression model,
although the testing of an epidemic alternative is relatively unexplored. Siegmund
(1988) and Yao (1989) developed large deviation approximations for the level of the
likelihood ratio test when \( \sigma^2 \) is known.

In Section 2, we study the generalized likelihood ratio test for hypothesis \( H \) against \( K_2 \)
with \( F = N(\mu, \sigma^2) \) and \( G = N(\mu + \delta, \sigma^2) \), where all \( \mu, \delta, \sigma^2 \) are unknown. To get an
approximation for the level of the test, the large deviations for some conditional non-
linear boundary-crossing probabilities are developed (see Theorem 1). Some results of
Monte Carlo experiments illustrate the accuracy of these approximations. The proof of
the Theorem 1 is somewhat cumbersome, and we deal with it separately in Section 3.
Section 4 is devoted to a simple linear model: \( y_i \) is assumed to be a \( N(\alpha + \beta x_i, \sigma^2) \)
variable for \( i = 1, \ldots, m \) under the null hypothesis, and the alternative specifies a drift
with height \( \delta \) in \( \alpha \) from time \( i + 1 \) to \( j \) for some \( 1 \leq i < j \leq m \). For the cases of both
known \( \sigma^2 \) and unknown \( \sigma^2 \), the large deviations for the levels of generalized likelihood
ratio tests are presented in Theorem 2. The accuracy of these approximations is also assessed by some Monte Carlo experiments. The proof of Theorem 2 is omitted. It is similar to the proof of Theorem 1 in principle but more complicated in detail.

2. Testing an epidemic alternative in a normal sequence

Let \( Y_i, i = 1, \cdots, m, \) be independent normal random variables with mean \( \mu_i \) and unknown variance \( \sigma^2 > 0 \). Consider the problem of testing

\[
H_0: \mu_1 = \cdots = \mu_m = \mu;
\]

against

\[
H_1: \text{for some } 1 \leq i < j \leq m, \mu_j = \cdots = \mu_i = \mu,
\]

\[
\mu_{i+1} = \cdots = \mu_j = \mu + \delta, \text{ and } \mu_{j+1} = \cdots = \mu_m = \mu;
\]

where \( \mu \) and \( \delta \neq 0 \) are nuisance parameters.

Let \( S_n = \sum_i^n Y_i, n = 1, \cdots, m, Q = (1/m) \sum_i^m (Y_i - S_m/m)^2 \). Some algebraic calculations show that the log likelihood ratio statistic is

\[
\max_{j \leq i < j \leq m} \left\{ \frac{1}{2} \ln \left( 1 - \left( S_j - S_i - \frac{j - i}{m} S_m \right)^2 \right) \right\} \frac{\left( j - i \right) \left( 1 - \frac{j - i}{m} \right) Q}.
\]

Since it is intrinsically difficult to do statistical inference on \((i, j)\) when one of \(j - i\) and \(m - (j - i)\) is sufficiently small (cf. Siegmund (1986)), we assume that both \(j - i\) and \(m - (j - i)\) are effectively infinitely large when the sample size \(m\) tends to infinity, more precisely \(m_0 \leq j - i \leq m_1\) with \(m_0/m \to t_0\), \(m_1/m \to t_1\) for some \(0 \leq t_0 < t_1 \leq 1\). Hence the generalized likelihood ratio test of \(H_0\) against \(H_1\) rejects \(H_0\) for large values of

\[
\max_{m_0 \leq j - i \leq m_1} \left| S_j - S_i - \frac{j - i}{m} S_m \right| \left/ \sqrt{(j - i) \left( 1 - \frac{j - i}{m} \right) Q} \right.
\]

Under \(H_0\) the distribution of the process

\[
\left( S_j - S_i - \frac{j - i}{m} S_m \right) \left/ \sqrt{Q} \right., \quad i, j = 1, \cdots, m,
\]

does not depend on \(\mu\) and \(\sigma^2\) and hence by Basu's theorem (Lehmann (1959), Theorem 5.2), the process is independent of the complete sufficient statistic \((S_m, U_m)\), where \(U_m = \sum_i^m Y_i^2\). Therefore the test level can be expressed as follows:

\[
p_{m_0} = \Pr_{H_0} \left\{ \left| S_j - S_i \right| \geq b \sqrt{(j - i) \left( 1 - \frac{j - i}{m} \right) Q} \right\}
\]

for some \(m_0 \leq j - i \leq m_1\) \(S_m = 0, U_m = m\),

where \(b\) is a positive constant which is determined by the assigned test level. Theorem 1 presents its large deviation approximation, which involves the special function
(3) \[ v(x) = 2x^{-2} \exp \left\{ -2 \sum_{i} \frac{1}{n} \Phi(-x \sqrt{n}/2) \right\}, \quad (x > 0), \]

where \( \Phi \) denotes the standard normal distribution function. For numerical purposes it often suffices to use the approximation for small \( x \) (cf. Siegmund (1985), §10.4)
\[ v(x) = \exp(-0.583x) + o(x^2). \]

