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BOUNDARY-CROSSING PROBABILITIES OF SOME RANDOM FIELDS RELATED TO LIKELIHOOD RATIO TESTS FOR EPIDEMIC ALTERNATIVES

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Abstract

We consider the likelihood ratio tests to detect an epidemic alternative in the following two cases of normal observations: (1) the alternative specifies a square wave drift in the mean value of an i.i.d. sequence; (2) the alternative permits a square wave drift in the intercept of a simple linear regression model. To develop the approximations for the significance levels leads us to consider boundary-crossing problems of some two-dimensional discrete-time Gaussian fields. By the method which was proposed originally by Woodroofe (1976) and adapted to study maxima of some random fields by Siegmund (1988), some large deviations for the conditional non-linear boundary-crossing probabilities are developed. Some results of Monte Carlo experiments confirm the accuracy of these approximations.

LARGE DEVIATION; GAUSSIAN FIELDS

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1. Introduction

In 1976 Woodroofe proposed a method of developing large deviation approximations for boundary-crossing probabilities of one-dimensional discrete-time random processes, which is to split the probability into a sum by means of the first crossing time, in which every summand is the probability that the first crossing occurs at a fixed time. Siegmund (1988) showed that this method could be modified to study maxima of some random fields with two-dimensional time. In this paper, we try to adapt this method to develop the large deviations for some complicated conditional boundary-crossing probabilities of Gaussian fields, which are related to the likelihood ratio tests for some epidemic alternatives in an independent identically distributed normal random variable sequence, and also in a simple normal linear regression model.

The problem of testing a change-point or an epidemic alternative has been widely discussed in different formulations with a variety of applications. One of the most distinguished topics is the likelihood ratio test (or the slightly generalized likelihood

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ratio tests in case that the model contains some nuisance parameters, cf. James et al. (1987)). To evaluate its significance level leads to some boundary-crossing probabilities, which can be precisely calculated on rare occasions. For most of the applications, some asymptotic approximation, for example, the large deviation, is unavoidable. Let $Y_1, \cdots, Y_m$ be independent random variables, F and G be two different distribution functions. The change-point testing problem is to test the null hypothesis

H: $Y_1, \cdots, Y_m$ are identically distributed with F;

against the alternative

K_1: there exist $1 \leqq n < m$ for which $Y_1, \cdots, Y_n$ are identically distributed with F while $Y_{n+1}, \cdots, Y_m$ are identically distributed with G.

The testing of an epidemic alternative is to test the null hypothesis H versus the alternative

K_2: there exist $1 \leqq i < j \leqq m$ for which $Y_1, \cdots, Y_i, Y_{j+1}, \cdots, Y_m$ are identically distributed with F while $Y_{i+1}, \cdots, Y_j$ are identically distributed with G.

The hypothesis K_2 has been called an epidemic alternative because an epidemic state G runs from time $i+1$ through j after which the normal state F is restored (cf. Levin and Kline (1985)). When F and G are completely specified, the problem is relatively easy since the related boundary-crossing problems are of constant boundaries (cf. Siegmund (1986), (1988), Hogan and Siegmund (1986), James et al. (1987)). However the case in which the distribution functions are of known form but contain some unknown parameters is more interesting and important in practice. In such a case, the significance level of the likelihood ratio test presents a non-linear boundary-crossing probability. A most widely utilized assumption for such a case is that $F = \mathrm{N}(\mu, \sigma^2)$ and $G = \mathrm{N}(\mu + \delta, \sigma^2)$ with unknown μ and δ. Siegmund (1986) developed the large deviation approximation for the significance level for the likelihood ratio test for a change point when σ^2 is known. James et al. (1988) got the parallel result with unknown σ^2. Kim and Siegmund (1989) extended the above results to a simple normal linear regression model, although the testing of an epidemic alternative is relatively unexplored. Siegmund (1988) and Yao (1989) developed large deviation approximations for the level of the likelihood ratio test when σ^2 is known.

In Section 2, we study the generalized likelihood ratio test for hypothesis H against K_2 with $F = \mathrm{N}(\mu, \sigma^2)$ and $G = \mathrm{N}(\mu + \delta, \sigma^2)$, where all μ, δ, σ^2 are unknown. To get an approximation for the level of the test, the large deviations for some conditional non-linear boundary-crossing probabilities are developed (see Theorem 1). Some results of Monte Carlo experiments illustrate the accuracy of these approximations. The proof of the Theorem 1 is somewhat cumbersome, and we deal with it separately in Section 3. Section 4 is devoted to a simple linear model: y_i is assumed to be a $\mathrm{N}(\alpha + \beta x_i, \sigma^2)$ variable for $i = 1, \cdots, m$ under the null hypothesis, and the alternative specifies a drift with height δ in α from time $i+1$ to j for some $1 \leqq i < j \leqq m$. For the cases of both known σ^2 and unknown σ^2, the large deviations for the levels of generalized likelihood ratio tests are presented in Theorem 2. The accuracy of these approximations is also assessed by some Monte Carlo experiments. The proof of Theorem 2 is omitted. It is similar to the proof of Theorem 1 in principle but more complicated in detail.

