A Power One Test for Unit Roots Based on Sample Autocovariances

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Abstract

We propose a new unit-root test for a stationary null hypothesis H_0 against a unit-root alternative H_1. Our approach is nonparametric as the null hypothesis only assumes that the process concerned is $I(0)$ without specifying any parametric forms. The new test is based on the fact that the sample autocovariance function (ACF) converges to the finite population ACF for an $I(0)$ process while it diverges to infinity with probability approaching one for a process with unit-roots. Therefore the new test rejects the null hypothesis for the large values of the sample ACF. To address the technical challenge 'how large is large', we split the sample and establish an appropriate normal approximation for the null-distribution of the test statistic. The substantial discriminative power of the new test statistic is rooted from the fact that it takes finite value under H_0 and diverges to infinity almost surely under H_1. This allows us to truncate the critical values of the test to make it with the asymptotic power one. It also alleviates the loss of power due to the sample-splitting. The finite sample properties of the test are illustrated by simulation which shows its stable and more powerful performance in comparison with the KPSS test (Kwiatkowski et al., 1992). The test is implemented in a user-friendly R-function.

Keywords: Autocovariance; Integrated processes; Normal approximation; Power-one test; Sample-splitting.

1 Introduction

Unit-root is one of the most frequently used settings for modeling nonstationary time series. Its importance stems from the fact that many economic, financial, business and social-domain time series data exhibit segmented trend-like or random wandering phenomena. While the random-walk-like behavior of stock prices was notified and recorded much earlier by, for example, Jules Regnault, a French broker, in 1863 and then by Louis Bachelier in his 1900 PhD thesis, the development of statistical inference for unit-roots only started in late 1970s. Nevertheless the literature on unit-root tests by now is immense and diverse. We only state a selection of some important developments below, which naturally leads to a new test to be presented in this paper.

The Dickey-Fuller tests (Dickey and Fuller, 1979, 1981) dealt with Gaussian random walks with independent error terms. Effort to relax the condition of independent Gaussian errors leads to, among others, the augmented Dickey-Fuller (ADF) tests (Said and Dickey, 1984; Elliott, Rothenberg and Stock, 1996; Xiao and Phillips, 1997) which replace the error term by an autoregressive process, the Phillips-Perron test (Phillips, 1987; Phillips and Perron, 1988) which estimates the long-run variance of the error process nonparametrically using the method of Newey and West (1987) and Andrews (1991). The ADF tests are further extended for dealing with structural breaks in trend (Zivot and Andrews, 1992), long memory processes (Robinson, 1994), seasonal unit roots (Hylleberg et al., 1990; Chan and Wei, 1988), bootstrap unit-root tests (Paparoditis and Politis, 2005), nonstationary volatility (Cavaliere and Taylor, 2007), panel data (Pesaran, 2007), and local stationary processes (Rho and Shao, 2019). We refer to survey papers Stock (1994) and Phillips and Xiao (1998), and monographs Hatanaka (1996) and Maddala and Kim (1998) for further information on the topic.

The Dickey-Fuller tests and their variants are based on the regression of a time series on its first lag in which the existence of unit root is postulated as a null hypothesis in the form of the regression coefficient being equal to one. This null hypothesis is tested against a stationary alternative hypothesis that the regression coefficient concerned is smaller than one. This setting leads to innate indecisive inference for ascertaining the existence of unit-roots, as a statistical test is incapable in accepting a null hypothesis. To make the assertion of unit-roots on a firmer ground, Kwiatkowski et al. (1992) adopted a different approach: the proposed KPSS test considers a stationary null hypothesis against a unit-root alternative. It is based on a plausible representation for possible nonstationary time series in which a unit-root is represented as an additive random walk component. Then under the null hypothesis the variance of the random walk component is zero. The KPSS test is the one-sided Lagrange multiplier test for testing the variance to be zero against greater than zero.

In spite of the many exciting developments stated above, testing for the existence of unit roots remains as a challenge in time series analysis, as most available methods suffer from the lack of accurate size control and poor power. In this paper we propose a new test, based on a radically different idea from the existing approaches. Our setting is similar in spirit to the KPSS test as we test for stationary hull hypothesis H_0 again a unit-root alternative H_1. However our approach is nonparametric as the null hypothesis only assumes that the process concerned is $I(0)$ without specifying any parametric forms. The new test is based on the simple fact that under H_0 the sample autocovariance function (ACF) converges to the finite population ACF while under H_1 it diverges to infinity with probability approaching one. Therefore we can reject the null hypothesis for large (absolute) values of the sample ACF. To address the technical challenge 'how large is large', we split the sample and establish an appropriate normal approximation for the null-distribution of the test statistic. Note that our sample ACF based test statistic offers substantial discriminative power as it takes finite value under H_0 or diverges to infinity almost surely under H_1. This allows us to truncate the critical values determined by the normal approximation under H_0 to make the test with the asymptotic power one. Furthermore, it also alleviates the loss of power due to the sample-splitting as it outperforms the KPSS test in the power comparison in simulation. Another advantage of the new method is that it has a remarkable discriminate power to tell the difference between, for example, a random walk and an AR(1) with the autoregressive coefficient close to (but still smaller than) one, for which most the available unit-root tests, including the KPSS method, suffer from weak discriminate power.

Admittedly the newly proposed test is technically sophisticated. To make it user-friendly, we have developed an R-function `ur.test` in the package `HDTSA` which implements the test in an automatic manner. See Section 2.4 below for more details. Note that the strong discriminative power of the test statistic also makes the choice of the two tuning parameters involved less sensitive, function `ur.test` incorporates some well-tested default values for the tuning parameters. Indeed the test performed competently and robustly on, for example, the 14 Nelson and Plosser time series (Nelson and Plosser, 1982) which were often used for testing unit-roots.

The rest of the paper is organised as follows. The main results including the newly proposed test, its asymptotic properties, and the implementation are presented in Section 2. The finite sample properties of the test are investigated by simulation in Section 3 which also includes the numerical comparison with the KPSS method. Technical proofs are collected in Section 4. The supplementary material contains the proof of Lemma 1 and some additional numerical results.

2 Main results

2.1 A power-one test

A time series $\{Y_t\}$ is said to be $I(0)$, denoted by $Y_t \sim I(0)$, if

$$\mathbb{E}(Y_t) \equiv \mu\,, \quad \mathbb{E}(Y_t^2) < \infty\,, \quad \gamma(k) \equiv \mathrm{Cov}(Y_{t+k}, Y_t) \text{ for all } t\,, \quad \text{and} \quad \sum_{k=0}^{\infty} |\gamma(k)| < \infty\,. \tag{1}$$

Let $\nabla Y_t = Y_t - Y_{t-1}$, and $\nabla^d Y_t = \nabla(\nabla^{d-1} Y_t)$ for any integer $d \geq 2$. Time series $\{Y_t\}$ is said to be $I(d)$, denoted by $Y_t \sim I(d)$, if $\{\nabla^d Y_t\}$ is $I(0)$ and $\{\nabla^{d-1} Y_t\}$ is not $I(0)$. An $I(d)$ process is also called a unit-root process with the integration order d. With the observations $Y_1, \ldots, Y_n$, we are interested in testing the unit-root hypotheses

$$H_0 : Y_t \sim I(0) \quad \text{versus} \quad H_1 : Y_t \sim I(d) \text{ for some integer } d \geq 1\,. \tag{2}$$

We propose a new test for (2) based on a simple fact that the sample autocovariances of a unit-root process diverge to infinity with probability approaching one while those of an $I(0)$ process converge to the true and finite autocovariances; see (1). More precisely we denote the sample ACF at lag k by

$$\hat{\gamma}(k) = \frac{1}{n} \sum_{t=1}^{n-k} (Y_{t+k} - \bar{Y})(Y_t - \bar{Y})\,, \tag{3}$$

which is a consistent estimate of $\gamma(k) = \mathrm{Cov}(Y_{t+k}, Y_t)$ under null hypothesis H_0, where $\bar{Y} = n^{-1} \sum_{t=1}^{n} Y_t$. However Proposition 1 below indicates that for $I(d)$ processes, $\hat{\gamma}(k)$ is at least as large as $O_{\mathrm{p}}(n^{2d-1})$. See also Peña and Pocela (2006). Therefore we can reject H_0 for large values of $|\hat{\gamma}(k)|$.

To state Proposition 1, we assume $Y_t \sim I(d)$ and

$$\nabla^d Y_t = \mu_d + \sum_{j=0}^{\infty} \psi_j \epsilon_{t-j} \tag{4}$$

where $\mu_d = \mathbb{E}(\nabla^d Y_t)$ is a constant, $\psi_0 = 1$, and $\{\epsilon_t\}$ are white noise innovations. Representation (4) is the Wold's decomposition for any purely non-deterministic $I(0)$ process. When $\{\epsilon_t\}$ are i.i.d., $\{\nabla^d Y_t\}$ is a linear process.

Proposition 1. *Let Y_t be defined by* (4) *in which $\epsilon_t \overset{\text{i.i.d.}}{\sim} (0, \sigma_\epsilon^2)$ and $\sum_{j=1}^{\infty} j|\psi_j| < \infty$. Let $k \geq 0$ be an integer.*

(i) *When $\mu_d = 0$, it holds that $n^{-(2d-1)}\hat{\gamma}(k) \xrightarrow{\text{D}} a^2\sigma_\epsilon^2 \int_0^1 V_{d-1}^2(t)\,\mathrm{d}t$, where $a = \sum_{j=0}^{\infty} \psi_j$, $V_{d-1}(t) = F_{d-1}(t) - \int_0^1 F_{d-1}(t)\,\mathrm{d}t$ and $F_{d-1}(t)$ is the scalar multi-fold integrated Brownian motion defined recursively as $F_j(t) = \int_0^t F_{j-1}(x)\,\mathrm{d}x$ for any $j \geq 1$ and $F_0(t) = W(t)$ is the standard Brownian motion.*

(ii) *When $\mu_d \neq 0$, it holds that $n^{-2d}\hat{\gamma}(k) \xrightarrow{\text{P}} \phi_{d,k}\mu_d^2$, where $\phi_{d,k}$ is a positive bounded constant only depending on d and k.*

Based on Proposition 1, we may reject H_0 for the large values of, for example, the test statistic $T_{\text{naive}} = \sum_{k=0}^{K_0} |\hat{\gamma}(k)|^2$, as under H_0 the test statistic T_{naive} converges to $\sum_{k=0}^{K_0} |\gamma(k)|^2$ which is finite, where $K_0 \geq 0$ is a prescribed integer. There are two obstacles preventing using such a test statistic. First to determine the critical values one has to derive the asymptotic distribution of $a_n\{T_{\text{naive}} - \sum_{k=0}^{K_0} |\gamma(k)|^2\}$ under H_0, where a_n is an appropriate normalized constant. Secondly, one needs a consistent estimator for $\sum_{k=0}^{K_0} |\gamma(k)|^2$ under H_0, which is not readily available as in practice we do not know if H_0 holds or not.

To overcome these obstacles, we apply the idea of 'data splitting'. Let $N = \lfloor n/2 \rfloor$, and

$$\hat{\gamma}_1(k) = \frac{1}{N}\sum_{t=1}^{N-k}(Y_{t+k} - \bar{Y})(Y_t - \bar{Y}) \quad \text{and} \quad \hat{\gamma}_2(k) = \frac{1}{N}\sum_{t=N+1}^{2N-k}(Y_{t+k} - \bar{Y})(Y_t - \bar{Y}), \tag{5}$$

i.e. $\hat{\gamma}_1(k)$ and $\hat{\gamma}_2(k)$ are, respectively, the sample autocovariance at lag k of $\{Y_t\}_{t=1}^{N}$ and $\{Y_t\}_{t=N+1}^{2N}$. The test statistic is defined as

$$T_n = \sum_{k=0}^{K_0} |\hat{\gamma}_2(k)|^2, \tag{6}$$

where $K_0 \geq 0$ is a prescribed integer which controls the amount of information from different time lags to be used in the test. Although our theoretical analysis allows K_0 to diverge with sample size n, the simulation results reported in Section 3 below indicate that the finite sample performance of the test is robust with respect to the different values of K_0. In fact the test works well even with small values of K_0. We suggest in practice to set $K_0 \in \{0, 1, 2, 3, 4\}$.

Formally we reject the null hypothesis H_0 at the significance level $\phi \in (0,1)$ if

$$T_n > \mathrm{cv}_\phi \,, \tag{7}$$

where cv_ϕ is the critical value satisfying $\mathbb{P}_{H_0}(T_n > \mathrm{cv}_\phi) \to \phi$ as $n \to \infty$. As we will see in (10) below, $\{\hat{\gamma}_1(k)\}_{k=0}^{K_0}$ are to be used to determine the critical value cv_ϕ. One obvious concern for splitting the sample into two halves is the loss in testing power. However the fact that T_n takes a finite value under H_0 and it diverges to ∞ (with probability approaching one) under H_1 implies that T_n has a strong discriminant power to tell apart H_1 from H_0, which is enough to sustain the adequate power in comparison to that of, for example, the KPSS test. Our simulation results indicate that the sample-splitting works well even for sample size $n = 80$ (i.e. $N = 40$).