**Theorem 1.** Suppose \( m \to \infty, m_0 \to \infty, m_1 \to \infty \) in such a way that for some \( 0 \leq t_0 < t_1 \leq 1, m_0/m \to t_0 \) and \( m_1/m \to t_1 \). Then for \( b = c \sqrt{m} \) with \( c \in (0, 1) \) fixed,
\[ p_{m, 1} \sim \frac{1}{2\sqrt{2\pi}} b^3 (1 - c^2)^{m/2 - 3} \int_{m/m_0}^{m/m_1} \frac{1}{(1 - t)^{3/2}} \left[ v(c \sqrt{t(1 - t)(1 - c^2)}) \right]^2 dt. \]

**Remark 1.** For the one-side drift alternative hypothesis, namely to assume \( \delta > 0 \) in \( H_1 \), the likelihood ratio statistic can be taken as
\[ \max_{m_0 \leq j - i \leq m_1} \left( S_j - S_i - \frac{j - i}{m} S_m \right) / \sqrt{(j - i) \left( 1 - \frac{j - i}{m} \right)} Q. \]

Hence the level of the test is
\[ \hat{p}_{m, 1} = P_{H_0} \left\{ S_j - S_i \geq b \sqrt{(j - i) \left( 1 - \frac{j - i}{m} \right)} \right\}. \]
\[ \qquad \text{for some } m_0 \leq j - i \leq m_1 ; S_m = 0, U_m = m. \]

From the proof of Theorem 1, one can see that \( \hat{p}_{m, 1} \) is asymptotically equivalent to one half of the right-hand side of (4) as \( m \to \infty \). That means \( p_{m, 1} \sim 2 \hat{p}_{m, 1} \).

**Remark 2.** There is also a version of Theorem 1 in the case that \( \sigma^2 \) is known. The significance level of the likelihood ratio test for \( H_0 \) against \( H_1 \) is
\[ p_{m, 2} = P_{H_0} \left\{ \frac{1}{\sigma} |S_j - S_i| \geq b \sqrt{(j - i) \left( 1 - \frac{j - i}{m} \right)} \right\}. \]
\[ \qquad \text{for some } m_0 \leq j - i \leq m_1 ; S_m = 0. \]

As in Remark 1, if we restrict \( \delta \) positive in \( H_1 \), the level of the test becomes
\[ \hat{p}_{m, 2} = P_{H_0} \left\{ \frac{1}{\sigma} (S_j - S_i) \geq b \sqrt{(j - i) \left( 1 - \frac{j - i}{m} \right)} \right\} \text{ for some } m_0 \leq j - i \leq m_1 ; S_m = 0. \]

They have the following asymptotic approximations:
\[ p_{m, 2} \sim 2 \hat{p}_{m, 2} \sim \frac{1}{2} b^3 \varphi(b) \int_{m_0/m}^{m_1/m} \frac{1}{(1 - t)^{3/2}} \left[ v(c \sqrt{t(1 - t)}) \right]^2 dt, \]
\[ \qquad (x > 0), \]

where \( \Phi \) denotes the standard normal distribution function. For numerical purposes it often suffices to use the approximation for small \( x \) (cf. Siegmund (1985), §10.4)
under the same assumptions as Theorem 1, where \( \phi \) denotes the standard normal density function (see Siegmund (1988), Yao (1989)).

Tables 1 and 2 give some indication of the accuracy of (4) and (7). For both cases of unknown and known \( \sigma^2 \), two 10,000 repetition Monte Carlo experiments with \( m = 25 \) offer some estimates \( \hat{p}_{m,1}, \hat{p}_{m,2} \) and \( \hat{p}_{m,2}, \hat{p}_{m,2} \). Table 1 shows that when the \( p \)-values are near 0.10, 0.05, and 0.01, the large deviations given in (4) and (7) offer quite good approximations. From Table 2 one can see that the approximations are also good for different values of \( m_0 \) and \( m_1 \) with a given value of \( m \). In addition, a simultaneous probability for \( \hat{p}_{m,1}, \hat{p}_{m,2} \) is always about one half of the corresponding simultaneous value for \( \hat{p}_{m,1}, \hat{p}_{m,2} \), which agrees with the arguments in Remark 1 or 2. Other simulation results, not reported here, show that the essential conclusions are unchanged over a range of the \( p \)-value and sample size, although the magnitude of the difference can be more or less.