2. Testing an epidemic alternative in a normal sequence

Let Y_i, $i = 1, \cdots, m$, be independent normal random variables with mean μ_i and unknown variance $\sigma^2 > 0$. Consider the problem of testing

$$\mathrm{H}_0\colon\ \mu_1 = \cdots = \mu_m = \mu;$$

against

$$\begin{aligned}\mathrm{H}_1\colon\ &\text{for some } 1 \leqq i < j \leqq m, \mu_1 = \cdots = \mu_i = \mu,\\ &\mu_{i+1} = \cdots = \mu_j = \mu + \delta, \text{ and } \mu_{j+1} = \cdots = \mu_m = \mu,\end{aligned}$$

where μ and $\delta \neq 0$ are nuisance parameters.

Let $S_n = \Sigma_1^n Y_k$, $n = 1, \cdots, m$, $Q = (1/m)\,\Sigma_1^m (Y_k - S_m/m)^2$. Some algebraic calculations show that the log likelihood ratio statistic is

$$\max_{1 \leqq i < j \leqq m} -\frac{1}{2} m \log \left\{1 - \left(S_j - S_i - \frac{j-i}{m} S_m\right)^2 \Big/ \left[(j-i)\left(1 - \frac{j-i}{m}\right) Q\right]\right\}.$$

Since it is intrinsically difficult to do statistical inference on (i, j) when one of $j - i$ and $m - (j - i)$ is sufficiently small (cf. Siegmund (1986)), we assume that both $j - i$ and $m - (j - i)$ are effectively infinitely large when the sample size m tends to infinity, more precisely $m_0 \leqq j - i \leqq m_1$ with $m_0/m \rightarrow t_0$, $m_1/m \rightarrow t_1$ for some $0 \leqq t_0 < t_1 \leqq 1$. Hence the generalized likelihood ratio test of H_0 against H_1 rejects H_0 for large values of

$$\max_{m_0 \leqq j - i \leqq m_1} \left|S_j - S_i - \frac{j-i}{m} S_m\right| \Big/ \sqrt{(j-i)\left(1 - \frac{j-i}{m}\right) Q}\,. \tag{1}$$

Under H_0 the distribution of the process

$$\left(S_j - S_i - \frac{j-i}{m} S_m\right) \Big/ \sqrt{Q}, \qquad i, j = 1, \cdots, m,$$

does not depend on μ and σ^2 and hence by Basu's theorem (Lehmann (1959), Theorem 5.2), the process is independent of the complete sufficient statistic (S_m, U_m), where $U_m = \Sigma_1^m Y_k^2$. Therefore the test level can be expressed as follows:

$$\begin{aligned}p_{m,1} \equiv P_{\mathrm{H}_0}\Bigg\{ |S_j - S_i| \geqq b \sqrt{(j-i)\left(1 - \frac{j-i}{m}\right)}\\ \text{for some } m_0 \leqq j - i \leqq m_1 \,|\, S_m = 0,\ U_m = m\Bigg\},\end{aligned} \tag{2}$$

where b is a positive constant which is determined by the assigned test level. Theorem 1 presents its large deviation approximation, which involves the special function

(3) $$v(x) = 2x^{-2} \exp\left\{ -2 \sum_1^\infty \frac{1}{n} \Phi(-x\sqrt{n}/2) \right\} \qquad (x > 0),$$

where Φ denotes the standard normal distribution function. For numerical purposes it often suffices to use the approximation for small x (cf. Siegmund (1985), §10.4)

$$v(x) = \exp(-0.583x) + o(x^2).$$

Theorem 1. Suppose $m \to \infty$, $m_0 \to \infty$, $m_1 \to \infty$ in such a way that for some $0 \leqq t_0 < t_1 \leqq 1$, $m_0/m \to t_0$ and $m_1/m \to t_1$. Then for $b = c\sqrt{m}$ with $c \in (0, 1)$ fixed,

(4) $$p_{m,1} \sim \frac{1}{2\sqrt{2\pi}} b^3 (1-c^2)^{m/2-3} \int_{m_0/m}^{m_1/m} \frac{1}{(1-t)t^2} [v(c/\sqrt{t(1-t)(1-c^2)})]^2 dt.$$

Remark 1. For the one-side drift alternative hypothesis, namely to assume $\delta > 0$ in H_1, the likelihood ratio statistic can be taken as

$$\max_{m_0 \leqq j-i \leqq m_1} \left(S_j - S_i - \frac{j-i}{m} S_m \right)^+ \Big/ \sqrt{(j-i)\left(1 - \frac{j-i}{m}\right) Q}.$$

Hence the level of the test is

(5) $$\tilde{p}_{m,1} \equiv P_{H_0}\left\{ S_j - S_i \geqq b \sqrt{(j-i)\left(1 - \frac{j-i}{m}\right)} \text{ for some } m_0 \leqq j-i \leqq m_1 \,\middle|\, S_m' = 0,\ U_m = m \right\}.$$

From the proof of Theorem 1, one can see that $\tilde{p}_{m,1}$ is asymptotically equivalent to one half of the right-hand side of (4) as $m \to \infty$. That means $p_{m,1} \sim 2\tilde{p}_{m,1}$.