For positive integers t and k, let

$$y_{t,k} = 2\{(Y_t - \mu)(Y_{t+k} - \mu) - \gamma(k)\}\mathrm{sgn}(k + t - N - 1/2)\,. \tag{8}$$

Define $\xi_{t,k} = 2y_{t,k}\gamma(k)$, $Q_t = \sum_{k=0}^{K_0} \xi_{t,k}$, and $B_k^2 = \mathbb{E}\{(\sum_{t=1}^{k} Q_t)^2\}$. Some regularity conditions are now in order.

Condition 1. Under the null hypothesis H_0, there exist uniform constants $s_1 \in (2,3]$ and $c_1 > 0$ such that $\max_{1\le t\le n} \mathbb{E}(|Y_t|^{2s_1}) \le c_1$.

Condition 2. Under the null hypothesis H_0, $\{Y_t\}$ is α-mixing in the sense that

$$\alpha(\tau) = \sup_t \sup_{A\in\mathcal{F}_{-\infty}^t,\, B\in\mathcal{F}_{t+\tau}^\infty} |\mathbb{P}(AB) - \mathbb{P}(A)\mathbb{P}(B)| \to 0 \quad \text{as} \quad \tau \to \infty\,,$$

where $\mathcal{F}_{-\infty}^t$ and $\mathcal{F}_{t+\tau}^\infty$ denote the σ-fields generated by $\{Y_u\}_{u\le t}$ and $\{Y_u\}_{u\ge t+\tau}$, respectively. There exist uniform constants $c_2 > 0$ and $\beta_1 > 2(s_1 - 1)s_1/(s_1 - 2)^2$ with s_1 specified in Condition 1 such that $\alpha(\tau) \le c_2\tau^{-\beta_1}$ for any $\tau \ge 1$.

Condition 3. Under the null hypothesis H_0, there exists a uniform constant $c_3 > 0$ such that $B_\ell^2 \ge c_3\ell$ for any $\ell \ge 1$.

Condition 1 requires that Y_t has more than four moments. The α-mixing assumption in Condition 2 is mild. Causal ARMA processes with continuous innovation distributions are α-mixing with exponentially decaying α-mixing coefficients. So are stationary Markov chains satisfying certain conditions. See Section 2.6.1 of Fan and Yao (2003) and the references within. In fact stationary GARCH models with finite second moments and continuous innovation distributions are also α-mixing with exponentially decaying α-mixing coefficients; see Proposition 12 of Carrasco and Chen (2002). Condition 3 requires the long-run covariance of the newly defined process $\{Q_t\}$ to be non-degenerated.

Theorem 1. *Let the null hypothesis H_0 hold with Conditions 1–3 being satisfied, and $K_0 =$*

$o\{n^{\xi(\beta,s_1)}\}$ *with*

$$\xi(\beta, s_1) = \min\left\{\frac{s_1-2}{4s_1}, \frac{(\beta-1)(s_1-2)}{(2\beta+2)s_1}\right\}, \tag{9}$$

where s_1 *and* β_1 *are specified, respectively, in Conditions* 1 *and* 2, *and* $\beta = \beta_1(s_1-2)^2/\{2s_1(s_1-1)\}$. *Then as* $n\to\infty$, *it holds that*

$$\sup_{u>0}\left|\mathbb{P}(\sqrt{n}T_n > u) - 1 + \Phi\left(\frac{2N\tilde{u}}{B_{2N-K_0}\sqrt{n}}\right)\right| \to 0\,,$$

where $\tilde{u} = u - \sqrt{n}\sum_{k=0}^{K_0}|\hat{\gamma}_1(k)|^2$, *and* $\Phi(\cdot)$ *is the cumulative distribution function of the standard normal distribution.*

By Theorem 1, one may select the critical value cv_ϕ of the test stated in (7) of the form $z_{1-\phi}\hat{B}_{2N-K_0}/(2N) + \sum_{k=0}^{K_0}|\hat{\gamma}_1(k)|^2$, where $z_{1-\phi}$ is the $(1-\phi)$-quantile of the standard normal distribution $\mathcal{N}(0,1)$, and $\hat{B}_{2N-K_0}$ is a consistent estimate of B_{2N-K_0} in the sense that $\hat{B}_{2N-K_0}/B_{2N-K_0} \xrightarrow{\mathrm{P}} 1$, as then

$$\mathbb{P}_{H_0}\left\{T_n > \frac{z_{1-\phi}\hat{B}_{2N-K_0}}{2N} + \sum_{k=0}^{K_0}|\hat{\gamma}_1(k)|^2\right\} \to \phi\,.$$

However a critical value specified in this manner is only valid under H_0, as $\sum_{k=0}^{K_0}|\hat{\gamma}_1(k)|^2$ diverges to infinity with probability approaching one under H_1. A non-trivial consequence of using this critical value is that the test suffers from the substantial power loss, as under H_1 the probability of the event $\{T_n > z_{1-\phi}\hat{B}_{2N-K_0}/(2N) + \sum_{k=0}^{K_0}|\hat{\gamma}_1(k)|^2\}$ is small. This is also confirmed in our simulation in Section 3 below. To rectify this defect, we apply here a truncation idea as in Section 2.3 of Chang et al. (2017). More precisely we set the critical value as

$$\mathrm{cv}_\phi = \begin{cases} \dfrac{z_{1-\phi}\hat{B}_{2N-K_0}}{2N} + \displaystyle\sum_{k=0}^{K_0}|\hat{\gamma}_1(k)|^2\,, & \text{if event } \mathcal{T} \text{ occurs}\,, \\ \kappa_n\,, & \text{if event } \mathcal{T}^c \text{ occurs}\,, \end{cases} \tag{10}$$

where $\kappa_n > 0$ is a constant satisfying condition $\kappa_n = o(n^{4d-2}/\log n)$, and the event $\mathcal{T}$ satisfies conditions $\mathbb{P}_{H_0}(\mathcal{T}) \to 1$ and $\mathbb{P}_{H_1}(\mathcal{T}^c) \to 1$ as $n\to\infty$. We state in Section 2.2 below how to specify $\mathcal{T}$. The intuition behind this truncation is as follows: Under H_0, $\mathrm{cv}_\phi = z_{1-\phi}\hat{B}_{2N-K_0}/(2N) + \sum_{k=0}^{K_0}|\hat{\gamma}_1(k)|^2$ with the probability approaching one, ensuring the correct size of the test asymptotically. See Theorem 2 below. Under H_1, $\mathrm{cv}_\phi = \kappa_n$ with probability approaching one. Proposition 1 implies that $|\hat{\gamma}_2(0)| \ge n^{2d-1}/\log^{1/2} n$ with probability approaching one under H_1. Consequently $T_n \ge |\hat{\gamma}_2(0)|^2 \ge n^{4d-2}/\log n > \kappa_n = \mathrm{cv}_\phi$ with probability approaching one under H_1, as $\kappa_n = o(n^{4d-2}/\log n)$. This entails that the test has the power one asymptotically. See Theorem 3 below.

Theorem 2. *Assume the conditions of Theorem 1 hold. Let $\hat{B}_{2N-K_0}/B_{2N-K_0} \xrightarrow{\mathrm{P}} 1$ under H_0. Then for cv_ϕ defined in (10), it holds that $\mathbb{P}_{H_0}(T_n > \mathrm{cv}_\phi) \to \phi$ as $n \to \infty$.*

An estimate $\hat{B}_{2N-K_0}$ satisfying condition $\hat{B}_{2N-K_0}/B_{2N-K_0} \xrightarrow{\mathrm{P}} 1$ under H_0 will be constructed in Section 2.3 below.

Theorem 3. *Consider the alternative hypothesis H_1 under which Y_t is defined by (4) with $\epsilon_t \overset{\text{i.i.d.}}{\sim} (0, \sigma_\epsilon^2)$ and $\sum_{j=1}^{\infty} j|\psi_j| < \infty$. Then for cv_ϕ defined in (10), it holds that $\mathbb{P}_{H_1}(T_n > \mathrm{cv}_\phi) \to 1$ as $n \to \infty$.*

2.2 Determining $\mathcal{T}$ and κ_n

The critical value cv_ϕ defined in (10) depends on event $\mathcal{T}$ and truncation parameter κ_n critically. As $\kappa_n = o(n^{4d-2}/\log n)$ (see Section 2.1), we set $\kappa_n = 0.1 \times \log N$ with $N = \lfloor n/2 \rfloor$. Let $X_t = \nabla Y_t$ for $t = 2, \ldots, n$, and

$$\hat{\gamma}_x(k) = \frac{1}{n-1} \sum_{t=2}^{n-k} (X_{t+k} - \bar{X})(X_t - \bar{X}) \tag{11}$$

for $k \geq 0$, where $\bar{X} = (n-1)^{-1} \sum_{t=2}^{n} X_t$. Recall $\hat{\gamma}(k)$ is defined as (3). Under H_0, it holds that

$$\frac{\hat{\gamma}(0) + \hat{\gamma}(1)}{\hat{\gamma}_x(0) + \hat{\gamma}_x(1)} = O_{\mathrm{p}}(1)\,,$$

which implies that for any fixed constant $C_* > 0$,

$$\mathbb{P}_{H_0}\bigg\{\frac{\hat{\gamma}(0) + \hat{\gamma}(1)}{\hat{\gamma}_x(0) + \hat{\gamma}_x(1)} < C_* N^{3/5}\bigg\} \to 1 \tag{12}$$

as $n \to \infty$. It follows from Proposition 1 that

$$\mathbb{P}_{H_1}\bigg\{\frac{\hat{\gamma}(0) + \hat{\gamma}(1)}{\hat{\gamma}_x(0) + \hat{\gamma}_x(1)} \geq C_* N^{3/5}\bigg\} \to 1 \tag{13}$$

as $n \to \infty$. Consequently we define $\mathcal{T}$ in (10) as follows:

$$\mathcal{T} = \bigg\{\frac{\hat{\gamma}(0) + \hat{\gamma}(1)}{\hat{\gamma}_x(0) + \hat{\gamma}_x(1)} < C_* N^{3/5}\bigg\}\,. \tag{14}$$

While the asymptotic properties (12) and (13) holds for any positive constant $C_* > 0$, to use $\mathcal{T}$ with finite samples C_* must be specified according to the underlying process. To specify such a constant C_*, we first assume $Y_t \sim I(1)$ defined by (4) with $\epsilon_t \overset{\text{i.i.d.}}{\sim} (0, \sigma_\epsilon^2)$. Then the proposition below holds.

Proposition 2. *Let $Y_t \sim I(1)$ be defined by (4) in which $\epsilon_t \overset{\text{i.i.d.}}{\sim} (0, \sigma_\epsilon^2)$ and $\sum_{j=1}^{\infty} j|\psi_j| < \infty$.*

(i) *When $\mu_1 = 0$, it holds that*

$$\frac{1}{n}\frac{\hat\gamma(0)+\hat\gamma(1)}{\hat\gamma_x(0)+\hat\gamma_x(1)} \xrightarrow{\mathrm{D}} \frac{2a^2\int_0^1 V_0^2(t)\,\mathrm{d}t}{\sum_{j=0}^\infty \psi_j^2 + \sum_{j=0}^\infty \psi_j\psi_{j+1}}\,,$$

where $a = \sum_{j=0}^\infty \psi_j$ and $V_0(t) = W(t) - \int_0^1 W(t)\,\mathrm{d}t$ with $W(t)$ being the standard Brownian motion.

(ii) *When $\mu_1 \neq 0$, it holds that*

$$\frac{1}{n^2}\frac{\hat\gamma(0)+\hat\gamma(1)}{\hat\gamma_x(0)+\hat\gamma_x(1)} \xrightarrow{\mathrm{P}} \frac{\mu_1^2}{6\sigma_\epsilon^2(\sum_{j=0}^\infty \psi_j^2 + \sum_{j=0}^\infty \psi_j\psi_{j+1})}\,.$$

Proposition 2 shows that the ratio $\{\hat\gamma(0)+\hat\gamma(1)\}/\{\hat\gamma_x(0)+\hat\gamma_x(1)\}$ with $\mu_1 \neq 0$ diverges to infinity faster than that with $\mu_1 = 0$. Thus for any given $C_* > 0$ the requirement $\mathbb{P}_{H_1}(\mathcal{T}^c) \to 1$ is satisfied more readily with $\mu_1 \neq 0$ than that with $\mu_1 = 0$. Therefore below we focus on the cases with $\mu_1 = 0$ only.