3. Proof of Theorem 1

Throughout this section we assume that \( Y_1, \ldots, Y_m \) are i.i.d. \( \mathcal{N}(0, 1) \) variables, and \( S_0 = 0, S_n = \sum_1^n Y_k, U_n = \sum_1^n Y_k^2, n = 1, \ldots, m \). We also assume that \( m \to \infty, m_0 \to \infty, \)
\[ m_1 \to \infty \] in such a way that for some \( 0 \leq t_0 < t_1 \leq 1, \ m_0/m \to t_0, \ m_1/m \to t_1, \) and \( b = c \sqrt{m} \) with \( c \in (0, 1) \). We use the notations
\[ t_n = n/m, \ \ \ \ \mu_n = \frac{1}{2} c / \sqrt{t_n(1 - t_n)}, \]
\[ J(p, q) = \{(i, j) : m_0 \leq j - i \leq m_1, \text{ and } j < q \} \cup \{(i, q) : q - m_1 < i \leq (q - m_0) \wedge p \} \]
for \( m_0 \leq n \leq m_1, \ 1 < p < q < m, \) and
\[ P_{\xi, \lambda}^{(m)}(A) = P(A \mid S_m = \xi, U_m = \lambda) \]
for \( A \in \sigma(Y_1, \ldots, Y_m) \). Hence
\[ p_{m_1} = P^{(m)}_{0, m_1} \left\{ |S_j - S_i| \geq b \sqrt{(j - i) \left( 1 - \frac{j - i}{m} \right)} \text{ for some } m_0 \leq j - i \leq m_1 \right\}, \]
\[ \hat{p}_{m_1} = P^{(m)}_{0, m_1} \left\{ S_j - S_i \geq b \sqrt{(j - i) \left( 1 - \frac{j - i}{m} \right)} \text{ for some } m_0 \leq j - i \leq m_1 \right\}. \]

The following lemmas are technical and will be used in the proof of Theorem 1.

**Lemma 1.** Assume \( m_0 \leq n \leq m_1, \ 0 \leq x < \log m \). As \( m \to \infty \),
\( (i) \ P^{(m)}_{0, m}(S_n \in b \sqrt{n(1 - n/m)} + dx) / dx \sim (2\pi)^{-1/2} n[(1 - n/m)]^{-1/2}(1 - c^2)^{m-n/2} \exp\{-2\mu_n x/(1 - c^2)\} \)
uniformly for \( x \) and \( n \);
\( (ii) \ P^{(m)}_{0, m}(S_n \geq b \sqrt{n(1 - n/m)} + \log m) = o(m^{-1/2}(1 - c^2)^{m/2}); \)
\( (ii) \ P^{(m)}_{0, m}(\left| U_m - m[t_n + c^2(1 - 2t_n)] \right| > m^{2/3} \left| S_n = b \sqrt{n(1 - n/m)} + x \right) = o(m^{-1/3}). \)

Lemma 1 follows from James et al. (1988), Lemma 1.

**Lemma 2.** As \( m \to \infty \),
\[ P \left\{ S_j - S_i < b \sqrt{(j - i) \left( 1 - \frac{j - i}{m} \right)} \left. \right\}, \text{ for all } 1 \leq j - i \leq m_1 - (\log m)^2 \mid S_m, \right\} \]
\[ = b \sqrt{m_1(1 - t_1) + x}, \ U_m = m[t_i + c^2(1 - 2t_i)] + y \to 1. \]

**Proof.** From the equality (2.3) of James et al. (1988), one obtains
\[ P^{(m)}_{\xi, \lambda}(S_n \in dx) = \pi^{-1/2} \frac{\Gamma((m - 1)/2)}{\Gamma(m/2 - 1)} (1 - y^2)^{m-n/2} dy, \]
where
\[ y = \sqrt{\frac{m}{n(m - n)}} (\lambda - \xi^2/m) - 1/2 \left( x - \frac{n}{m} \xi \right). \]
From this one can show that for every \( m_0 < j \leq m_1 - (\log m)^2 \),
\[ P(S_j \geq b \sqrt{j(1 - j/m)} \mid S_{m_0} = b \sqrt{m_0(1 - t_i)} + x, U_{m_1} = m[t_1 + c^2(1 - 2t)] + y) \]

\[ \leq \pi^{-1/2} \frac{\Gamma((m_1 - 1)/2)}{\Gamma(m_1/2 - 1)} \int_{d_m} (1 - y^2)^{(m_1 - 2)/2} dy (1 + o(1)), \]

where

\[ d_m = \frac{\log m}{\sqrt{m_1}} c/(1 + \sqrt{1 - \tau_i}) \sqrt{t_i(1 - c^2)}. \]

For any \( a \in (0, 1) \), it is easy to prove that

\[ \int_a^1 (1 - y^2)^{(m - 4)/2} dy \leq (1 - a^2)^{(m - 2)/2}/[a(m - 2)]. \]

Consequently,

\[ 1 - \text{LHS of (9)} \leq \left( m^2 \pi^{-1/2} \frac{\Gamma((m_1 - 1)/2)}{\Gamma(m_1/2 - 1)} (1 - d_m^2)^{(m_1 - 2)/2}/[d_m(m_1 - 2)] \right) (1 + o(1)) \to 0, \]

as \( m \to \infty \), which completes the proof.