Remark 2. There is also a version of Theorem 1 in the case that σ^2 is known. The significance level of the likelihood ratio test for H_0 against H_1 is

(6) $$p_{m,2} \equiv P_{H_0}\left\{ \frac{1}{\sigma} |S_j - S_i| \geqq b \sqrt{(j-i)\left(1 - \frac{j-i}{m}\right)} \text{ for some } m_0 \leqq j-i \leqq m_1 \,\middle|\, S_m = 0 \right\}.$$

As in Remark 1, if we restrict δ positive in H_1, the level of the test becomes

$$\tilde{p}_{m,2} \equiv P_{H_0}\left\{ \frac{1}{\sigma} (S_j - S_i) \geqq b \sqrt{(j-i)\left(1 - \frac{j-i}{m}\right)} \text{ for some } m_0 \leqq j-i \leqq m_1 \,\middle|\, S_m = 0 \right\}.$$

They have the following asymptotic approximations:

(7) $$p_{m,2} \sim 2\tilde{p}_{m,2} \sim \frac{1}{2} b^3 \varphi(b) \int_{m_0/m}^{m_1/m} \frac{1}{(1-t)t^2} [v(c/\sqrt{t(1-t)})]^2 dt,$$

TABLE 1
$m = 25, m_0 = 1, m_1 = 24$

b	σ^2 unknown: Approximation (4)	Monte Carlo $p_{m,1}$	Monte Carlo $\tilde{p}_{m,1}$	σ^2 known: Approximation (7)	Monte Carlo $p_{m,2}$	Monte Carlo $\tilde{p}_{m,2}$
3.06	0.133	0.132	0.073	0.176	0.169	0.097
3.13	0.100	0.103	0.057	0.145	0.140	0.079
3.26	0.056	0.059	0.033	0.100	0.100	0.055
3.28	0.051	0.053	0.030	0.094	0.095	0.051
3.48	0.018	0.021	0.011	0.050	0.053	0.028
3.58	0.010	0.011	0.005	0.036	0.032	0.017
3.94	0.001	0.001	0.001	0.010	0.010	0.005

TABLE 2
$m = 25, b = 3.13$

(m_0, m_1)	σ^2 unknown: Approximation (4)	Monte Carlo $p_{m,1}$	Monte Carlo $\tilde{p}_{m,1}$	σ^2 known: Approximation (7)	Monte Carlo $p_{m,2}$	Monte Carlo $\tilde{p}_{m,2}$
(1, 24)	0.100	0.097	0.052	0.145	0.133	0.075
(1, 21)	0.097	0.095	0.052	0.139	0.129	0.073
(4, 24)	0.063	0.063	0.034	0.082	0.084	0.047
(4, 21)	0.059	0.060	0.033	0.076	0.079	0.045
(4, 18)	0.053	0.056	0.031	0.069	0.074	0.042
(7, 21)	0.039	0.044	0.023	0.049	0.058	0.032
(7, 18)	0.034	0.040	0.021	0.042	0.052	0.029

under the same assumptions as Theorem 1, where φ denotes the standard normal density function (see Siegmund (1988), Yao (1989)).

Tables 1 and 2 give some indication of the accuracy of (4) and (7). For both cases of unknown and known σ^2, two 10 000 repetition Monte Carlo experiments with $m = 25$ offer some estimates $p_{m,1}$, $\tilde{p}_{m,1}$ and $p_{m,2}$, $\tilde{p}_{m,2}$. Table 1 shows that when the p-values are near 0.10, 0.05, and 0.01, the large deviations given in (4) and (7) offer quite good approximations. From Table 2 one can see that the approximations are also good for different values of m_0 and m_1 with a given value of m. In addition, a simulant probability for $\tilde{p}_{m,1}$, $\tilde{p}_{m,2}$ is always about one half of the corresponding simulant value for $p_{m,1}$, $p_{m,2}$, which agrees with the arguments in Remark 1, or 2. Other simulation results, not reported here, show that the essential conclusions are unchanged over a range of the p-value and sample size, although the magnitude of the difference can be more or less.

3. Proof of Theorem 1

Throughout this section we assume that $Y_1, \cdots, Y_m$ are i.i.d. N(0, 1) variables, and $S_0 = 0$, $S_n = \Sigma_1^n Y_k$, $U_n = \Sigma_1^n Y_k^2$, $n = 1, \cdots, m$. We also assume that $m \to \infty$, $m_0 \to \infty$,

$m_1 \to \infty$ in such a way that for some $0 \leqq t_0 < t_1 \leqq 1$, $m_0/m \to t_0$, $m_1/m \to t_1$, and $b = c\sqrt{m}$ with $c \in (0, 1)$. We use the notations

$$t_n = n/m, \qquad \mu_n = \tfrac{1}{2}c/\sqrt{t_n(1-t_n)},$$

$$J(p, q) = \{(i, j): m_0 \leqq j - i \leqq m_1, \text{ and } j < q\} \cup \{(i, q): q - m_1 < i \leqq (q - m_0) \wedge p\} \tag{8}$$

for $m_0 \leqq n \leqq m_1$, $1 < p < q < m$, and

$$P^{(m)}_{\xi,\lambda}(A) = \boldsymbol{P}(A \mid S_m = \xi,\ U_m = \lambda)$$

for $A \in \sigma(Y_1, \cdots, Y_m)$. Hence

$$p_{m,1} = P^{(m)}_{0,m}\left\{ |S_j - S_i| \geqq b\sqrt{(j-i)\left(1 - \frac{j-i}{m}\right)} \text{ for some } m_0 \leqq j - i \leqq m_1 \right\},$$

$$\tilde{p}_{m,1} = P^{(m)}_{0,m}\left\{ S_j - S_i \geqq b\sqrt{(j-i)\left(1 - \frac{j-i}{m}\right)} \text{ for some } m_0 \leqq j - i \leqq m_1 \right\}.$$

The following lemmas are technical and will be used in the proof of Theorem 1.