Recall $X_t = \nabla Y_t = \mu_1 + \sum_{j=0}^\infty \psi_j \epsilon_{t-j}$. It holds that

$$\frac{2a^2\int_0^1 V_0^2(t)\,\mathrm{d}t}{\sum_{j=0}^\infty \psi_j^2 + \sum_{j=0}^\infty \psi_j\psi_{j+1}} = \frac{2\int_0^1 V_0^2(t)\,\mathrm{d}t}{\lambda(1+\rho)}\,,$$

where

$$\rho = \frac{\mathbb{E}\{(X_{t+1}-\mu_1)(X_t-\mu_1)\}}{\mathbb{E}\{(X_t-\mu_1)^2\}} = \frac{\sum_{j=0}^\infty \psi_j\psi_{j+1}}{\sum_{j=0}^\infty \psi_j^2} \tag{15}$$

is the first order autocorrelation coefficient, $\lambda = \sigma_S^2/\sigma_L^2$, and σ_S^2 and σ_L^2 are, respectively, the short-run variance and the long-run variance:

$$\sigma_S^2 = \sigma_\epsilon^2 \sum_{j=0}^\infty \psi_j^2 \quad \text{and} \quad \sigma_L^2 = \sigma_\epsilon^2 \bigg(\sum_{j=0}^\infty \psi_j\bigg)^2.$$

We estimate $\sigma_S^2 = \mathrm{Var}(X_t)$ by $\hat\sigma_S^2 = \hat\gamma_x(0)$, and $\sigma_L^2 = \lim_{n\to\infty} \mathrm{Var}(n^{-1/2}\sum_{t=2}^n X_t)$ by the kernel-type method suggested in Section 2.3 based on the sequence $\{X_t - \bar X\}_{t=2}^n$. We denote by $\hat\sigma_L^2$ the estimation of σ_L^2. Then we estimate λ by

$$\hat\lambda = \frac{\hat\sigma_S^2}{\hat\sigma_L^2}\,. \tag{16}$$

Finally we estimate ρ by

$$\hat\rho = \frac{\hat\gamma_x(1)}{\hat\gamma_x(0)}\,. \tag{17}$$

As $\mathbb{E}\{\int_0^1 V_0^2(t)\,\mathrm{d}t\} = 1/6$, we now specify the model-dependent constant C_* in (14) as

$$C_* = \frac{2c_\kappa}{\hat{\lambda}(1+\hat{\rho})}$$

for some uniform constant c_κ greater than 1/6, where $\hat{\lambda}$ and $\hat{\rho}$ are given in (16) and (17), respectively. Consequently, the critical value cv_ϕ admits the following representation

$$\mathrm{cv}_\phi = \begin{cases} \dfrac{z_{1-\phi}\hat{B}_{2N-K_0}}{2N} + \displaystyle\sum_{k=0}^{K_0} |\hat{\gamma}_1(k)|^2\,, & \text{if } \dfrac{\hat{\gamma}(0)+\hat{\gamma}(1)}{\hat{\gamma}_x(0)+\hat{\gamma}_x(1)} < \dfrac{2c_\kappa N^{3/5}}{\hat{\lambda}(1+\hat{\rho})}\,; \\ 0.1 \times \log N\,, & \text{if } \dfrac{\hat{\gamma}(0)+\hat{\gamma}(1)}{\hat{\gamma}_x(0)+\hat{\gamma}_x(1)} \geq \dfrac{2c_\kappa N^{3/5}}{\hat{\lambda}(1+\hat{\rho})}\,. \end{cases} \tag{18}$$

Our extensive simulation results indicate that this specification of cv_ϕ with c_κ between 0.45 and 0.65 works well across a variety of models. See Table 1 in Section 2.4, Tables 2 and 3 in Section 3 and also the supplementary material for details.

Though the above specification was derived for $Y_t \sim I(1)$, our simulation results indicate that it also works fine for $I(2)$ processes; see Table 3 in Section 3 and Tables S5–S8 and S14–S18 in the supplementary material. Note that testing $I(0)$ against $I(d)$ with $d > 1$ is easier than that with $d = 1$, as the autocovariances are of the order at least n^{2d-1} for $I(d)$ processes. Therefore the difference between the values of test statistic T_n under H_1 and those under H_0 increases as d increases.

2.3 Estimation of $B^2_{2N-K_0}$

Let $m = 2N - K_0$. Recall that

$$B^2_{2N-K_0} = m\mathrm{Var}\bigg(\frac{1}{\sqrt{m}}\sum_{t=1}^m Q_t\bigg) =: mV_m$$

where V_m is the long-run variance of $\{Q_t\}_{t=1}^m$. We only need to estimate the long-run variance V_m. There exist various estimation methods for long-run variances, including the kernel-type estimators (Andrews, 1991) and the estimators utilizing moving block bootstraps (Lahiri, 2003). See also Den Haan and Levin (1997) and Kiefer, Vogelsang and Bunzel (2000).

Recall $Q_t = \sum_{k=0}^{K_0} \xi_{t,k}$ with $\xi_{t,k} = 2y_{t,k}\gamma(k)$ and $y_{t,k}$ defined in (8). Let

$$\begin{cases} \tilde{y}_{t,k} = 2\{(Y_t - \bar{Y})(Y_{t+k} - \bar{Y}) - \hat{\gamma}(k)\}\mathrm{sgn}(k+t-N-1/2)\,, \\ \tilde{\xi}_{t,k} = 2\tilde{y}_{t,k}\hat{\gamma}(k)\,, \end{cases} \tag{19}$$

and $\tilde{Q}_t = \sum_{k=0}^{K_0} \tilde{\xi}_{t,k}$, where $\bar{Y} = n^{-1}\sum_{t=1}^n Y_t$ and $\hat{\gamma}(k)$ is defined in (3). We adopt the following kernel-type estimator for V_m based on $\{\tilde{Q}_t\}_{t=1}^m$:

$$\tilde{V}_m = \sum_{j=-m+1}^{m-1} \mathcal{K}\bigg(\frac{j}{b_m}\bigg)\tilde{G}_j \,, \tag{20}$$

where $\mathcal{K}(\cdot)$ is a symmetric kernel function that is continuous at 0 with $\mathcal{K}(0) = 1$, b_m is the bandwidth, $\tilde{G}_j = m^{-1}\sum_{t=j+1}^m \tilde{Q}_t\tilde{Q}_{t-j}$ if $j \geq 0$ and $\tilde{G}_j = m^{-1}\sum_{t=-j+1}^m \tilde{Q}_{t+j}\tilde{Q}_t$ otherwise. Let

$$\hat{B}_{2N-K_0} = (m\tilde{V}_m)^{1/2} \,. \tag{21}$$

Andrews (1991) systematically investigated the theoretical properties of this estimation method. It shows that the Quadratic Spectral kernel

$$\mathcal{K}_{QS}(u) = \frac{25}{12\pi^2u^2}\bigg\{\frac{\sin(6\pi u/5)}{6\pi u/5} - \cos(6\pi u/5)\bigg\}$$

is optimal in the sense of minimizing the asymptotic truncated mean square error of the estimator. In our numerical work reported in both Section 3 and the supplementary material, we always use this kernel function by calling function `lrvar` from the R-package `sandwich` with the default bandwidth specified in the function.

To state the required asymptotic property for $\hat{B}_{2N-K_0}$, we introduce some regularity conditions first.

Condition 4. The kernel function $\mathcal{K}(\cdot) : \mathbb{R} \to [-1, 1]$ is continuously differentiable on $\mathbb{R}$ and satisfies (i) $\mathcal{K}(0) = 1$, (ii) $\mathcal{K}(x) = \mathcal{K}(-x)$ for any $x \in \mathbb{R}$, and (iii) $\int_{-\infty}^{\infty} |\mathcal{K}(x)| \,\mathrm{d}x < \infty$. Let $K_* = 1 + K_0$ satisfying $K_*^{13} \log K_* = o(n^{1-2/s_2})$. The bandwidth $b_m \to \infty$ satisfies $b_m = o\{n^{1/2-1/s_2}(K_*^5 \log K_*)^{-1/2}\}$ and $K_*^4 = o(b_m)$.

Condition 5. Under the null hypothesis H_0, there exist uniform constants $s_2 > 4$, $c_4 > 0$, $c_5 > 0$ and $\beta_2 > \max\{2s_2/(s_2 - 2), s_2/(s_2 - 4)\}$ such that $\max_{1\leq t\leq n} \mathbb{E}(|Y_t|^{2s_2}) \leq c_4$, and the α-mixing coefficients $\{\alpha(\tau)\}_{\tau\geq 1}$ satisfy $\alpha(\tau) \leq c_5\tau^{-\beta_2}$, where $\alpha(\tau)$ is defined in Condition 2.

Theorem 4. *Let Conditions* 4 *and* 5 *hold. Then as* $n \to \infty$*, it holds under the null hypothesis* H_0 *that* $\hat{B}_{2N-K_0}/B_{2N-K_0} \xrightarrow{\mathrm{P}} 1$.

2.4 Implementation of the test

Based on Sections 2.2 and 2.3 above, Algorithm 1 outlines the steps to be taken in order to perform the proposed test. The algorithm is implemented in an R-function `ur.test` contained in the package `HDTSA` which is available via 'github':

```
devtoolsi::install_github('ghghgh2020/HDTSA/HDTSA')
```

To perform the test using function `ur.test`, one merely needs to input time series $\{Y_t\}_{t=1}^n$ and significance level ϕ. The package sets the default value $c_\kappa = 0.55$ and returns the five testing results for $K_0 = 0, 1, \ldots, 4$ respectively. One can also set the values of c_κ and K_0 subjectively. We recommend to use $c_\kappa \in [0.45, 0.65]$ and $K_0 \in \{0, 1, 2, 3, 4\}$.

Algorithm 1 Sample ACF-based unit-root test

1: Given $\{Y_t\}_{t=1}^n$ and K_0, compute $\hat\gamma(k)$ defined as in (3) for $k = 0, 1$, and also $\hat\gamma_1(k)$ and $\hat\gamma_2(k)$ defined as in (5) for $k = 0, 1, \ldots, K_0$.

2: Let $X_t = \nabla Y_t$. Compute $\hat\gamma_x(k)$ defined as (11) for $k = 0, 1$, and put $\hat\rho = \hat\gamma_x(1)/\hat\gamma_x(0)$.

3: Call function `lrvar` from the R-package `sandwich` (with the default value of the bandwidth specified in the function) to compute the long-run covariances of $\{\tilde Q_t\}$ and $\{X_t\}$, denoted by $\tilde V_{2N-K_0}$ and $\hat\sigma_L^2$, respectively, where $\tilde Q_t$ is defined immediately below (19).

4: Given significant level $\phi \in (0, 1)$, calculate the test statistic $T_n = \sum_{k=0}^{K_0} |\hat\gamma_2(k)|^2$ and the critical value

$$\mathrm{cv}_\phi = \begin{cases} \dfrac{z_{1-\phi}\hat B_{2N-K_0}}{2N} + \displaystyle\sum_{k=0}^{K_0} |\hat\gamma_1(k)|^2 \,, & \text{if } \dfrac{\hat\gamma(0)+\hat\gamma(1)}{\hat\gamma_x(0)+\hat\gamma_x(1)} < \dfrac{2c_\kappa N^{3/5}}{\hat\lambda(1+\hat\rho)}\,; \\ 0.1 \times \log N \,, & \text{if } \dfrac{\hat\gamma(0)+\hat\gamma(1)}{\hat\gamma_x(0)+\hat\gamma_x(1)} \geq \dfrac{2c_\kappa N^{3/5}}{\hat\lambda(1+\hat\rho)}\,; \end{cases}$$

where $c_\kappa \in (0.45, 0.65)$ is a prescribed constant, $z_{1-\phi}$ is the $(1-\phi)$-quantile of $\mathcal N(0, 1)$, $\hat B_{2N-K_0} = (2N - K_0)^{1/2} \tilde V_{2N-K_0}^{1/2}$ and $\hat\lambda = \hat\gamma_x(0)/\hat\sigma_L^2$.

5: Reject null hypothesis H_0 if $T_n > \mathrm{cv}_\phi$.

To illustrate the robustness of the test with respect to the values of c_κ and K_0, we applied the test to the 14 US annual economic time serie initially analyzed by Nelson and Plosser (1982), and subsequently by many others including Perron (1988); DeJong et al. (1989); Kwiatkowski et al. (1992). The length n for those 14 series varies between 62 and 111. The conventional wisdom is that a unit root is present in most of the Nelson-Plosser series. The analysis from the aforementioned papers indicates that the unit-root hypothesis cannot be validated for only one or two series such as unemployment rates. The results from the proposed new test corroborates those findings; see Table 1. The new test rejects the null hypothesis $Y_t \sim I(0)$ in favour of $H_1 : Y_t \sim I(d)$ at the 5% significance level for the 12 out of the 14 series. The two series for which H_0 cannot be rejected are the unemployment rate series and the velocity series. Note that the new test is robust in the sense that taking $c_\kappa = 0.45, 0.55$ or 0.65, and $K_0 = 0, 1, 2, 3$ or 4 leads to exactly the same results.

3 Simulation study

We illustrate the finite sample properties of the proposed test T_n by simulation. Various versions of the proposed test with $K_0 \in \{0, 1, 2, 3, 4\}$ and $c_\kappa \in \{0.45, 0.55, 0.65\}$ are considered; see (18). We also consider the proposed test with untruncated critical value, i.e. $c_\kappa = \infty$ in (18). For the comparison purpose, we also include the KPSS test (Kwiatkowski et al., 1992) in our experiments. We set $n = 80, 140, 200$ (i.e. $N = 40, 70, 100$) and repeat each setting 2000 times.

Unless specified otherwise, we always assume that $\epsilon_t \overset{\text{i.i.d.}}{\sim} \mathcal N(0, \sigma_\epsilon^2)$ with $\sigma_\epsilon^2 = 1$ or 2. To examine the sizes of the tests, we consider the following four models.

Table 1: Testing results for the 14 Nelson-Plosser time series by the proposed new test at the 5% significance level with $c_\kappa = 0.45, 0.55$ and 0.65 respectively: "+" indicates the stationary null hypothesis is rejected in favour of a unit-root alternative, while "−" indicates the null hypothesis cannot be rejected. The results are unchanged with $K_0 = 0, 1, 2, 3$ or 4.