**Lemma 3.** Let \( p \geq \sqrt{m}, n = q - p \geq m_0 + \sqrt{m}, \) and \( t_n = n/m \to t' \in (t_0, t_1) \) as \( m \to \infty \). Then uniformly in such \((p, q)\) and \( x \in (0, \log m) \), \(|y| < m^{1/2}\), the following relation holds when \( m \to \infty \):

\[ P_{0, m}^{(p, q)} \left\{ S_j - S_i < b \sqrt{(j - i) \left( 1 - \frac{j - i}{m} \right)} \right\}, \text{ for all } (i, j) \in J(p, q) \mid S_q - S_p \]

\[ = b \sqrt{n(1 - n/m)} + x, U_q - U_p = m[t_n + c^2(1 - 2t_n)] + y \]

\[ \sim P \left\{ \min_{k \geq 1} \tilde{S}_{k} > x \right\} \left( \min_{k \geq 1} \tilde{S}_{k} + \min_{k \geq 0} \tilde{S}_{k} > x \right), \]

where \( \tilde{S}_k = (1 - c)^{1/2}(S_k + k\mu_n) \) for \( k \geq 0 \), \( \{\tilde{S}_k, k \geq 0\} \) is an independent copy of \( \{S_k, k \geq 0\} \), and \( \mu_n \) and \( J(p, q) \) are defined in (8).

**Proof.** To simplify the notation, let \( P^{(p, q)} \) denote the conditional probability measure on the left-hand side of (10).

Since \( \sqrt{k(1 - k/m)} \) is a convex function of \( k \), the following inequality holds for \( i, j \geq 1, n + i \leq m, \) and \( n - j \geq 1 \).

\[ \sqrt{n \left( 1 - \frac{n}{m} \right)} + \sqrt{(n - j + i) \left( 1 - \frac{n - j + i}{m} \right)} \]

\[ > \sqrt{(n - j) \left( 1 - \frac{n - j}{m} \right)} + \sqrt{(n + i) \left( 1 - \frac{n + i}{m} \right)}. \]

Consequently
\[
\left\{ S_j - S_i < b \sqrt{(j - i) \left( 1 - \frac{j - i}{m} \right)} \right\}, \text{ for all } (i, j) \in J(p, q), \quad (11)
\]

\[
\cap \left\{ S_q - S_i < b \sqrt{(q - i) \left( 1 - \frac{q - i}{m} \right)} \right\}, \text{ for all } q - m_1 < i \leq (q - m_0) \cap p.
\]

By Lemma 2, one can see that under the measure \( P^{(p, q)} \) the inequalities on the right-hand side of the above expression are asymptotically almost surely valid for some indices, especially for all \( 1 \leq i < j \leq p \). Therefore

\[
\text{LHS of (15)} = P^{(p, q)} \left\{ S_q - S_{q-j} < b \sqrt{(n + i) \left( 1 - \frac{n + i}{m} \right)} \right\} \text{ for all } 1 \leq i < (\log m)^2,
\]

and

\[
S_{q-j} - S_{q-i} < b \sqrt{(n - j + i) \left( 1 - \frac{n - j + i}{m} \right)},
\]

for all \( i \leq 0, j \geq 1 \), and \( 0 < j - i < (\log m)^2 \) + o(1)

\[
(12)
= P^{(p, q)}(S_{q-i} - S_p > x - i\mu_n(1 - 2t_n)) \text{ for all } 1 \leq i < (\log m)^2; \quad \text{and}
\]

\[
S_q - S_{q-j} + S_{q-i} - S_p > x + (j - i)\mu_n(1 - 2t_n) \quad \text{for all}
\]

\[
i \leq 0, j \geq 1 \text{ and } 0 < j - i < (\log m)^2 \} + o(1).
\]

The last equality follows the asymptotic relation

\[
b \sqrt{n \left( 1 - \frac{n}{m} \right)} - b \sqrt{(n - k) \left( 1 - \frac{n - k}{m} \right)} = k\mu_n(1 - 2t_n) + o(1)
\]

for \( 1 \leq k \leq (\log m)^2 \).