Lemma 1. Assume $m_0 \leqq n \leqq m_1$, $0 \leqq x < \log m$. As $m \to \infty$,

(i) $P^{(m)}_{0,m}\{S_n \in b\sqrt{n(1 - n/m)} + dx\}/dx$
$\sim (2\pi)^{-1/2}[n(1 - n/m)]^{-1/2}(1 - c^2)^{(m-4)/2}\exp\{-2\mu_n x/(1 - c^2)\}$
uniformly for x and n;

(ii) $P^{(m)}_{0,m}\{S_n \geqq b\sqrt{n(1 - n/m)} + \log m\} = o(m^{-1/2}(1 - c^2)^{m/2})$;

(ii) $P^{(m)}_{0,m}\{|U_n - m[t_n + c^2(1 - 2t_n)]| > m^{2/3} \mid S_n = b\sqrt{n(1 - n/m)} + x\} = o(m^{-1/3})$.

Lemma 1 follows from James et al. (1988), Lemma 1.

Lemma 2. As $m \to \infty$,

$$\boldsymbol{P}\left\{ S_j - S_i < b\sqrt{(j-i)\left(1 - \frac{j-i}{m}\right)}, \text{ for all } 1 \leqq j - i \leqq m_1 - (\log m)^2 \,\middle|\, S_{m_1} = b\sqrt{m_1(1 - t_1)} + x,\ U_{m_1} = m[t_1 + c^2(1 - 2t_1)] + y \right\} \to 1. \tag{9}$$

Proof. From the equality (2.3) of James et al. (1988), one obtains

$$P^{(m)}_{\xi,\lambda}(S_n \in dx) = \pi^{-1/2}\frac{\Gamma((m-1)/2)}{\Gamma(m/2 - 1)}(1 - y^2)^{(m-4)/2}dy,$$

where

$$y = \sqrt{\frac{m}{n(m-n)}}(\lambda - \xi^2/m)^{-1/2}\left(x - \frac{n}{m}\xi\right).$$

From this one can show that for every $m_0 < j \leqq m_1 - (\log m)^2$,

$$P\{S_j \geqq b\sqrt{j(1-j/m)}\,|\,S_{m_1} = b\sqrt{m_1(1-t_1)} + x,\ U_{m_1} = m[t_1 + c^2(1-2t_1)] + y\}$$

$$\leqq \pi^{-1/2}\frac{\Gamma((m_1-1)/2)}{\Gamma(m_1/2-1)}\int_{d_m}^{1}(1-y^2)^{(m_1-4)/2}dy(1+o(1)),$$

where

$$d_m = \frac{\log m}{\sqrt{m_1}}c/[(1+\sqrt{1-t_1})\sqrt{t_1(1-c^2)}].$$

For any $a \in (0, 1)$, it is easy to prove that

$$\int_a^1 (1-y^2)^{(m-4)/2}dy \leqq (1-a^2)^{(m-2)/2}/[a(m-2)].$$

Consequently,

$$1 - \text{LHS of (9)} \leqq \left\{m^2\pi^{-1/2}\frac{\Gamma((m_1-1)/2)}{\Gamma(m_1/2-1)}(1-d_m^2)^{(m_1-2)/2}/[d_m(m_1-2)]\right\}(1+o(1)) \to 0,$$

as $m \to \infty$, which completes the proof.

Lemma 3. Let $p \geqq \sqrt{m}$, $n \equiv q - p \geqq m_0 + \sqrt{m}$, and $t_n = n/m \to t' \in (t_0, t_1)$ as $m \to \infty$. Then uniformly in such (p, q) and $x \in (0, \log m)$, $|y| < m^{2/3}$, the following relation holds when $m \to \infty$:

$$P_{0,m}^{(m)}\left\{S_j - S_i < b\sqrt{(j-i)\left(1-\frac{j-i}{m}\right)},\ \text{for all } (i,j) \in J(p,q)\,|\,S_q - S_p\right.$$

$$(10) \qquad \left. = b\sqrt{n(1-n/m)} + x,\ U_q - U_p = m[t_n + c^2(1-2t_n)] + y\right\}$$

$$\sim P\left\{\min_{k\geqq 1}\tilde{S}_k > x\right\}P\left\{\min_{k\geqq 1}\hat{S}_k + \min_{k\geqq 0}\hat{S}'_k > x\right\},$$

where $\tilde{S}_k = (1-c^2)^{1/2}(S_k + k\mu_n)$ for $k \geqq 0$, $\{\tilde{S}'_k, k \geqq 0\}$ is an independent copy of $\{\tilde{S}_k, k \geqq 0\}$, and μ_n and $J(p, q)$ are defined in (8).

Proof. To simplify the notation, let $P^{(p,q)}$ denote the conditional probability measure on the left-hand side of (10).