	$c_\kappa = 0.45$	$c_\kappa = 0.55$	$c_\kappa = 0.65$
Real GNP	+	+	+
Nominal GNP	+	+	+
Real per capital GNP	+	+	+
Industrial production	+	+	+
Employment	+	+	+
Unemployment rate	−	−	−
GNP deflator	+	+	+
Consumer prices	+	+	+
Wages	+	+	+
Real wages	+	+	+
Money stock	+	+	+
Velocity	−	−	−
Bond yield	+	+	+
S&P500 stock prices	+	+	+

- Model 1: $Y_t = \rho Y_{t-1} + \epsilon_t$.
- Model 2: $Y_t = \epsilon_t + \phi_1 \epsilon_{t-1} + \phi_2 \epsilon_{t-2}$.
- Model 3: $Y_t - \rho_1 Y_{t-1} - \rho_2 Y_{t-2} = \epsilon_t + 0.5\epsilon_{t-1} + 0.3\epsilon_{t-2}$.
- Model 4: $Y_t = \epsilon_t + \sum_{i=1}^{5} \phi \epsilon_{t-i}$.

We set the nominal size of the tests at $\phi = 5\%$. The KPSS test is implemented by calling function `kpss.test` in R-package `tseries`. The results with $K_0 = 0$ are listed in Table 2, and the results with $K_0 \in \{1, 2, 3, 4\}$ are reported in Tables S1–S4 in the supplementary material. Note that the results with different c_κ and K_0 are similar; indicating once again that the test T_n is robust with respect to the choice of tuning parameters c_κ and K_0. We also consider the cases that $\epsilon_t \overset{\text{i.i.d.}}{\sim} t(2)$ and $\epsilon_t \overset{\text{i.i.d.}}{\sim} t(5)$ and report the results in Tables S9–S13 in the supplementary material.

Overall the proposed test provides reasonable approximations for the size of the test especially with large n (eg. $N = 100$), and truncation (10) has little impact on the achieved significance levels, as indicated in Theorem 2. The performance of the new test is stable across different models with different parameters, different K_0 and different innovation distributions, while that of the KPSS test varies and is adequate only for some settings.

Table 2 indicates that for Model 1 the new test controls the size well for both positive and negative ρ, while the KPSS test performs poorly when $\rho < 0$, and even worse when $\rho > 0$. In fact the KPSS test completely fails when $\rho = 0.9$, as the empirical sizes are at least 46.7%. This is due to the fact that when ρ is close to 1, the KPSS test has difficulties in distinguishing it from 1 which is unit-root. See also Table 3 of Kwiatkowski et al. (1992). Our new method does not suffer from this closeness to 1, as for which the order of the magnitude of ACF matters. Table 2 also shows that for Model 2 both the new test and the KPSS test provide comparable approximations for the size of the test. For Models 3 and 4, the new test provides adequate approximations for the size of the test. Unfortunately the KPSS test does not work for Models 3 and 4 as the empirical sizes range from 15.4% to 26.2% for Model 3, and 13.8% to 20.4% for Model 4.

The performance of empirical powers is based on the following models.

- Model 5: $\nabla Y_t = Z_t$, $Z_t = \rho Z_{t-1} + \epsilon_t$.
- Model 6: $\nabla Y_t = Z_t$, $Z_t = \epsilon_t + \phi_1 \epsilon_t + \phi_2 \epsilon_{t-1}$.
- Model 7: $\nabla Y_t = Z_t$, $Z_t - \rho_1 Z_{t-1} - \rho_2 Z_{t-2} = \epsilon_t + 0.5\epsilon_t + 0.3\epsilon_{t-1}$.
- Model 8: $\nabla^2 Y_t = Z_t$, $Z_t = \epsilon_t + \phi_1 \epsilon_t + \phi_2 \epsilon_{t-1}$.

The corresponding results are reported in Table 3 for $K_0 = 0$ with normal distributed innovations, and in Tables S5–S8 in the supplementary material for $K_0 \in \{1, 2, 3, 4\}$. The KPSS test shows impressive power under the models above. Note that the KPSS test has a tendency to overestimate test levels, leading to inflated power. Nevertheless the new test displays greater power in most cases. Noticeably the power one property of the new test, presented in Theorem 3, is observable in the simulation as the empirical power tends to 1 when N increases. Comparing the results of Models 6 and 8, we found that the proposed new tests show off the asymptotic power one property more distinctly as the test statistic T_n has more discriminate power between $I(2)$ and $I(0)$ than that between $I(1)$ and $I(0)$. We also simulated the power of the tests with $t(2)$ and $t(5)$ innovations in Models 5–8. The results are presented in Tables S14–S18 in the supplementary material; showing similar profiles as those in Tables 3 and S5–S8.

4 Technical proofs

4.1 Proof of Proposition 1

For any $0 \leq j \leq d$, we write $Y_t^{(d-j)} = \nabla^j Y_t$. Assume $Y_{-(d-1)} = \cdots = Y_0 = 0$. Let $\{F_g(\cdot)\}$ be the multi-fold integrated Brownian motion considered in Chan and Wei (1988), which is defined recursively as $F_g(t) = \int_0^t F_{g-1}(x)\,\mathrm{d}x$ for any $g \geq 1$ and $F_0(t) = W(t)$ is the scalar Brownian motion.

We first consider the case with $\mu_d = 0$ and $k \geq 1$. Due to $Y_t \sim I(d)$, we can reformulate Y_t

Table 2: Empirical sizes ($\times 10^2$) of the proposed test T_n defined as (6) for $K_0 = 0$ with the untruncated critical value ($c_\kappa = \infty$) and the truncated critical values defined as (18) with $c_\kappa = 0.45, 0.55, 0.65$, and the KPSS test in a simulation with 2000 replications. Constant c_κ determines the level of truncation for the critical values of T_n. The innovations $\epsilon_t \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \sigma_\epsilon^2)$. The nominal size of the tests is 5%.

			$\sigma_\epsilon^2 = 1$					$\sigma_\epsilon^2 = 2$				
	Setting	N	∞	0.45	0.55	0.65	KPSS	∞	0.45	0.55	0.65	KPSS
Model 1	0.5	40	6.0	6.0	6.0	6.0	10.4	6.5	6.5	6.5	6.5	10.8
		70	6.9	6.9	6.9	6.9	10.1	5.8	5.8	5.8	5.8	10.7
		100	6.1	6.1	6.1	6.1	10.2	5.6	5.6	5.6	5.6	8.5
	0.9	40	7.2	41.9	30.0	20.3	51.2	8.8	40.8	27.4	19.3	50.5
		70	7.8	23.7	14.6	10.4	46.7	8.0	23.4	13.9	10.5	49.5
		100	8.5	12.7	9.4	8.6	49.2	8.6	14.3	10.1	8.9	49.3
	−0.5	40	7.4	7.4	7.4	7.4	1.8	7.5	7.5	7.5	7.5	1.8
		70	6.9	6.9	6.9	6.9	2.5	7.2	7.2	7.2	7.2	2.0
		100	6.4	6.4	6.4	6.4	1.8	7.0	7.0	7.0	7.0	2.6
Model 2	(0.8, 0.3)	40	6.2	6.2	6.2	6.2	7.6	6.8	6.8	6.8	6.8	7.7
		70	6.4	6.4	6.4	6.4	6.2	6.7	6.7	6.7	6.7	7.0
		100	7.2	7.2	7.2	7.2	7.0	6.0	6.0	6.0	6.0	7.0
	(0.9, 0.5)	40	6.7	6.7	6.7	6.7	8.5	7.5	7.5	7.5	7.5	7.8
		70	6.5	6.5	6.5	6.5	8.1	6.9	6.9	6.9	6.9	7.1
		100	5.6	5.6	5.6	5.6	7.4	6.6	6.6	6.6	6.6	8.2
	(0.95, 0.9)	40	7.2	7.2	7.2	7.2	9.0	7.1	7.1	7.1	7.1	8.3
		70	7.1	7.1	7.1	7.1	7.3	6.9	6.9	6.9	6.9	7.3
		100	5.5	5.5	5.5	5.5	8.1	5.9	5.9	5.9	5.9	8.2
Model 3	(0.4, 0.2)	40	7.2	8.2	7.4	7.3	22.5	6.9	8.0	7.0	6.9	20.8
		70	7.7	7.7	7.7	7.7	17.3	7.3	7.3	7.3	7.3	17.5
		100	7.2	7.2	7.2	7.2	18.0	6.0	6.0	6.0	6.0	17.4
	(0.5, 0.1)	40	8.5	8.9	8.5	8.5	19.6	7.1	7.5	7.1	7.1	19.6
		70	8.0	8.0	8.0	8.0	16.6	6.2	6.2	6.2	6.2	17.4
		100	6.3	6.3	6.3	6.3	17.4	6.3	6.3	6.3	6.3	15.4
	(0.6, 0.1)	40	8.5	12.7	9.6	8.7	26.2	8.8	11.7	9.5	8.9	24.2
		70	7.3	7.3	7.3	7.3	22.4	9.1	9.2	9.1	9.1	22.5
		100	7.6	7.6	7.6	7.6	20.3	7.0	7.0	7.0	7.0	23.6
Model 4	0.4	40	9.7	10.3	9.8	9.7	20.4	8.9	9.6	8.9	8.9	17.7
		70	7.5	7.5	7.5	7.5	15.7	8.8	8.8	8.8	8.8	15.2
		100	7.6	7.6	7.6	7.6	15.0	8.8	8.8	8.8	8.8	15.3
	0.5	40	8.2	9.2	8.3	8.2	20.2	8.6	9.9	8.9	8.8	19.1
		70	8.3	8.3	8.3	8.3	15.4	8.5	8.5	8.5	8.5	16.9
		100	7.8	7.8	7.8	7.8	15.8	6.8	6.8	6.8	6.8	15.8
	0.6	40	8.5	10.2	8.8	8.5	19.6	7.3	8.9	7.8	7.3	20.0
		70	8.9	8.9	8.9	8.9	13.8	9.2	9.2	9.2	9.2	16.2
		100	7.2	7.2	7.2	7.2	15.9	8.1	8.1	8.1	8.1	15.8

Table 3: Empirical powers ($\times 10^2$) of the proposed test T_n defined as (6) for $K_0 = 0$ with the untruncated critical value ($c_\kappa = \infty$) and the truncated critical values defined as (18) with $c_\kappa = 0.45, 0.55, 0.65$, and the KPSS test in a simulation with 2000 replications. Constant c_κ determines the level of truncation for the critical values of T_n. The innovations $\epsilon_t \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \sigma_\epsilon^2)$. The nominal size of the tests is 5%.

			$\sigma_\epsilon^2 = 1$					$\sigma_\epsilon^2 = 2$				
	Setting	N	∞	0.45	0.55	0.65	KPSS	∞	0.45	0.55	0.65	KPSS
Model 5	0.5	40	11.7	94.2	88.4	84.0	84.2	12.4	93.8	89.5	84.5	83.2
		70	11.7	96.5	92.9	88.4	90.9	12.6	97.1	93.8	90.3	91.0
		100	11.3	98.0	95.5	92.2	95.5	11.8	98.1	95.5	92.5	95.5
	0.9	40	13.1	99.2	97.3	94.6	91.1	14.1	99.4	98.2	96.2	92.8
		70	14.8	99.8	99.1	97.9	95.3	14.5	99.9	99.7	99.1	95.0
		100	16.4	99.9	99.5	99.1	97.2	15.4	100.0	99.9	99.6	97.4
	-0.5	40	5.6	82.2	75.1	67.6	81.5	6.2	84.2	75.6	68.2	81.8
		70	6.3	92.1	86.1	80.0	90.1	6.5	90.8	85.7	79.5	89.2
		100	5.8	94.2	89.5	85.2	94.5	5.9	94.5	90.6	86.0	94.2
Model 6	(0.8, 0.3)	40	11.8	94.3	88.8	82.3	82.0	12.4	93.0	87.1	81.2	81.3
		70	11.8	96.6	92.7	88.3	90.1	11.2	96.5	92.6	88.1	89.9
		100	12.1	98.4	95.4	91.8	95.3	10.8	98.0	95.8	92.1	95.0
	(0.9, 0.5)	40	11.8	95.3	90.0	84.2	83.5	12.2	95.3	88.8	83.3	82.6
		70	12.2	97.2	93.8	89.8	89.2	11.3	97.3	93.8	90.0	90.6
		100	11.6	98.6	96.4	92.7	94.8	11.2	97.8	95.2	91.2	93.8
	(0.95, 0.9)	40	13.1	95.0	90.0	83.9	83.0	12.0	95.4	90.7	85.8	84.5
		70	11.6	97.3	93.8	89.7	90.2	12.8	97.5	94.4	89.8	90.1
		100	13.7	99.0	96.4	92.3	95.2	11.6	98.2	95.7	92.5	94.8
Model 7	(0.4, 0.2)	40	14.8	98.0	95.2	90.6	85.9	14.8	98.3	95.4	91.1	85.8
		70	15.4	99.1	97.0	93.8	92.0	14.1	99.3	97.2	94.3	91.6
		100	16.6	99.6	98.8	96.5	96.5	15.7	99.6	98.5	96.3	96.0
	(0.5, 0.1)	40	14.2	99.1	95.9	91.3	84.7	15.8	98.7	95.9	91.2	85.3
		70	14.8	99.4	97.2	94.0	91.2	14.3	99.4	97.7	94.8	91.3
		100	15.0	99.6	98.5	96.2	95.5	15.3	99.3	98.2	96.5	95.0
	(0.6, 0.1)	40	14.5	99.2	97.1	93.3	87.2	15.3	98.9	96.7	93.2	86.8
		70	15.7	99.7	98.5	96.2	93.5	17.6	99.7	98.7	97.0	92.0
		100	16.4	99.8	99.1	97.7	95.7	15.9	100.0	99.5	98.6	96.0
Model 8	(0.8, 0.3)	40	6.7	100.0	100.0	99.9	98.5	6.2	100.0	100.0	100.0	98.2
		70	6.3	100.0	100.0	100.0	99.7	6.0	100.0	100.0	100.0	99.2
		100	7.0	100.0	100.0	100.0	99.8	6.1	100.0	100.0	100.0	99.9
	(0.9, 0.5)	40	7.0	100.0	100.0	100.0	98.4	7.0	100.0	100.0	100.0	98.5
		70	5.5	100.0	100.0	100.0	99.5	6.0	100.0	100.0	100.0	99.3
		100	5.9	100.0	100.0	100.0	99.9	6.2	100.0	100.0	100.0	99.8
	(0.95, 0.9)	40	8.0	100.0	100.0	100.0	98.5	6.8	100.0	100.0	100.0	98.6
		70	7.3	100.0	100.0	100.0	99.2	6.3	100.0	100.0	100.0	99.7
		100	6.1	100.0	100.0	100.0	99.9	5.3	100.0	100.0	100.0	100.0