On the other hand, one can show (directly, or by Lemma 1 in Chapter 4 of Hu (1985)) that as \( m \to \infty \), the ratio of the \( P^{(p, q)} \)-joint density of random variables

\[
S_{p-i} - S_p + i\mu_n(1 - 2t_n), \quad i = 1, \ldots, (\log m)^2,
\]

or

\[
S_q - S_{q-j} - j\mu_n(1 - 2t_n), \quad j = 1, \ldots, (\log m)^2,
\]

or

\[
S_{p-i} - S_p + i\mu_n(1 - 2t_n), \quad i = -1, \ldots, -(\log m)^2
\]

to the joint density of \( \mathcal{S}_k, k = 1, \ldots, (\log m)^2 \), converges to 1, and furthermore asymptotically these three collections of random variables are stochastically independent. Hence
the right-hand side of (12) is asymptotically equivalent to the right-hand side of (10). The proof is completed.

Lemma 4. Let \(\{\hat{S}_k, k \geq 0\}\) and \(\{\check{S}_k, k \geq 0\}\) be the same as in Lemma 3. Then

\[
\int_0^\infty \exp \left( -\frac{2\mu x}{1-c} \right) \mathbb{P} \left\{ \min_{k \geq 1} \hat{S}_k > x \right\} \mathbb{P} \left\{ \min_{k \geq 1} \check{S}_k + \min_{k \geq 0} \check{S}_k > x \right\} \, dx
= 2\mu^2 (1-c)^{-4} \left[ \nu (2\mu_x/(1-c)^{1/2}) \right]^2.
\]


Proof of Theorem 1. The proof proceeds in two steps. At first one proves that \(\hat{p}_{m,1}\) is asymptotically one half of the right-hand side of (4). The second is to show \(\hat{p}_{m,1} \sim \frac{1}{2} \check{p}_{m,1}\). We split \(\hat{p}_{m,1}\) into the following sum:

\[
\hat{p}_{m,1} = \left( \sum_{a = (m_0 + \sqrt{m})}^{m_1} \sum_{q-p=n} \sum_{p < \sqrt{m}} \sum_{a = (m_0 + \sqrt{m})}^{|m_0 + \sqrt{m}| - 1} \sum_{q-p=n} \right) \mathbb{P}^{(m)}_{\check{p},m} \left\{ S_q - S_p \geq b \sqrt{n(1-n/m)}; S_j - S_i < b \sqrt{(j-i) \left( 1 - \frac{i-j}{m} \right)} \right\}
\]

\[\forall (i,j) \in J(p,q)\}

\[= p_1 + p_2 + p_3.\]

First of all, we try to calculate the main part \(p_1\). For any \((p,q)\) with \(p \geq \sqrt{m}\), and \(q-p=n\) between \(m_0\) and \(m_1\):

\[
\mathbb{P}^{(m)}_{\check{p},m} \left\{ S_q - S_p \geq b \sqrt{n(1-n/m)}; S_j - S_i < b \sqrt{(j-i) \left( 1 - \frac{i-j}{m} \right)} \right\}
= \int_{A_{m,n}} \mathbb{P}^{(m)}_{\check{p},m} \{ S_q - S_p \geq b \sqrt{n(1-n/m)} + dx \}
\]

\[
\times \mathbb{P}^{(m)}_{\check{p},m} \left\{ S_j - S_i < b \sqrt{(j-i) \left( 1 - \frac{i-j}{m} \right)} \right\} \forall (i,j) \in J(p,q)\}
\]

\[
\left( S_q - S_p \geq b \sqrt{n(1-n/m)} + x, U_q - U_p \in [m[t_n + c^2(1 - 2t_n)] + y] \right) \}
\]

\[= b \sqrt{n(1-n/m)} + x, U_q - U_p \in [m[t_n + c^2(1 - 2t_n)] + y] \}
\]

where

\[A_{m,n} = \{(x,y): x \geq 0, y \geq 0, b \sqrt{n(1-n/m)} + x
\]

\[< \sqrt{nm[t_n + c^2(1 - 2t_n)] + ny} \times \sqrt{(m-n)[m - m[t_n + c^2(1 - 2t_n)] - y}]\}.\]

Lemma 1 (ii) indicates that the \(\mathbb{P}^{(m)}_{\check{p},m}\)-probability of \(S_q \geq b \sqrt{n(1-n/m)} + \log m\) is of higher order, which can be neglected. Hence it seems plausible that the range of the value
of $x$ in the integral in (1.3) can be restricted to the interval $[0, \log m]$. Similarly by Lemma 1 (iii), we can also restrict the range of the value of $y$ to, say, $|y| < m^{3/2}$. Furthermore using Lemmas 1 (i), 3, and 4, we have

$$p_1 \sim \frac{c^3}{4\sqrt{2\pi}} \sum_{n = m_0 + m_1}^{m_1} \frac{1}{n \sqrt{(1 - t_n)(1 - t_{n-1})(1 - c^2))}} \int_{m_0/m}^{m_1/m} \frac{1}{(1 - t)^2} [\phi(c(1 - t)(1 - c^2))]^2 dt,$$

which is just half of the right-hand side of (4).