Since $\sqrt{k(1-k/m)}$ is a convex function of k, the following inequality holds for $i, j \geqq 1$, $n + i \leqq m$, and $n - j \geqq 1$.

$$\sqrt{n\left(1-\frac{n}{m}\right)} + \sqrt{(n-j+i)\left(1-\frac{n-j+i}{m}\right)}$$

$$> \sqrt{(n-j)\left(1-\frac{n-j}{m}\right)} + \sqrt{(n+i)\left(1-\frac{n+i}{m}\right)}.$$

Consequently

$$\left\{S_j - S_i < b\sqrt{(j-i)\left(1-\frac{j-i}{m}\right)},\ \text{for all } (i,j)\in J(p,q)\right\}$$

$$= \left\{S_q - S_i < b\sqrt{(q-i)\left(1-\frac{q-i}{m}\right)},\ \text{for all } q-m_1 < i \leqq (q-m_0)\wedge p\right\}$$

(11)

$$\cap \left\{S_j - S_i < b\sqrt{(j-i)\left(1-\frac{j-i}{m}\right)},\ \text{for all } p \leqq i < j \leqq q, \text{ and } m_0 < j-i < m_1\right\}$$

$$\cap \left\{S_j - S_i < b\sqrt{(j-i)\left(1-\frac{j-i}{m}\right)},\ \text{for all } 1 \leqq i < j \leqq p, \text{ and } m_0 < j-i < m_1\right\}.$$

By Lemma 2, one can see that under the measure $P^{(p,q)}$ the inequalities on the right-hand side of the above expression are asymptotically almost surely valid for some indices, especially for all $1 \leqq i < j \leqq p$. Therefore

$$\text{LHS of (15)} = P^{(p,q)}\left\{S_q - S_{p-i} < b\sqrt{(n+i)\left(1-\frac{n+i}{m}\right)} \text{ for all } 1 \leqq i < (\log m)^2;\right.$$

$$\text{and } S_{q-j} - S_{p-i} < b\sqrt{(n-j+i)\left(1-\frac{n-j+i}{m}\right)}$$

$$\left.\text{for all } i \leqq 0, j \geqq 1, \text{ and } 0 < j-i < (\log m)^2\right\} + o(1)$$

(12)

$$= P^{(p,q)}\{S_{p-i} - S_p > x - i\mu_n(1-2t_n) \text{ for all } 1 \leqq i < (\log m)^2; \text{ and}$$

$$S_q - S_{q-j} + S_{p-i} - S_p > x + (j-i)\mu_n(1-2t_n) \text{ for all}$$

$$i \leqq 0, j \geqq 1 \text{ and } 0 < j-i < (\log m)^2\} + o(1).$$

The last equality follows the asymptotic relation

$$b\sqrt{n\left(1-\frac{n}{m}\right)} - b\sqrt{(n-k)\left(1-\frac{n-k}{m}\right)} = k\mu_n(1-2t_n) + o(1)$$

for $1 \leqq k \leqq (\log m)^2$.

On the other hand, one can show (directly, or by Lemma 1 in Chapter 4 of Hu (1985)) that as $m \to \infty$, the ratio of the $P^{(p,q)}$-joint density of random variables

$$S_{p-i} - S_p + i\mu_n(1-2t_n), \qquad i = 1, \cdots, (\log m)^2;$$

or

$$S_q - S_{q-j} - j\mu_n(1-2t_n), \qquad j = 1, \cdots, (\log m)^2;$$

or

$$S_{p-i} - S_p + i\mu_n(1-2t_n), \qquad i = -1, \cdots, -(\log m)^2$$

to the joint density of $\tilde{S}_k$, $k = 1, \cdots, (\log m)^2$, converges to 1, and furthermore asymptotically these three collections of random variables are stochastically independent. Hence the right-hand side of (12) is asymptotically equivalent to the right-hand side of (10). The proof is completed.

Lemma 4. Let $\{\tilde{S}_k, k \geqq 0\}$ and $\{\tilde{S}'_k, k \geqq 0\}$ be the same as in Lemma 3. Then

$$\int_0^\infty \exp\left(-\frac{2\mu_n x}{1-c^2}\right) \boldsymbol{P}\left\{\min_{k\geqq 1} \tilde{S}_k > x\right\} \boldsymbol{P}\left\{\min_{k\geqq 1} \tilde{S}_k + \min_{k\geqq 0} \tilde{S}'_k > x\right\} dx$$

$$= 2\mu_n^3(1-c^2)^{-1}[\nu(2\mu_n/(1-c^2)^{1/2})]^2.$$

Lemma 4 follows from Siegmund (1988) Lemma 7 and Siegmund (1985) §8.5.

Proof of Theorem 1. The proof proceeds in two steps. At first one proves that $\tilde{p}_{m,1}$ is asymptotically one half of the right-hand side of (4). The second is to show $\tilde{p}_{m,1} \sim \frac{1}{2} p_{m,1}$. We split $\tilde{p}_{m,1}$ into the following sum:

$$\tilde{p}_{m,1} = \left(\sum_{n=[m_0+\sqrt{m}]}^{m_1} \sum_{\substack{q-p=n \\ p \geqq \sqrt{m}}} + \sum_{n=[m_0+\sqrt{m}]}^{m_1} \sum_{\substack{q-p=n \\ p < \sqrt{m}}} + \sum_{n=m_0}^{[m_0+\sqrt{m}]-1} \sum_{q-p=n} \right)$$

$$P_{0,m}^{(m)} \left\{ S_q - S_p \geqq b\sqrt{n(1-n/m)};\, S_j - S_i < b \sqrt{(j-i)\left(1-\frac{j-i}{m}\right)},\right.$$

$$\left. \forall (i,j) \in J(p,q) \right\}$$

$$\equiv p_1 + p_2 + p_3.$$

First of all, we try to calculate the main part p_1. For any (p,q) with $p \geqq \sqrt{m}$, and $q-p=n$ between m_0 and m_1,