as $Y_t = Y_t^{(d)} = Y_{t-1}^{(d)} + Y_t^{(d-1)} = \cdots = Y_{t-k}^{(d)} + \sum_{j=0}^{k-1} Y_{t-j}^{(d-1)}$ for any $k \geq 1$, which implies that

$$
\begin{aligned}
\frac{\hat{\gamma}(k)}{n^{2d-1}} &= \frac{1}{n^{2d}} \sum_{t=k+1}^{n} \{Y_{t-k}^{(d)} - \bar{Y}\}\{Y_t^{(d)} - \bar{Y}\} \\
&= \frac{1}{n^{2d}} \sum_{t=k+1}^{n} \{Y_t^{(d)} - \bar{Y}\}^2 - \frac{1}{n^{2d}} \sum_{t=k+1}^{n} \{Y_t^{(d)} - \bar{Y}\}\{Y_{t-k+1}^{(d-1)} + \cdots + Y_t^{(d-1)}\}\,.
\end{aligned} \tag{22}
$$

Meanwhile, for each $0 \leq i \leq k-1$, it holds that

$$
\sum_{t=k+1}^{n} Y_{t-i}^{(d-1)} Y_t^{(d)} = \sum_{t=k+1}^{n} Y_{t-i}^{(d-1)} \{Y_{t-i-1}^{(d)} + Y_{t-i}^{(d-1)} + \cdots + Y_t^{(d-1)}\} \tag{23}
$$

and

$$
\begin{aligned}
\sum_{t=k+1}^{n} Y_{t-i}^{(d-1)} Y_{t-i-1}^{(d)} = \sum_{j=k+1-i}^{n-i} Y_j^{(d-1)} Y_{j-1}^{(d)} &= \sum_{j=k+1-i}^{n_i} \{Y_j^{(d)} - Y_{j-1}^{(d)}\} Y_{j-1}^{(d)} \\
&= \frac{1}{2}\{Y_{n_i}^{(d)}\}^2 - \frac{1}{2}\{Y_{k-i}^{(d)}\}^2 - \frac{1}{2} \sum_{j=k+1-i}^{n_i} \{Y_j^{(d)} - Y_{j-1}^{(d)}\}^2\,,
\end{aligned}
$$

where $n_i = n - i$. Note that $Y_j^{(d)} - Y_{j-1}^{(d)} = Y_j^{(d-1)}$. If $d \geq 2$, by (2.87), (2.142) and Theorem 2.17 in Tanaka (2017), we have $(n_i^{2d-2})^{-1} \sum_{j=1}^{n_i} \{Y_j^{(d)} - Y_{j-1}^{(d)}\}^2 \xrightarrow{\mathrm{D}} a^2\sigma_\epsilon^2 \int_0^1 F_{d-2}^2(s)\,\mathrm{d}s$ and $(n_i^{2d-1})^{-1}\{Y_{n_i}^{(d)}\}^2 \xrightarrow{\mathrm{D}} a^2\sigma_\epsilon^2 F_{d-1}^2(1)$ as $n_i \to \infty$. Due to $Y_{k-i}^{(d)} = O_{\mathrm{p}}(1)$, then we have

$$
\frac{1}{n^{2d-1}} \sum_{t=k+1}^{n} Y_{t-i}^{(d-1)} Y_{t-i-1}^{(d)} \xrightarrow{\mathrm{D}} \frac{a^2\sigma_\epsilon^2}{2} F_{d-1}^2(1)\,. \tag{24}
$$

For any $0 \leq j \leq i$, by the Cauchy-Schwarz inequality, it holds that

$$
\begin{aligned}
\bigg| \sum_{t=k+1}^{n} Y_{t-i}^{(d-1)} Y_{t-j}^{(d-1)} \bigg| &\leq \bigg[\sum_{t=k+1}^{n} \{Y_{t-i}^{(d-1)}\}^2 \bigg]^{1/2} \bigg[\sum_{t=k+1}^{n} \{Y_{t-j}^{(d-1)}\}^2 \bigg]^{1/2} \\
&= \bigg[\sum_{t=k+1}^{n} \{Y_{t-i}^{(d)} - Y_{t-i-1}^{(d)}\}^2 \bigg]^{1/2} \bigg[\sum_{t=k+1}^{n} \{Y_{t-j}^{(d)} - Y_{t-j-1}^{(d)}\}^2 \bigg]^{1/2} \\
&= O_{\mathrm{p}}(n^{2d-2})\,.
\end{aligned}
$$

When $d \geq 2$, together with (24), (23) leads to

$$
\sum_{t=k+1}^{n} Y_{t-i}^{(d-1)} Y_t^{(d)} = O_{\mathrm{p}}(n^{2d-1})\,, \tag{25}
$$

for any $0 \le i \le k-1$. Since $n^{-1}\sum_{t=k+1}^{n} Y_{t-i}^{(0)} Y_{t-i-1}^{(1)} = n^{-1}\sum_{j=k+1-i}^{n_i}\{Y_j^{(1)} - Y_{j-1}^{(1)}\}Y_{j-1}^{(1)}$, by (2.89) in Tanaka (2017), $n^{-1}\sum_{t=k+1}^{n} Y_{t-i}^{(0)} Y_{t-i-1}^{(1)} \xrightarrow{\mathrm{D}} a^2\sigma_\epsilon^2\{\int_0^1 W(t)\,\mathrm{d}W(t) + (1-\lambda)/2\}$ with $\lambda = a^{-2}\sum_{j=0}^{\infty}\psi_j^2$, which implies that $\sum_{t=k+1}^{n} Y_{t-i}^{(0)} Y_{t-i-1}^{(1)} = O_{\mathrm{p}}(n)$. Hence, (25) holds for any $d \ge 1$ and $0 \le i \le k-1$.

Since $n^{-d-1/2}\sum_{t=1}^{n} Y_t^{(d)} \xrightarrow{\mathrm{D}} a\sigma_\epsilon \int_0^1 F_{d-1}(s)\,\mathrm{d}s$ for any $d \ge 1$, then $\sum_{t=k+1}^{n} Y_{t-i}^{(d-1)}\bar{Y} = O_{\mathrm{p}}(n^{2d-1})$ for any $d \ge 1$ and $0 \le i \le k-1$. Hence, (22) leads to

$$
\begin{aligned}
\frac{\hat{\gamma}(k)}{n^{2d-1}} &= \frac{1}{n^{2d}}\sum_{t=k+1}^{n}\{Y_t^{(d)} - \bar{Y}\}^2 + O_{\mathrm{p}}(n^{-1}) \\
&= \frac{1}{n^{2d}}\sum_{t=k+1}^{n}\{Y_t^{(d)}\}^2 - 2\bar{Y}\cdot\frac{1}{n^{2d}}\sum_{t=k+1}^{n} Y_t^{(d)} + \frac{n-k}{n^{2d}}\bar{Y}^2 + O_{\mathrm{p}}(n^{-1})\,.
\end{aligned}
$$

Also notice that $n^{-2d}\sum_{t=1}^{n}\{Y_t^{(d)}\}^2 \xrightarrow{\mathrm{D}} a^2\sigma_\epsilon^2\int_0^1 F_{d-1}^2(s)\,\mathrm{d}s$. For any $k \ge 1$, it follows from the continuous mapping theorem that $n^{-2d+1}\hat{\gamma}(k) \xrightarrow{\mathrm{D}} a^2\sigma_\epsilon^2\int_0^1 V_{d-1}^2(s)\,\mathrm{d}s$, where $V_{d-1}(s) = F_{d-1}(s) - \int_0^1 F_{d-1}(s)\,\mathrm{d}s$. For $k = 0$, since $n^{-2d+1}\hat{\gamma}(0) = n^{-2d}\sum_{t=1}^{n}\{Y_t^{(d)} - \bar{Y}\}^2$, we then also have $n^{-2d+1}\hat{\gamma}(0) \xrightarrow{\mathrm{D}} a^2\sigma_\epsilon^2\int_0^1 V_{d-1}^2(s)\,\mathrm{d}s$. We complete the proof of part (i) of Proposition 1.

We now consider the case with $\mu_d \ne 0$. Let $U_t^{(0)} = \sum_{j=0}^{\infty}\psi_j\epsilon_{t-j}$ with $\{\psi_j\}$ and $\{\epsilon_t\}$ specified in (4). Recall $Y_t = Y_{t-k}^{(d)} + \sum_{j=0}^{k-1} Y_{t-j}^{(d-1)}$ for any $k \ge 1$. Then $Y_t = Y_0^{(d)} + \sum_{j_1=0}^{t-1} Y_{t-j_1}^{(d-1)}$. Recursively, we have

$$
\begin{aligned}
Y_t = Y_t^{(d)} &= Y_0^{(d)} + \sum_{j_1=0}^{t-1} Y_{t-j_1}^{(d-1)} \\
&= Y_0^{(d)} + \sum_{j_1=0}^{t-1}\left\{Y_0^{(d-1)} + \sum_{j_2=0}^{t-j_1-1} Y_{t-j_1-j_2}^{(d-2)}\right\} \\
&= Y_0^{(d)} + \sum_{h=1}^{d-1} Y_0^{(d-h)}\left(\sum_{j_1=0}^{t-1}\cdots\sum_{j_h=0}^{t-j_1-\cdots-j_{h-1}-1} 1\right) + \sum_{j_1=0}^{t-1}\cdots\sum_{j_d=0}^{t-j_1-\cdots-j_{d-1}-1}\mu_d \\
&\quad + \sum_{j_1=0}^{t-1}\cdots\sum_{j_d=0}^{t-j_1-\cdots-j_{d-1}-1} U_{t-j_1-\cdots-j_d}^{(0)}\,.
\end{aligned}
$$

Define $\omega_t = \sum_{j_1=0}^{t-1}\cdots\sum_{j_d=0}^{t-j_1-\cdots-j_{d-1}-1} 1$ and $r_t = \sum_{j_1=0}^{t-1}\cdots\sum_{j_d=0}^{t-j_1-\cdots-j_{d-1}-1} U_{t-j_1-\cdots-j_d}^{(0)}$. Notice that $Y_{-(d-1)} = \cdots = Y_0 = 0$. Then $Y_t = Y_t^{(d)} = \mu_d\cdot\omega_t + r_t$. For any $k \ge 1$, we have

$$
\begin{aligned}
\frac{\hat{\gamma}(k)}{n^{2d}} &= \frac{1}{n^{2d+1}}\sum_{t=k+1}^{n}(Y_{t-k} - \bar{Y})(Y_t - \bar{Y}) \\
&= \frac{\mu_d^2}{n^{2d+1}}\sum_{t=k+1}^{n}(\omega_{t-k} - \bar{\omega})(\omega_t - \bar{\omega}) + \frac{1}{n^{2d+1}}\sum_{t=k+1}^{n}(r_{t-k} - \bar{r})(r_t - \bar{r})
\end{aligned}
$$

$$+\frac{\mu_d}{n^{2d+1}}\sum_{t=k+1}^{n}(\omega_{t-k}-\bar{\omega})(r_t-\bar{r})+\frac{\mu_d}{n^{2d+1}}\sum_{t=k+1}^{n}(r_{t-k}-\bar{r})(\omega_t-\bar{\omega})\,,$$

where $\bar{\omega}=n^{-1}\sum_{t=1}^{n}\omega_t$ and $\bar{r}=n^{-1}\sum_{t=1}^{n}r_t$. Notice that

$$\nabla r_t=\sum_{j_2=0}^{t-1}\sum_{j_3=0}^{t-j_2-1}\cdots\sum_{j_d=0}^{t-j_2-\cdots-j_{d-1}-1}U^{(0)}_{t-j_2-\cdots-j_d}\,,$$

we have $r_t\sim I(d)$ without the drift term μ_d. Applying the result of part (i), we know that $n^{-(2d+1)}\sum_{t=k+1}^{n}(r_{t-k}-\bar{r})(r_t-\bar{r})=o_{\mathrm{p}}(1)$, $n^{-(2d+1)}\sum_{t=k+1}^{n}(\omega_{t-k}-\bar{\omega})(r_t-\bar{r})=o_{\mathrm{p}}(1)$ and $n^{-(2d+1)}\sum_{t=k+1}^{n}(r_{t-k}-\bar{r})(\omega_t-\bar{\omega})=o_{\mathrm{p}}(1)$. Notice that $\omega_t=O(t^d)$. Thus $n^{-2d}\hat{\gamma}(k)\xrightarrow{\mathrm{P}}\phi_{d,k}\mu_d^2$, where $\phi_{d,k}=\lim_{n\to\infty}n^{-(2d+1)}\sum_{t=k+1}^{n}(\omega_{t-k}-\bar{\omega})(\omega_t-\bar{\omega})$. Similarly, for $k=0$, we also have $Y_t=w_t+r_t$, then $n^{-2d}\hat{\gamma}(0)=\mu_d^2n^{-2d-1}\sum_{t=1}^{n}(\omega_t-\bar{\omega})(\omega_t-\bar{\omega})+2\mu_dn^{-2d-1}\sum_{t=1}^{n}(\omega_t-\bar{\omega})(r_t-\bar{r})+n^{-2d-1}\sum_{t=1}^{n}(r_t-\bar{r})(r_t-\bar{r})$, we therefore conclude $n^{-2d}\hat{\gamma}(0)\xrightarrow{\mathrm{P}}\phi_{d,0}\mu_d^2$. We complete the proof of part (ii). □