On the other hand, it follows from Lemma 1 easily that

$$p_2 \sim \sqrt{m} \sum_{n = m_0}^{m_1} P_{0,m}^{(m)} \{ S_n \geq b \sqrt{n(1 - n/m)} \} = O(m(1 - c^2)^{m/2});$$

$$p_3 \sim m \sum_{n = m_0}^{m_1} P_{0,m}^{(m)} \{ S_n \leq b \sqrt{n(1 - n/m)} \} = O(m(1 - c^2)^{m/2}),$$

that means that $p_{m,1}$ is asymptotically equivalent to $p_1$.

To show $p_{m,1} \sim 2p_{m,1}$, one only needs to prove that

$$(14) \quad 2p_{m,1} - p_{m,1} = o(m^{3/2}(1 - c^2)^{m/2}),$$

since obviously $2p_{m,1} > p_{m,1}$. For any $(p, q)$ with $q - p = n$ between $m_0$ and $m_1$,

$$2P_{0,m}^{(m)} \{ S_q - S_p \geq b \sqrt{n(1 - n/m)} \};$$

and

$$S_j - S_i < b \sqrt{(j - i)\left(1 - \frac{j - i}{m}\right)}, \forall (i, j) \in J(p, q) \}$$

$$(15) \quad -P_{0,m}^{(m)} \{ |S_q - S_p| \geq b \sqrt{n(1 - n/m)} ;$$

and

$$S_j - S_i < b \sqrt{(j - i)\left(1 - \frac{j - i}{m}\right)}, \forall (i, j) \in J(p, q) \}$$

$$= 2 \int_{A_{m,t}} f_m(p, q)P_{0,m}^{(m)} \{ S_n \leq b \sqrt{n(1 - n/m)} + dx, U_n \in m[t_n + c^2(1 - 2t_n)] + dy \},$$

where

$$f_m(p, q) = P_{0,m}^{(m)} \left\{ S_j - S_i \leq -b \sqrt{(j - i)\left(1 - \frac{j - i}{m}\right)} \right\},$$

for some $(i, j) \in J(p, q) \} S_q - S_p = b \sqrt{n(1 - n/m)} + x,$

$$U_q - U_p \in m[t_n + c^2(1 - 2t_n)] + y \}.$$
\[ P \left[ S_k < b \sqrt{n(1 - n/m)} + x + b \sqrt{(n-k) \left(1 - \frac{n-k}{m}\right)} \right], \]

\[ \forall 1 \leq k < n - m \mid S_n = b \sqrt{n(1 - n/m)} + x, \quad U_n = m[t_n + c^2(1 - 2t_n)] + y \]

\[ \times P \left[ S_k < b \sqrt{n(1 - n/m)} + x + b \sqrt{(n+k) \left(1 - \frac{n+k}{m}\right)} \right], \]

\[ \forall 1 \leq k < m - n \mid S_{m-n} = b \sqrt{n(1 - n/m)} + x, \]

\[ U_{m-n} = m[t_{m-n} + c^2(1 - 2t_{m-n})] - y. \]

With some similar arguments as in the proof of Lemma 2, one can easily show that both of the probabilities in the above product tend to 1 uniformly for \(0 \leq x < \log m\), and \(|y| < m^{1/2}\). It follows from Lemma 1 that the left-hand side of (15) is \(o(m^{-12}(1 - c^2)^{m^2})\), which entails the validity of relation (14). This completes the proof.

4. Testing an epidemic alternative in a simple linear model

Suppose that \(Y_1, \ldots, Y_m\) are independent and normally distributed with common variance \(\sigma^2\). This section concerns the likelihood ratio tests of the null hypothesis

\[ H_0: \ EY_k = \alpha + \beta x_k, \quad k = 1, \ldots, m \]

against the alternative

\[ H_1: \text{there exist } 1 \leq i < j \leq m \text{ such that} \]

\[ EY_k = \begin{cases} \alpha + \beta x_k, & k = 1, \ldots, i, j + 1, \ldots, m; \\ \alpha + \delta + \beta x_k, & k = i + 1, \ldots, j, \end{cases} \]