$$P_{0,m}^{(m)} \left\{ S_q - S_p \geqq b\sqrt{n(1-n/m)};\, S_j - S_i < b \sqrt{(j-i)\left(1-\frac{j-i}{m}\right)},\, \forall (i,j) \in J(p,q) \right\}$$

$$= \int_{A_{m,n}} P_{0,m}^{(m)} \{ S_n \in b\sqrt{n(1-n/m)} + dx \}$$

$$\times P_{0,m}^{(m)} \{ U_n \in m[t_n + c^2(1-2t_n)] + dy \mid S_n = b\sqrt{n(1-n/m)} + x \} \tag{13}$$

$$\times P_{0,m}^{(m)} \left\{ S_j - S_i < b \sqrt{(j-i)\left(1-\frac{j-i}{m}\right)},\, \forall (i,j) \in J(p,q) \mid S_q - S_p \right.$$

$$\left. = b\sqrt{n(1-n/m)} + x,\, U_q - U_p \in m[t_n + c^2(1-2t_n)] + y \right\}$$

where

$$A_{m,n} = \{(x,y) : x \geqq 0, y \geqq 0, b\sqrt{n(1-n/m)} + x$$

$$< \sqrt{nm[t_n + c^2(1-2t_n)] + ny} \wedge \sqrt{(m-n)\{m - m[t_n + c^2(1-2t_n)] - y\}}\,\}.$$

Lemma 1 (ii) indicates that the $P_{0,n}^{(m)}$-probability of $S_n \geqq b\sqrt{n(1-n/m)} + \log m$ is of higher order, which can be neglected. Hence it seems plausible that the range of the value of x in the integral in (13) can be restricted to the interval $[0, \log m]$. Similarly by Lemma 1 (iii), we can also restrict the range of the value of y to, say, $|y| < m^{2/3}$. Furthermore using Lemmas 1 (i), 3, and 4, we have

$$p_1 \sim \frac{c^3\sqrt{m}}{4\sqrt{2\pi}}(1-c^2)^{m/2-3} \sum_{n=[m_0+\sqrt{m}]}^{m_1} \frac{1}{(1-t_n)t_n^2}[\nu(c/\sqrt{t_n(1-t_n)(1-c^2)})]^2$$

$$\sim \frac{c^3 m^{3/2}}{4\sqrt{2\pi}}(1-c^2)^{m/2-3} \int_{m_0/m}^{m_1/m} \frac{1}{(1-t)t^2}[\nu(c/\sqrt{t(1-t)(1-c^2)})]^2 dt,$$

which is just half of the right-hand side of (4).

On the other hand, it follows from Lemma 1 easily that

$$p_2 \leqq \sqrt{m} \sum_{n=m_0}^{m_1} P_{0,m}^{(m)}\{S_n \geqq b\sqrt{n(1-n/m)}\} = O(m(1-c^2)^{m/2});$$

$$p_3 \leqq m \sum_{n=m_0}^{[m_0+\sqrt{m}]} P_{0,m}^{(m)}\{S_n \geqq b\sqrt{n(1-n/m)}\} = O(m(1-c^2)^{m/2}),$$

that means that $\tilde{p}_{m,1}$ is asymptotically equivalent to p_1.

To show $p_{m,1} \sim 2\tilde{p}_{m,1}$, one only needs to prove that

$$2\tilde{p}_{m,1} - p_{m,1} = o(m^{3/2}(1-c^2)^{m/2}), \tag{14}$$

since obviously $2\tilde{p}_{m,1} > p_{m,1}$. For any (p, q) with $q - p = n$ between m_0 and m_1,

$$2P_{0,m}^{(m)}\Big\{S_q - S_p \geqq b\sqrt{n(1-n/m)};\ \text{and}$$

$$S_j - S_i < b\sqrt{(j-i)\left(1-\frac{j-i}{m}\right)},\ \forall (i,j) \in J(p,q)\Big\}$$

$$-P_{0,m}^{(m)}\Big\{|S_q - S_p| \geqq b\sqrt{n(1-n/m)};\ \text{and} \tag{15}$$

$$S_j - S_i < b\sqrt{(j-i)\left(1-\frac{j-i}{m}\right)},\ \forall (i,j) \in J(p,q)\Big\}$$

$$= 2\int_{A_{m,n}} f_m(p,q) P_{0,m}^{(m)}\{S_n \in b\sqrt{n(1-n/m)} + dx,\ U_n \in m[t_n + c^2(1-2t_n)] + dy\},$$

where

$$f_m(p,q) = P_{0,m}^{(m)}\Big\{S_j - S_i \leqq -b\sqrt{(j-i)\left(1-\frac{j-i}{m}\right)},$$

$$\text{for some } (i,j) \in J(p,q) \,|\, S_q - S_p = b\sqrt{n(1-n/m)} + x,$$

$$U_q - U_p \in m[t_n + c^2(1-2t_n)] + y\Big\}.$$

Using equality (11) with $-b$ instead of b, one can see that $1 - f_m(p,q)$ is equal to

$$
\begin{aligned}
&\boldsymbol{P}\Big\{S_k < b\sqrt{n(1-n/m)} + x + b\sqrt{(n-k)\Big(1-\frac{n-k}{m}\Big)},\\
&\qquad \forall\, 1 \leqq k < n - m_0 \,\big|\, S_n = b\sqrt{n(1-n/m)} + x,\ U_n = m[t_n + c^2(1-2t_n)] + y\Big\}\\
&\times \boldsymbol{P}\Big\{S_k < b\sqrt{n(1-n/m)} + x + b\sqrt{(n+k)\Big(1-\frac{n+k}{m}\Big)},\\
&\qquad \forall\, 1 \leqq k < m_1 - n \,\big|\, S_{m-n} = b\sqrt{n(1-n/m)} + x,\\
&\qquad U_{m-n} = m[t_{m-n} + c^2(1-2t_{m-n})] - y\Big\}.
\end{aligned}
$$