4.2 Proof of Proposition 2

Since $X_t=\mu_1+\sum_{j=0}^{\infty}\psi_j\epsilon_{t-j}$ and $\epsilon_t\overset{\text{i.i.d.}}{\sim}(0,\sigma_\epsilon^2)$, then we have $\hat{\gamma}_x(0)\xrightarrow{\mathrm{P}}\gamma_x(0)=\sigma_\epsilon^2\sum_{j=0}^{\infty}\psi_j^2$ and $\hat{\gamma}_x(1)\xrightarrow{\mathrm{P}}\gamma_x(1)=\sigma_\epsilon^3\sum_{j=0}^{\infty}\psi_j\psi_{j+1}$.

We first consider the case $\mu_1=0$. Recall that we have shown in the proof of Proposition 1 that

$$\frac{1}{n}\{\hat{\gamma}(0)+\hat{\gamma}(1)\}=\frac{2}{n^2}\sum_{t=1}^{n}(Y_t-\bar{Y})^2+O_{\mathrm{p}}(n^{-1})\xrightarrow{\mathrm{D}}2a^2\sigma_\epsilon^2\int_0^1V_0^2(s)\,\mathrm{d}s\,.$$

It follows from the Slutsky's Theorem that

$$\frac{1}{n}\frac{\hat{\gamma}(0)+\hat{\gamma}(1)}{\hat{\gamma}_x(0)+\hat{\gamma}_x(1)}\xrightarrow{\mathrm{D}}\frac{2a^2\int_0^1V_0^2(t)\,\mathrm{d}t}{\sum_{j=0}^{\infty}\psi_j^2+\sum_{j=0}^{\infty}\psi_j\psi_{j+1}}\,.$$

We complete the proof of part (i).

We begin to consider the case with $\mu_1\neq0$. As we have shown in the proof of Proposition that $n^{-2}\hat{\gamma}(1)\xrightarrow{\mathrm{P}}\phi_{1,1}\mu_1^2$ and $n^{-2}\hat{\gamma}(0)\xrightarrow{\mathrm{P}}\phi_{1,0}\mu_1^2$, it then holds that

$$\frac{1}{n^2}\{\hat{\gamma}(0)+\hat{\gamma}(1)\}\xrightarrow{\mathrm{P}}(\phi_{1,0}+\phi_{1,1})\mu_1^2\,,$$

which implies that

$$\frac{1}{n^2}\frac{\hat{\gamma}(0)+\hat{\gamma}(1)}{\hat{\gamma}_x(0)+\hat{\gamma}_x(1)}\xrightarrow{\mathrm{P}}\frac{(\phi_{1,0}+\phi_{1,1})\mu_1^2}{\sigma_\epsilon^2(\sum_{j=0}^{\infty}\psi_j^2+\sum_{j=0}^{\infty}\psi_j\psi_{j+1})}\,.$$

Notice that $\phi_{1,0}=\lim_{n\to\infty}n^{-3}\sum_{t=1}^{n}(\omega_t-\bar{\omega})^2$ and $\phi_{1,1}=\lim_{n\to\infty}n^{-3}\sum_{t=2}^{n}(\omega_{t-1}-\bar{\omega})(\omega_t-\bar{\omega})$

with $\omega_t = \sum_{j_1=0}^{t-1} 1 = t$ and $\bar{\omega} = n^{-1}\sum_{t=1}^{n} \omega_t = (n+1)/2$. Then $\phi_{1,0} = \phi_{1,1} = 4^{-1}\int_0^1 s^2(2-s)^2\,\mathrm{d}s = 1/12$. We complete the proof of part (ii). $\square$

4.3 A useful proposition

Define

$$T_n^* = \sum_{k=0}^{K_0} 2\{\hat{\gamma}_2(k) - \hat{\gamma}_1(k)\}\gamma(k)\,. \tag{26}$$

Notice that

$$\begin{aligned}
\hat{\gamma}_2(k) - \hat{\gamma}_1(k) &= \frac{1}{N}\sum_{t=N+1}^{2N-k}\{(Y_t-\mu)(Y_{t+k}-\mu) - \gamma(k)\} \\
&\quad - \frac{1}{N}\sum_{t=1}^{N-k}\{(Y_t-\mu)(Y_{t+k}-\mu) - \gamma(k)\} \\
&\quad + (\bar{Y}-\mu)\cdot\frac{1}{N}\bigg(\sum_{t=N+1}^{N+k} + \sum_{t=2N-k+1}^{2N} - \sum_{t=1}^{k} - \sum_{t=N-k+1}^{N}\bigg)(Y_t-\mu) \\
&\quad + (\bar{Y}-\mu)\cdot\frac{2}{N}\bigg(\sum_{t=1}^{N} - \sum_{t=N+1}^{2N}\bigg)(Y_t-\mu)\,,
\end{aligned}$$

where $N = \lfloor n/2 \rfloor$. Recall that $y_{t,k} = 2\{(Y_t-\mu)(Y_{t+k}-\mu) - \gamma(k)\}\mathrm{sgn}(k+t-N-1/2)$ for each t and k. Then

$$\begin{aligned}
\hat{\gamma}_2(k) - \hat{\gamma}_1(k) &= \frac{1}{2N}\sum_{t=1}^{2N-k} y_{t,k} - \frac{1}{2N}\sum_{t=N-k+1}^{N} y_{t,k} \\
&\quad + (\bar{Y}-\mu)\cdot\underbrace{\frac{1}{N}\bigg(\sum_{t=N+1}^{N+k} + \sum_{t=2N-k+1}^{2N} - \sum_{t=1}^{k} - \sum_{t=N-k+1}^{N}\bigg)(Y_t-\mu)}_{R_{1,k}} \\
&\quad + (\bar{Y}-\mu)\cdot\underbrace{\frac{2}{N}\bigg(\sum_{t=1}^{N} - \sum_{t=N+1}^{2N}\bigg)(Y_t-\mu)}_{R_{2,k}} \\
&= \frac{1}{2N}\sum_{t=1}^{2N-k} y_{t,k} - \frac{1}{2N}\sum_{t=N-k+1}^{N} y_{t,k} + (\bar{Y}-\mu)R_k
\end{aligned}$$

with $R_k = R_{1,k} + R_{2,k}$. Notice that $Q_t = \sum_{k=0}^{K_0} \xi_{t,k}$ with $\xi_{t,k} = 2y_{t,k}\gamma(k)$ for each $t = 1, \ldots, 2N - K_0$. It holds that

$$T_n^* = \underbrace{\frac{1}{2N}\sum_{k=0}^{K_0}\sum_{t=1}^{2N-k} \xi_{t,k} - \frac{1}{2N}\sum_{k=0}^{K_0}\sum_{t=N-k+1}^{N} \xi_{t,k}}_{T_n^{**}} + 2(\bar{Y} - \mu)\sum_{k=0}^{K_0} R_k\gamma(k)\,, \tag{27}$$

where

$$\begin{aligned} T_n^{**} &= \frac{1}{2N}\sum_{t=1}^{2N-K_0}\sum_{k=0}^{K_0} \xi_{t,k} + \frac{1}{2N}\sum_{k=0}^{K_0}\bigg(\sum_{t=2N-K_0+1}^{2N-k} \xi_{t,k} - \sum_{t=N-k+1}^{N} \xi_{t,k}\bigg) \\ &= \underbrace{\frac{1}{2N}\sum_{t=1}^{2N-K_0} Q_t}_{\mathrm{I}} + \underbrace{\frac{1}{2N}\sum_{k=0}^{K_0}\bigg(\sum_{t=2N-K_0+1}^{2N-k} \xi_{t,k} - \sum_{t=N-k+1}^{N} \xi_{t,k}\bigg)}_{\mathrm{II}}\,. \end{aligned} \tag{28}$$

In the sequel, we use C to denote a generic positive finite constant that may be different in different uses. For two sequences of positive numbers $\{a_q\}$ and $\{b_q\}$, we write $a_q \lesssim b_q$ or $b_q \gtrsim a_q$ if there exists a positive uniform constant c such that $a_q/b_q \le c$ for any q. We write $a_q \asymp b_q$ if and only if $a_q \lesssim b_q$ and $b_q \lesssim a_q$ hold simultaneously. We first present the following result.

Proposition 3. *Let $K_* = 1 + K_0$. Under the null hypothesis H_0 with Conditions 1–3 being satisfied, if $K_* = o\{n^{\xi(\beta,s_1)}\}$ with $\xi(\beta, s_1)$ defined as (9), then it holds that*

$$d_n := \sup_{u\in\mathbb{R}}\bigg|\mathbb{P}(\sqrt{n}T_n^{**} \le u) - \Phi\bigg(\frac{2Nu}{B_{2N-K_0}\sqrt{n}}\bigg)\bigg| \to 0$$

as $n \to \infty$, where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution.

4.3.1 Two auxiliary lemmas

To construct Proposition 3, we need to analyze the two terms I and II in (28), respectively. Recall $\mathbb{E}(Q_t) = 0$ and $\mathbb{E}(\xi_{t,k}) = 0$ under H_0. Lemma 1 gives the Berry-Esseen bound for

$$\Delta_m = \sup_{u\in\mathbb{R}}\bigg|\mathbb{P}\bigg(\frac{1}{B_m}\sum_{t=1}^{m} Q_t \le u\bigg) - \Phi(u)\bigg| \tag{29}$$

with $B_m^2 = \mathbb{E}\{(\sum_{t=1}^m Q_t)^2\}$, and Lemma 2 gives an upper bound for the tail probability of the term II in (28). Notice that $\mathrm{II} = 0$ when $K_0 = 0$. Our Lemma 2 focuses on the non-trivial case with $K_0 \ge 1$.

Lemma 1. *Let $K_* = 1 + K_0$. Under the null hypothesis H_0 with Conditions 1–3 being satisfied,* *then it holds that*

$$\Delta_m \lesssim K_*^{2s_1-1}m^{-(s_1-2)/2} + K_*^{s_1}m^{-(\beta-1)(s_1-2)/(2\beta+2)}$$

for any $m \geq 1$*, provided that* $K_*^{s_1}m^{-(\beta-1)(s_1-2)/(2\beta+2)} = o(1)$*, where* $\beta = \beta_1(s_1-2)^2/\{2s_1(s_1-1)\}$.