where \(x_1, \ldots, x_m\) are given constants, and \(\alpha, \beta, \delta (\neq 0)\) play the role of nuisance parameters. The hypothesis \(H_0\) specifies a usual straight-line regression model, and under this model the maximum likelihood estimators for \(\alpha, \beta, \) and \(\sigma^2\) are

\[ \hat{\alpha} = \bar{Y} - \hat{\beta} \bar{x}; \]

\[ \hat{\beta} = \frac{\sum_i (Y_i - \bar{Y})(x_i - \bar{x})}{\sum_i (x_i - \bar{x})^2}; \]

\[ \hat{\sigma}^2 = m^{-1} \left\{ \sum_i (Y_i - \bar{Y})^2 - \beta \sum_i (Y_i - \bar{Y})(x_i - \bar{x}) \right\} \]

respectively, where \(\bar{Y} = (1/m) \sum_i Y_i\), and \(\bar{x} = (1/m) \sum_i x_i\). Some tedious calculation shows that the generalized likelihood ratio test rejects \(H_0\) for large values of \(\max_{m \leq i \leq m} \sigma^{-1} U_m(i, j)\) when \(\sigma^2\) is known, or for large values of \(\max_{m \leq i \leq m} \hat{\alpha}^{-1} U_m(i, j)\) when \(\sigma^2\) is unknown, where
\[ U_m(i, j) = \left| S_j - S_i - (j - i)\hat{\alpha} - \hat{\beta} \sum_{k=i+1}^{j} x_k \right| / \sqrt{(j - i) \left( 1 - \frac{j - i}{m} \right) D_m(i, j)} , \]
\[ D_m(i, j) = 1 - \left( \sum_{k=i+1}^{j} x_k - (j - i)\hat{x} \right)^2 / \left[ (j - i) \left( 1 - \frac{j - i}{m} \right) \sum_{k=1}^{m} (x_k - \bar{x})^2 \right] , \]

and \( S_k \) denotes the partial sum of \( Y_k \)'s. By Basu's theorem, the same arguments as in Section 2 entail that the process \( U_m(i, j) \), \( i, j = 1, \cdots, m \), is independent of a complete sufficient statistic, which is \( (\hat{\alpha}, \hat{\beta}) \) when \( \sigma^2 \) is known, or \( (\hat{\alpha}, \hat{\beta}, \hat{\sigma}^2) \) when \( \sigma^2 \) is unknown. Hence, for the two cases the significance level can be expressed as

\[ p_{m, i} = P_{\theta_0} \left[ \left| S_j - S_i \right| \geq b \sqrt{(j - i) \left( 1 - \frac{j - i}{m} \right) U_m(i, j)} \right] ; \]

for some \( m_0 \leq j - i \leq m_1 \mid \hat{\alpha} = 0, \hat{\beta} = 0 \};

\[ p_{m, a} = P_{\theta_0} \left[ \left| S_j - S_i \right| \geq b \sqrt{(j - i) \left( 1 - \frac{j - i}{m} \right) U_m(i, j)} \right] , \]

for some \( m_0 \leq j - i \leq m_1 \mid \hat{\alpha} = 0, \hat{\beta} = 0, \hat{\sigma}^2 = 1 \}

respectively, where \( b \) is a positive constant. Theorem 2 presents some large deviation approximations for these probabilities in a special case, say \( x_k = k/m \), which can be thought of as the time at which the \( k \)th of equally spaced observations is made.

**Theorem 2.** Suppose \( m \to \infty, m_0 \to \infty, m_1 \to \infty \) in such a way that for some \( 0 \leq t_0 < t_1 \leq 1 \), \( m_0/m \to t_0 \) and \( m_1/m \to t_1 \). Then for \( b = c \sqrt{m} \) with \( c > 0 \) fixed,

(i) if \( \sigma^2 \) is known,

\[ p_{m, i} \sim \frac{1}{2} b^2 \rho(b) \int_{m_0/m}^{m_1/m} dt \int_0^{1-t} \left[ \mu(t, s) \nu(c \sqrt{\mu(t, s)}) \right]^2 ds ; \]

(ii) if \( \sigma^2 \) is unknown, and \( c \in (0, 1) \),

\[ p_{m, a} \sim \frac{1}{2\sqrt{2\pi}} b^2 (1 - c^2)^{m-\gamma/2} \]

\[ \times \int_{m_0/m}^{m_1/m} dt \int_0^{1-t} \left[ \mu(t, s) \nu(c \sqrt{\mu(t, s)/(1 - c^2)}) \right]^2 ds , \]

where \( \nu(x) \) is given in (3), and

\[ \mu(t, s) = \left[ t(1 - t) \left( 1 - \frac{3t}{1 - t} (1 - t - 2s)^2 \right) \right]^{-1} . \]