With some similar arguments as in the proof of Lemma 2, one can easily show that both of the probabilities in the above product tend to 1 uniformly for $0 \leqq x < \log m$, and $|y| < m^{2/3}$. It follows from Lemma 1 that the left-hand side of (15) is $o(m^{-1/2}(1-c^2)^{m/2})$, which entails the validity of relation (14). This completes the proof.

4. Testing an epidemic alternative in a simple linear model

Suppose that $Y_1, \cdots, Y_m$ are independent and normally distributed with common variance σ^2. This section concerns the likelihood ratio tests of the null hypothesis

$$
\mathrm{H}_0:\ \boldsymbol{E}Y_k = \alpha + \beta x_k, \qquad k = 1, \cdots, m
$$

against the alternative

$$
\mathrm{H}_1:\ \text{there exist } 1 \leqq i < j \leqq m \text{ such that}
$$

$$
\boldsymbol{E}Y_k = \begin{cases} \alpha + \beta x_k, & k = 1, \cdots, i, j+1, \cdots, m; \\ \alpha + \delta + \beta x_k, & k = i+1, \cdots, j, \end{cases}
$$

where $x_1, \cdots, x_m$ are given constants, and α, β, δ $(\neq 0)$ play the role of nuisance parameters. The hypothesis H_0 specifies a usual straight-line regression model, and under this model the maximum likelihood estimators for α, β, and σ^2 are

$$
\begin{aligned}
\hat\alpha &= \bar Y - \hat\beta \bar x;\\
\hat\beta &= \sum_1^m (Y_k - \bar Y)(x_k - \bar x) \Big/ \sum_1^m (x_k - \bar x)^2;\\
\hat\sigma^2 &= m^{-1}\Big\{\sum_1^m (Y_k - \bar Y)^2 - \hat\beta \sum_1^m (Y_k - \bar Y)(x_k - \bar x)\Big\}
\end{aligned}
$$

respectively, where $\bar Y = (1/m)\Sigma_1^m Y_k$, and $\bar x = (1/m)\,\Sigma_1^m x_k$. Some tedious calculation shows that the generalized likelihood ratio test rejects H_0 for large values of $\max_{m_0 \leqq j-i \leqq m_1} \sigma^{-1} U_m(i,j)$ when σ^2 is known, or for large values of $\max_{m_0 \leqq j-i \leqq m_1} \hat\sigma^{-1} U_m(i,j)$ when σ^2 is unknown, where

$$U_m(i,j)=\left|S_j-S_i-(j-i)\hat{\alpha}-\hat{\beta}\sum_{k=i+1}^{j}x_k\right|\Big/\sqrt{(j-i)\left(1-\frac{j-i}{m}\right)D_m(i,j)},$$

$$D_m(i,j)=1-\left(\sum_{k=i+1}^{j}x_k-(j-i)\bar{x}\right)^2\Big/\left[(j-i)\left(1-\frac{j-i}{m}\right)\sum_{1}^{m}(x_k-\bar{x})^2\right],$$

and S_k denotes the partial sum of Y_k's. By Basu's theorem, the same arguments as in Section 2 entail that the process $U_m(i,j)$, $i,j=1,\cdots,m$, is independent of a complete sufficient statistic, which is $(\hat{\alpha},\hat{\beta})$ when σ^2 is known, or $(\hat{\alpha},\hat{\beta},\hat{\sigma}^2)$ when σ^2 is unknown. Hence, for the two cases the significance level can be expressed as

$$p_{m,3}\equiv P_{\mathrm{H}_0}\left\{\frac{1}{\sigma}|S_j-S_i|\geqq b\sqrt{(j-i)\left(1-\frac{j-i}{m}\right)}U_m(i,j),\right.$$
$$\left.\text{for some } m_0\leqq j-i\leqq m_1\,|\,\hat{\alpha}=0,\hat{\beta}=0\right\};$$

$$p_{m,4}\equiv P_{\mathrm{H}_0}\left\{|S_j-S_i|\geqq b\sqrt{(j-i)\left(1-\frac{j-i}{m}\right)}U_m(i,j),\right.$$
$$\left.\text{for some } m_0\leqq j-i\leqq m_1\,|\,\hat{\alpha}=0,\hat{\beta}=0,\hat{\sigma}^2=1\right\}$$

respectively, where b is a positive constant. Theorem 2 presents some large deviation approximations for these probabilities in a special case, say $x_k=k/m$, which can be thought of as the time at which the kth of equally spaced observations is made.