For each t and $\tau \geq 0$, denote by $\tilde{\mathcal{F}}_{-\infty}^t$ and $\tilde{\mathcal{F}}_{t+\tau}^\infty$ the σ-fields generated by $\{Q_u\}_{u\leq t}$ and $\{Q_u\}_{u\geq t+\tau}$, respectively. Recall $\mathcal{F}_{-\infty}^t$ and $\mathcal{F}_{t+\tau}^\infty$ be the σ-fields generated by $\{Y_u\}_{u\leq t}$ and $\{Y_u\}_{u\geq t+\tau}$, respectively. It follows from the definition of the new process $\{Q_t\}$ that $\tilde{\mathcal{F}}_{-\infty}^t \subset \mathcal{F}_{-\infty}^{t+K_0}$ and $\tilde{\mathcal{F}}_{t+\tau}^\infty \subset \mathcal{F}_{t+\tau}^\infty$. For each $\tau \geq 0$, it holds that

$$\begin{aligned} \alpha_Q(\tau) := &\sup_t \sup_{A\in\tilde{\mathcal{F}}_{-\infty}^t, B\in\tilde{\mathcal{F}}_{t+\tau}^\infty} |\mathbb{P}(AB) - \mathbb{P}(A)\mathbb{P}(B)| \\ \leq &\sup_t \sup_{A\in\mathcal{F}_{-\infty}^{t+K_0}, B\in\mathcal{F}_{t+\tau}^\infty} |\mathbb{P}(AB) - \mathbb{P}(A)\mathbb{P}(B)| \\ = &\,\alpha(|\tau - K_0|_+)\,, \end{aligned} \tag{30}$$

where $|\cdot|_+$ denotes the positive part of $\cdot$. Since $\alpha(\tau) \to 0$ as $\tau \to \infty$, then $\alpha_Q(\tau) \to 0$ as $\tau \to \infty$, which indicates that the new process $\{Q_t\}$ is also α-mixing with α-mixing coefficients $\alpha_Q(\cdot)$. By Condition 1 and Cauchy-Schwarz inequality, we have $\mathbb{E}(|(Y_t-\mu)(Y_{t+k}-\mu)|^{s_1}) \leq c_1$. When K_0 is finite, Theorem 2 of Sunklodas (1984) shows that $\Delta_m \lesssim m^{-(\beta-1)(s_1-2)/(2\beta+2)}$. Since the tuning parameter K_0 may diverge with n, Theorem 2 of Sunklodas (1984) cannot be applied directly. Lemma 1 here extends Theorem 2 of Sunklodas (1984) to the more general triangular array case. The proof of Lemma 1 is given in the supplementary material.

Lemma 2. *Let* $K_* = 1 + K_0$*. Under the null hypothesis* H_0 *with Conditions* 1 *and* 2 *being satisfied, then*

$$\begin{aligned} \mathbb{P}\bigg\{\bigg|\frac{1}{2N}\sum_{k=0}^{K_0}\bigg(\sum_{t=2N-K_0+1}^{2N-k}\xi_{t,k} - \sum_{t=N-k+1}^{N}\xi_{t,k}\bigg)\bigg| > \varepsilon\bigg\} \\ \lesssim K_*\exp\{-CK_*^{-4}(\varepsilon n)^2\} + K_*^{2s_1}n(\varepsilon n)^{-s_1} \\ + K_*^{(\beta_1+1)s_1/(\beta_1+s_1)+1}n(\varepsilon n)^{-(\beta_1+1)s_1/(\beta_1+s_1)} \end{aligned}$$

for any $\varepsilon > 0$ *such that* $\varepsilon n/K_* \to \infty$.

Proof. By the triangular inequality, it holds that

$$\begin{aligned} &\mathbb{P}\bigg\{\bigg|\frac{1}{2N}\sum_{k=0}^{K_0}\bigg(\sum_{t=2N-K_0+1}^{2N-k}\xi_{t,k} - \sum_{t=N-k+1}^{N}\xi_{t,k}\bigg)\bigg| > \varepsilon\bigg\} \\ &\leq \sum_{k=0}^{K_0}\mathbb{P}\bigg(\bigg|\sum_{t=2N-K_0+1}^{2N-k}\xi_{t,k}\bigg| > \frac{\varepsilon N}{K_*}\bigg) + \sum_{k=0}^{K_0}\mathbb{P}\bigg(\bigg|\sum_{t=N-k+1}^{N}\xi_{t,k}\bigg| > \frac{\varepsilon N}{K_*}\bigg) \end{aligned} \tag{31}$$

for any $\varepsilon > 0$. We will use the Fuk-Nagaev inequality to bound the terms on the right-hand side of (31). For each fixed $k = 0, \ldots, K_0$, similar to (30), we know $\{\xi_{t,k}\}$ is also an α-mixing process with α-mixing coefficients $\alpha_k(\tau) \le \alpha(|\tau - k|_+)$ for all $\tau \ge 0$. It follows from Condition 1 and Lemma 2 of Chang, Tang and Wu (2013) that $\mathbb{P}(|\xi_{t,k}| > x) \le Cx^{-s_1}$ for any $x > 0$. By Theorem 6.2 of Rio (2017), we have

$$\mathbb{P}\bigg(\bigg|\sum_{t=2N-K_0+1}^{2N-k} \xi_{t,k}\bigg| > \frac{\varepsilon N}{K_*}\bigg) \lesssim \bigg\{1 + \frac{(\varepsilon n)^2}{CrK_*^4}\bigg\}^{-r/2} + \frac{K_*^{2s_1-1} r^{s_1-1} n}{(\varepsilon n)^{s_1}} + \frac{K_*^{(\beta_1+1)s_1/(\beta_1+s_1)} r^{\beta_1(s_1-1)/(\beta_1+s_1)} n}{(\varepsilon n)^{(\beta_1+1)s_1/(\beta_1+s_1)}} \tag{32}$$

and

$$\mathbb{P}\bigg(\bigg|\sum_{t=N-k+1}^{N} \xi_{t,k}\bigg| > \frac{\varepsilon N}{K_*}\bigg) \lesssim \bigg\{1 + \frac{(\varepsilon n)^2}{CrK_*^4}\bigg\}^{-r/2} + \frac{K_*^{2s_1-1} r^{s_1-1} n}{(\varepsilon n)^{s_1}} + \frac{K_*^{(\beta_1+1)s_1/(\beta_1+s_1)} r^{\beta_1(s_1-1)/(\beta_1+s_1)} n}{(\varepsilon n)^{(\beta_1+1)s_1/(\beta_1+s_1)}}$$

for any $r \ge 1$ and $\varepsilon > 0$ satisfying $\varepsilon n/(rK_*) \ge c_*$, where c_* is a uniform positive constant. Therefore, (31) leads to

$$\mathbb{P}\bigg\{\bigg|\frac{1}{2N}\sum_{k=0}^{K_0}\bigg(\sum_{t=2N-K_0+1}^{2N-k} \xi_{t,k} - \sum_{t=N-k+1}^{N} \xi_{t,k}\bigg)\bigg| > \varepsilon\bigg\} \lesssim K_*\bigg\{1 + \frac{(\varepsilon n)^2}{CrK_*^4}\bigg\}^{-r/2} + \frac{K_*^{2s_1} r^{s_1-1} n}{(\varepsilon n)^{s_1}} + \frac{K_*^{(\beta_1+1)s_1/(\beta_1+s_1)+1} r^{\beta_1(s_1-1)/(\beta_1+s_1)} n}{(\varepsilon n)^{(\beta_1+1)s_1/(\beta_1+s_1)}}\,.$$

Notice that $(1 + x^{-1})^{-x} \to e^{-1}$ as $x \to \infty$. With a sufficiently large but fixed r, we have

$$\mathbb{P}\bigg\{\bigg|\frac{1}{2N}\sum_{k=0}^{K_0}\bigg(\sum_{t=2N-K_0+1}^{2N-k} \xi_{t,k} - \sum_{t=N-k+1}^{N} \xi_{t,k}\bigg)\bigg| > \varepsilon\bigg\} \lesssim K_* \exp\{-CK_*^{-4}(\varepsilon n)^2\} + K_*^{2s_1} n(\varepsilon n)^{-s_1} + K_*^{(\beta_1+1)s_1/(\beta_1+s_1)+1} n(\varepsilon n)^{-(\beta_1+1)s_1/(\beta_1+s_1)}$$

for any $\varepsilon > 0$ such that $\varepsilon n/K_* \to \infty$. We complete the proof of Lemma 2. $\square$

4.3.2 Proof of Proposition 3

Now we begin to prove Proposition 3. By (28), Lemmas 1 and 2 imply that

$$\mathbb{P}(\sqrt{n}T_n^{**} > u) \le \mathbb{P}\bigg(\frac{\sqrt{n}}{2N}\sum_{t=1}^{2N-K_0} Q_t > u - \varepsilon\bigg)$$

$$
\begin{aligned}
&+\mathbb{P}\bigg\{\frac{\sqrt{n}}{2N}\sum_{k=0}^{K_0}\bigg(\sum_{t=2N-K_0+1}^{2N-k}\xi_{t,k}-\sum_{t=N-k+1}^{N}\xi_{t,k}\bigg)>\varepsilon\bigg\}\\
\le{}& 1-\Phi\bigg\{\frac{2N(u-\varepsilon)}{B_{2N-K_0}\sqrt{n}}\bigg\}+\Delta_{2N-K_0} \qquad (33)\\
&+CK_*\exp(-CK_*^{-4}\varepsilon^2 n)+CK_*^{2s_1}n(\varepsilon^2 n)^{-s_1/2}\\
&+CK_*^{(\beta_1+1)s_1/(\beta_1+s_1)+1}n(\varepsilon^2 n)^{-(\beta_1+1)s_1/(2\beta_1+2s_1)}\\
\le{}& 1-\Phi\bigg(\frac{2Nu}{B_{2N-K_0}\sqrt{n}}\bigg)+C\varepsilon+\Delta_{2N-K_0}\\
&+CK_*\exp(-CK_*^{-4}\varepsilon^2 n)+CK_*^{2s_1}n(\varepsilon^2 n)^{-s_1/2}\\
&+CK_*^{(\beta_1+1)s_1/(\beta_1+s_1)+1}n(\varepsilon^2 n)^{-(\beta_1+1)s_1/(2\beta_1+2s_1)}
\end{aligned}
$$

for any $\varepsilon > 0$ such that $\varepsilon\sqrt{n}/K_* \to \infty$. On the other hand, analogous to (33), we have

$$
\begin{aligned}
\mathbb{P}(\sqrt{n}T_n^{**}>u)\ge{}& 1-\Phi\bigg(\frac{2Nu}{B_{2N-K_0}\sqrt{n}}\bigg)-C\varepsilon-\Delta_{2N-K_0}\\
&-CK_*\exp(-CK_*^{-4}\varepsilon^2 n)-CK_*^{2s_1}n(\varepsilon^2 n)^{-s_1/2}\\
&-CK_*^{(\beta_1+1)s_1/(\beta_1+s_1)+1}n(\varepsilon^2 n)^{-(\beta_1+1)s_1/(2\beta_1+2s_1)}
\end{aligned}
$$

for any $\varepsilon > 0$ such that $\varepsilon\sqrt{n}/K_* \to \infty$. Therefore,

$$
\begin{aligned}
d_n ={}& \sup_{u\in\mathbb{R}}\bigg|\mathbb{P}(\sqrt{n}T_n^{**}\le u)-\Phi\bigg(\frac{2Nu}{B_{2N-K_0}\sqrt{n}}\bigg)\bigg|\\
\lesssim{}& \varepsilon+\Delta_{2N-K_0}+K_*\exp(-CK_*^{-4}\varepsilon^2 n)+K_*^{2s_1}n(\varepsilon^2 n)^{-s_1/2}\\
&+K_*^{(\beta_1+1)s_1/(\beta_1+s_1)+1}n(\varepsilon^2 n)^{-(\beta_1+1)s_1/(2\beta_1+2s_1)}
\end{aligned}
$$

for any $\varepsilon > 0$ such that $\varepsilon\sqrt{n}/K_* \to \infty$. Since $K_* = o\{n^{\xi(\beta,s_1)}\}$ with $\xi(\beta, s_1)$ defined as (9), there exists suitable selection of $\varepsilon = o(1)$ such that $d_n = o(1)$. We complete the proof of Proposition 3. □

4.4 Proof of Theorem 1

4.4.1 An auxiliary lemma

To prove Theorem 1, we need the following lemma.

Lemma 3. *Let $K_* = 1 + K_0$. Under the null hypothesis H_0 with Conditions 1 and 2 being satisfied, then*

$$
\begin{aligned}
\max_{0\le k\le K_0}\mathbb{P}\big\{|\hat\gamma_1(k)-\gamma(k)|>\varepsilon\big\}\lesssim{}& \exp(-CK_*^{-1}n\varepsilon^2)+K_*^{s_1-1}n(\varepsilon n)^{-s_1}\\
&+n(\varepsilon n)^{-(\beta_1+1)s_1/(\beta_1+s_1)}
\end{aligned}
$$

and

$$\max_{0\le k\le K_0}\mathbb{P}\big\{|\hat\gamma_2(k)-\gamma(k)|>\varepsilon\big\}\lesssim\ \exp(-CK_*^{-1}n\varepsilon^2)+K_*^{s_1-1}n(\varepsilon n)^{-s_1}$$
$$+\,n(\varepsilon n)^{-(\beta_1+1)s_1/(\beta_1+s_1)}$$

for any $\varepsilon=o(K_*)$ such that $\varepsilon n\to\infty$ and $\varepsilon=o(n^{\beta_1/s_1})$.