**Remark 3.** If we restrict \( \delta \) to be positive in \( H_0 \), the level of the likelihood ratio test would be
TABLE 3
\(n = 25, m_0 = 1, m_1 = 24\)

<table>
<thead>
<tr>
<th>(b)</th>
<th>(\sigma^2) known</th>
<th>(\sigma^2) unknown</th>
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<td>(\text{Monte Carlo})</td>
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<td>0.027</td>
</tr>
<tr>
<td>3.97</td>
<td>0.010</td>
<td>0.009</td>
</tr>
</tbody>
</table>

\[
\hat{p}_{m,1} = p_{m,1} \begin{bmatrix} 1 \\ \sigma \end{bmatrix} \begin{bmatrix} S_j - S_i \end{bmatrix} \geq b \sqrt{(j-i) \left( 1 - \frac{j-i}{m} \right) U_m(i,j)},
\]

for some \(m_0 \leq j - i \leq m_1 \mid \alpha = 0, \beta = 0\),

when \(\sigma^2\) is known; and

\[
\hat{p}_{m,4} = p_{m,4} \begin{bmatrix} 1 \\ \sigma \end{bmatrix} \begin{bmatrix} S_j - S_i \end{bmatrix} \geq b \sqrt{(j-i) \left( 1 - \frac{j-i}{m} \right) U_m(i,j)},
\]

for some \(m_0 \leq j - i \leq m_1 \mid \alpha = 0, \beta = 0, \sigma^2 = 1\),

when \(\sigma^2\) is unknown. Similar to Remark 1 and 2, one can show that under the same assumptions as in Theorem 2, \(2\hat{p}_{m,3} \sim p_{m,3}\), and \(2\hat{p}_{m,4} \sim p_{m,4}\) when \(m \to \infty\).

Remark 4. The proof of Theorem 2 is omitted here since it is in principle similar to the proof of Theorem 1. One thing which is worth mentioning is that when the hypothesis \(H_0\) holds and also \(\alpha = 0, \beta = 0\),

\[
Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_{m_1} \end{pmatrix} = (I - P)Y,
\]

where \(I\) is the \(m \times m\) identity matrix, \(P\) denotes the projection matrix on the linear space spanned by \(1 = (1, \cdots , 1)'\) and \(x = (x_1, \cdots , x_m)'\). Consequently under such conditions, \(Y\) is a \(m\)-dimensional normal random vector with mean zero and variance \(\sigma^2(I - P)\) when \(\sigma^2\) is known. When \(\sigma^2\) is unknown, \(\hat{\sigma}^2 = m^{-1} \|Y\|_2^2\), from which one can easily get the conditional distribution of \(Y\).

Tables 3 and 4 present some results of two 10,000 repetition Monte Carlo experiments, which assess the accuracy of Theorem 2, and are also agreeable to the asymptotic relations \(2\hat{p}_{m,3} \sim p_{m,3}\), and \(2\hat{p}_{m,4} \sim p_{m,4}\).
Table 4
\( m = 25, \ h = 3.30 \)

| \((m_0, m_1)\) | \(\sigma^2\) known | | | \(\sigma^2\) unknown | | |
|----------------|--------------------|-----------------|----------------|--------------------|-----------------|
|                | Approximation \((16)\) | Monte Carlo \(\beta_{m,j}\) | Monte Carlo \(\tilde{\beta}_{m,j}\) | Approximation \((17)\) | Monte Carlo \(p_{m,i}\) | Monte Carlo \(\tilde{p}_{m,i}\) |
| (1, 24)        | 0.106              | 0.100           | 0.051          | 0.071              | 0.065           | 0.034          |
| (1, 21)        | 0.102              | 0.094           | 0.051          | 0.069              | 0.064           | 0.033          |
| (4, 24)        | 0.066              | 0.059           | 0.033          | 0.048              | 0.043           | 0.023          |
| (4, 21)        | 0.062              | 0.057           | 0.032          | 0.054              | 0.041           | 0.023          |
| (4, 18)        | 0.057              | 0.055           | 0.031          | 0.042              | 0.040           | 0.022          |
| (7, 21)        | 0.043              | 0.042           | 0.023          | 0.031              | 0.029           | 0.015          |
| (7, 18)        | 0.037              | 0.039           | 0.022          | 0.028              | 0.027           | 0.014          |

Acknowledgements

A part of the results was derived during the author’s stay at the Institute of Mathematical Statistics in the University of Freiburg and the Sonderforschungsbereich 123 in the University of Heidelberg. The hospitality of the institutes, especially the help and encouragement of Professor H. R. Lerche, is gratefully mentioned. The author is also indebted to Professor D. Siegmund for many helpful suggestions.

References