Theorem 2. Suppose $m\to\infty$, $m_0\to\infty$, $m_1\to\infty$ in such a way that for some $0\leqq t_0<t_1\leqq 1$, $m_0/m\to t_0$ and $m_1/m\to t_1$. Then for $b=c\sqrt{m}$ with $c>0$ fixed,

(i) if σ^2 is known,

$$p_{m,3}\sim\frac{1}{2}b^3\varphi(b)\int_{m_0/m}^{m_1/m}dt\int_0^{1-t}[\mu(t,s)\nu(c\sqrt{\mu(t,s)})]^2ds; \tag{16}$$

(ii) if σ^2 is unknown, and $c\in(0,1)$,

$$p_{m,4}\sim\frac{1}{2\sqrt{2\pi}}b^3(1-c^2)^{(m-7)/2}$$
$$\times\int_{m_0/m}^{m_1/m}dt\int_0^{1-t}[\mu(t,s)\nu(c\sqrt{\mu(t,s)/(1-c^2)})]^2ds, \tag{17}$$

where $\nu(x)$ is given in (3), and

$$\mu(t,s)=\left[t(1-t)\left(1-\frac{3t}{1-t}(1-t-2s)^2\right)\right]^{-1}.$$

Remark 3. If we restrict δ to be positive in H_1, the level of the likelihood ratio test would be

TABLE 3
$m = 25, m_0 = 1, m_1 = 24$

	σ^2 known			σ^2 unknown		
b	Approximation (16)	Monte Carlo $p_{m,3}$	$\tilde{p}_{m,3}$	Approximation (17)	Monte Carlo $p_{m,4}$	$\tilde{p}_{m,4}$
3.18	0.152	0.134	0.074	0.123	0.109	0.059
3.22	0.135	0.118	0.066	0.103	0.093	0.049
3.30	0.106	0.100	0.051	0.071	0.065	0.034
3.37	0.085	0.074	0.042	0.051	0.047	0.026
3.51	0.054	0.044	0.026	0.024	0.023	0.012
3.67	0.032	0.027	0.014	0.009	0.007	0.004
3.97	0.010	0.009	0.005	0.001	0.001	0.000

$$\tilde{p}_{m,3} \equiv P_{H_0}\left\{\frac{1}{\sigma}(S_j - S_i) \geqq b\sqrt{(j-i)\left(1-\frac{j-i}{m}\right)U_m(i,j)},\right.$$
$$\left.\text{for some } m_0 \leqq j-i \leqq m_1 \mid \hat{\alpha}=0, \hat{\beta}=0\right\}$$

when σ^2 is known; and

$$\tilde{p}_{m,4} \equiv P_{H_0}\left\{S_j - S_i \geqq b\sqrt{(j-i)\left(1-\frac{j-i}{m}\right)U_m(i,j)},\right.$$
$$\left.\text{for some } m_0 \leqq j-i \leqq m_1 \mid \hat{\alpha}=0, \hat{\beta}=0, \hat{\sigma}^2=1\right\}$$

when σ^2 is unknown. Similar to Remark 1 and 2, one can show that under the same assumptions as in Theorem 2, $2\tilde{p}_{m,3} \sim p_{m,3}$, and $2\tilde{p}_{m,4} \sim p_{m,4}$ when $m \to \infty$.

Remark 4. The proof of Theorem 2 is omitted here since it is in principle similar to the proof of Theorem 1. One thing which is worth mentioning is that when the hypothesis H_0 holds and also $\hat{\alpha}=0, \hat{\beta}=0$,

$$\boldsymbol{Y} \equiv \begin{pmatrix} Y_1 \\ \vdots \\ Y_m \end{pmatrix} = (\boldsymbol{I}-\boldsymbol{P})\boldsymbol{Y},$$

where $\boldsymbol{I}$ is the $m \times m$ identity matrix, $\boldsymbol{P}$ denotes the projection matrix on the linear space spanned by $\mathbf{1} \equiv (1, \cdots, 1)'$ and $\boldsymbol{x} \equiv (x_1, \cdots, x_m)'$. Consequently under such conditions, $\boldsymbol{Y}$ is a m-dimensional normal random vector with mean zero and variance $\sigma^2(\boldsymbol{I}-\boldsymbol{P})$ when σ^2 is known. When σ^2 is unknown, $\hat{\sigma}^2 = m^{-1}\|\boldsymbol{Y}\|^2$, from which one can easily get the conditional distribution of $\boldsymbol{Y}$.

Tables 3 and 4 present some results of two 10 000 repetition Monte Carlo experiments, which assess the accuracy of Theorem 2, and are also agreeable to the asymptotic relations $2\tilde{p}_{m,3} \sim p_{m,3}$, and $2\tilde{p}_{m,4} \sim p_{m,4}$.

TABLE 4
$m = 25, b = 3.30$

(m_0, m_1)	σ^2 known			σ^2 unknown		
	Approximation (16)	Monte Carlo $p_{m,3}$	$\tilde{p}_{m,3}$	Approximation (17)	Monte Carlo $p_{m,4}$	$\tilde{p}_{m,4}$
(1, 24)	0.106	0.100	0.051	0.071	0.065	0.034
(1, 21)	0.102	0.094	0.051	0.069	0.064	0.033
(4, 24)	0.066	0.059	0.033	0.048	0.043	0.023
(4, 21)	0.062	0.057	0.032	0.045	0.041	0.023
(4, 18)	0.057	0.055	0.031	0.042	0.040	0.022
(7, 21)	0.043	0.042	0.023	0.031	0.029	0.015
(7, 18)	0.037	0.039	0.022	0.028	0.027	0.014

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