Proof. Recall that

$$\hat\gamma_1(k)-\gamma(k)=\frac{1}{N}\sum_{t=1}^{N-k}(Y_{t+k}-\mu)(Y_t-\mu)-\gamma(k)$$
$$-(\bar Y-\mu)\bigg\{\frac{1}{N}\sum_{t=1}^{N-k}(Y_t-\mu)+\frac{1}{N}\sum_{t=1}^{N-k}(Y_{t+k}-\mu)\bigg\}+\frac{N-k}{N}(\bar Y-\mu)^2\,.$$

Same as (32), it holds that

$$\max_{0\le k\le K_0}\mathbb{P}\bigg\{\bigg|\frac{1}{N}\sum_{t=1}^{N-k}(Y_{t+k}-\mu)(Y_t-\mu)-\gamma(k)\bigg|>\varepsilon\bigg\}$$
$$\lesssim\bigg\{1+\frac{(\varepsilon n)^2}{CrK_*n}\bigg\}^{-r/2}+\frac{K_*^{s_1-1}r^{s_1-1}n}{(\varepsilon n)^{s_1}}+\frac{nr^{\beta_1(s_1-1)/(\beta_1+s_1)}}{(\varepsilon n)^{(\beta_1+1)s_1/(\beta_1+s_1)}}$$

for any $r\ge 1$ and $\varepsilon>0$ satisfying $\varepsilon n/r\ge c_*$, where c_* is a uniform positive constant. With sufficiently large r, we have

$$\max_{0\le k\le K_0}\mathbb{P}\bigg\{\bigg|\frac{1}{N}\sum_{t=1}^{N-k}(Y_{t+k}-\mu)(Y_t-\mu)-\gamma(k)\bigg|>\varepsilon\bigg\}$$
$$\lesssim\exp(-CK_*^{-1}n\varepsilon^2)+K_*^{s_1-1}n(\varepsilon n)^{-s_1}+n(\varepsilon n)^{-(\beta_1+1)s_1/(\beta_1+s_1)}\,.$$

Applying Theorem 6.2 of Rio (2017) again, we have

$$\mathbb{P}(|\bar Y-\mu|\ge\varepsilon)\lesssim\bigg(1+\frac{n\varepsilon^2}{Cr}\bigg)^{-r/2}+\frac{nr^{\beta_1(2s_1-1)/(\beta_1+2s_1)}}{(\varepsilon n)^{2s_1(\beta_1+1)/(\beta_1+2s_1)}}$$

for any $r\ge 1$ and $\varepsilon>0$. With sufficiently large r, we have

$$\mathbb{P}(|\bar Y-\mu|\ge\varepsilon)\lesssim\exp(-Cn\varepsilon^2)+n(\varepsilon n)^{-2s_1(\beta_1+1)/(\beta_1+2s_1)}\tag{34}$$

for any $\varepsilon>0$. Analogously, we have

$$\mathbb{P}\bigg\{\bigg|\frac{1}{N}\sum_{t=1}^{N-k}(Y_t-\mu)\bigg|\ge\varepsilon\bigg\}\lesssim\exp(-Cn\varepsilon^2)+n(\varepsilon n)^{-2s_1(\beta_1+1)/(\beta_1+2s_1)}$$

and

$$\mathbb{P}\bigg\{\bigg|\frac{1}{N}\sum_{t=1}^{N-k}(Y_{t+k}-\mu)\bigg|\geq\varepsilon\bigg\}\lesssim\exp(-Cn\varepsilon^2)+n(\varepsilon n)^{-2s_1(\beta_1+1)/(\beta_1+2s_1)}$$

for any $\varepsilon>0$. Therefore, it holds that

$$\max_{0\leq k\leq K_0}\mathbb{P}\big\{|\hat{\gamma}_1(k)-\gamma(k)|>\varepsilon\big\}\lesssim\exp(-CK_*^{-1}n\varepsilon^2)+K_*^{s_1-1}n(\varepsilon n)^{-s_1}+n(\varepsilon n)^{-(\beta_1+1)s_1/(\beta_1+s_1)}$$

for any $\varepsilon=o(K_*)$ such that $\varepsilon n\to\infty$ and $\varepsilon=o(n^{\beta_1/s_1})$. Similarly, we can prove the other result. We complete the proof of Lemma 3. □

4.4.2 Proof of Theorem 1

Now, we begin to show Theorem 1. Recall $T_n=\sum_{k=0}^{K_0}|\hat{\gamma}_2(k)|^2$. For each $u\in\mathbb{R}$, we have

$$\mathbb{P}(\sqrt{n}T_n>u)=\mathbb{P}\bigg[\sqrt{n}\sum_{k=0}^{K_0}\{|\hat{\gamma}_2(k)|^2-|\hat{\gamma}_1(k)|^2\}>\tilde{u}\bigg]$$

with $\tilde{u}=u-\sqrt{n}\sum_{k=0}^{K_0}|\hat{\gamma}_1(k)|^2$. Notice that

$$\sum_{k=0}^{K_0}\{|\hat{\gamma}_2(k)|^2-|\hat{\gamma}_1(k)|^2\}=T_n^*+\sum_{k=0}^{K_0}\{\hat{\gamma}_2(k)-\gamma(k)\}^2-\sum_{k=0}^{K_0}\{\hat{\gamma}_1(k)-\gamma(k)\}^2\,,$$

where T_n^* is defined as (26). It follows from (27) that

$$\begin{aligned}\mathbb{P}(\sqrt{n}T_n>u)&\leq\mathbb{P}(\sqrt{n}T_n^*>\tilde{u}-\delta)+\mathbb{P}\bigg[\sum_{k=0}^{K_0}\{\hat{\gamma}_2(k)-\gamma(k)\}^2>\delta\bigg]\\&\leq\mathbb{P}(\sqrt{n}T_n^{**}>\tilde{u}-2\delta)+\mathbb{P}\bigg\{\bigg|(\bar{Y}-\mu)\sum_{k=0}^{K_0}R_k\gamma(k)\bigg|>\frac{\delta}{2}\bigg\}\\&\quad+\mathbb{P}\bigg[\sum_{k=0}^{K_0}\{\hat{\gamma}_2(k)-\gamma(k)\}^2>\delta\bigg]\end{aligned}\tag{35}$$

for any $\delta>0$. Write $\theta=\max_{0\leq k\leq K_0}|\gamma(k)|$. Same as (34), we have

$$\begin{aligned}\mathbb{P}\bigg\{\bigg|\sum_{k=0}^{K_0}R_k\gamma(k)\bigg|>K_*\theta\epsilon\bigg\}&\leq\sum_{k=0}^{K_0}\mathbb{P}(|R_k|>\epsilon)\\&\leq K_*\exp(-Cn\epsilon^2)+K_*n(\epsilon n)^{-2s_1(\beta_1+1)/(\beta_1+2s_1)}\end{aligned}$$

for any $\epsilon > 0$. Let $\delta = K_*\theta\epsilon$ with

$$\max[n^{-1/2}\log^{1/2} K_*, K_*^{(\beta_1+2s_1)/\{2s_1(\beta_1+1)\}} n^{-\beta_1(2s_1-1)/\{2s_1(\beta_1+1)\}}] = o(\epsilon)\,.$$

Then we have

$$\mathbb{P}\bigg\{\bigg|(\bar{Y}-\mu)\sum_{k=0}^{K_0} R_k\gamma(k)\bigg| > \frac{\delta}{2}\bigg\} = o(1)\,.$$

Since $K_* = o\{n^{\xi(\beta,s_1)}\}$ with $\xi(\beta, s_1)$ defined as (9), there exists $\epsilon = o(1)$ satisfying $\delta = o(1)$ and $\max[n^{-1/2}\log^{1/2} K_*, K_*^{(\beta_1+2s_1)/\{2s_1(\beta_1+1)\}} n^{-\beta_1(2s_1-1)/\{2s_1(\beta_1+1)\}}] = o(\epsilon)$. Hence,

$$\mathbb{P}(\sqrt{n}T_n > u) \le \mathbb{P}(\sqrt{n}T_n^{**} > \tilde{u} - 2\delta) + \mathbb{P}\bigg[\sum_{k=0}^{K_0}\{\hat{\gamma}_2(k) - \gamma(k)\}^2 > \delta\bigg] + o(1) \tag{36}$$

for some $\delta = o(1)$. On the other hand, by Lemma 3, we also have

$$\mathbb{P}\bigg[\sum_{k=0}^{K_0}\{\hat{\gamma}_2(k) - \gamma(k)\}^2 > \delta\bigg] = o(1)$$

with such suitable selection of ε. From (36) and Proposition 3, we have

$$\begin{aligned}
\mathbb{P}(\sqrt{n}T_n > u) &\le \mathbb{P}(\sqrt{n}T_n^{**} > \tilde{u} - 2\delta) + o(1) \\
&\le 1 - \Phi\bigg\{\frac{2N(\tilde{u} - 2\delta)}{B_{2N-K_0}\sqrt{n}}\bigg\} + d_n + o(1) \\
&\le 1 - \Phi\bigg(\frac{2N\tilde{u}}{B_{2N-K_0}\sqrt{n}}\bigg) + C\delta + d_n + o(1)
\end{aligned}$$

for d_n defined in Proposition 3. Letting $\delta \to 0$, we have

$$\mathbb{P}(\sqrt{n}T_n > u) \le 1 - \Phi\bigg(\frac{2N\tilde{u}}{B_{2N-K_0}\sqrt{n}}\bigg) + d_n + o(1)\,.$$

Analogously, we can show

$$\mathbb{P}(\sqrt{n}T_n > u) \ge 1 - \Phi\bigg(\frac{2N\tilde{u}}{B_{2N-K_0}\sqrt{n}}\bigg) - d_n - o(1)\,.$$

We complete the proof of Theorem 1. □

4.5 Proof of Theorem 4

4.5.1 An auxiliary lemma

Lemma 4. *Under the null hypothesis H_0 with Conditions 4 and 5 being satisfied, then*

$$\bigg|\sum_{j=0}^{m-1}\mathcal{K}\bigg(\frac{j}{b_m}\bigg)\bigg[\frac{1}{m}\sum_{t=j+1}^{m}\{Q_tQ_{t-j}-\mathbb{E}(Q_tQ_{t-j})\}\bigg]\bigg| = O_{\rm p}\bigg(\frac{b_mK_*^{5/2}}{m^{1/2}}\bigg)\,.$$

Proof. Let $\zeta_{t,j}=Q_{t+j}Q_t-\mathbb{E}(Q_{t+j}Q_t)$. It follows from the Markov inequality that

$$\sup_{0\le j\le m-1}\sup_{1\le t\le m-j}\mathbb{P}(|\zeta_{t,j}|>x)\le\frac{\mathbb{E}(|\zeta_{t,j}|^{s_2/2})}{x^{s_2/2}}$$

for any $x>0$. By the Jensen's inequality and the Cauchy-Schwarz inequality, Condition 1 leads to $\mathbb{E}(|\zeta_{t,j}|^{s_2/2})\lesssim\mathbb{E}(|Q_{t+j}Q_t|^{s_2/2})\lesssim\{\mathbb{E}(|Q_{t+j}|^{s_2})\}^{1/2}\{\mathbb{E}(|Q_t|^{s_2})\}^{1/2}\lesssim K_*^{s_2}$. By the triangle inequality and the Davydov's inequality, we have

$$\begin{aligned}\mathrm{Var}\bigg(\frac{1}{m}\sum_{t=1}^{m-j}\zeta_{t,j}\bigg)&\le\frac{1}{m^2}\sum_{t_1=1}^{m-j}\sum_{t_2=1}^{m-j}\big|\mathrm{Cov}(\zeta_{t_1,j},\zeta_{t_2,j})\big|\\&\lesssim\frac{1}{m^2}\sum_{t=1}^{m-j}\mathbb{E}\big(|\zeta_{t,j}|^2\big)+\frac{1}{m^2}\sum_{t_1<t_2}\big|\mathrm{Cov}(\zeta_{t_1,j},\zeta_{t_2,j})\big|\\&\lesssim\frac{K_*^4}{m}+\underbrace{\frac{1}{m^2}\sum_{\substack{1\le t_2-t_1\le j\\1\le t_1,t_2\le m-j}}\big|\mathrm{Cov}(\zeta_{t_1,j},\zeta_{t_2,j})\big|}_{\mathrm{I}(j)}\\&\qquad+\underbrace{\frac{1}{m^2}\sum_{\substack{j+1\le t_2-t_1\le m-j-1\\1\le t_1,t_2\le m-j}}\big|\mathrm{Cov}(\zeta_{t_1,j},\zeta_{t_2,j})\big|}_{\mathrm{II}(j)}\,.\end{aligned}\tag{37}$$

Here we adopt the convention $\mathrm{II}(j)=0$ if $j+1>m-j-1$. If $1\le t_2-t_1\le j$, by the triangle inequality, it holds that

$$\begin{aligned}\big|\mathrm{Cov}(\zeta_{t_1,j},\zeta_{t_2,j})\big|&\le\big|\mathbb{E}(Q_{t_1}Q_{t_1+j}Q_{t_2}Q_{t_2+j})-\mathbb{E}(Q_{t_1}Q_{t_2})\mathbb{E}(Q_{t_1+j}Q_{t_2+j})\big|\\&\quad+\big|\mathbb{E}(Q_{t_1}Q_{t_2})\mathbb{E}(Q_{t_1+j}Q_{t_2+j})-\mathbb{E}(Q_{t_1}Q_{t_1+j})\mathbb{E}(Q_{t_2}Q_{t_2+j})\big|\\&\le\big|\mathbb{E}(Q_{t_1}Q_{t_1+j}Q_{t_2}Q_{t_2+j})-\mathbb{E}(Q_{t_1}Q_{t_2})\mathbb{E}(Q_{t_1+j}Q_{t_2+j})\big|\\&\quad+\big|\mathbb{E}(Q_{t_1}Q_{t_2})\mathbb{E}(Q_{t_1+j}Q_{t_2+j})\big|+\big|\mathbb{E}(Q_{t_1}Q_{t_1+j})\mathbb{E}(Q_{t_2}Q_{t_2+j})\big|\,.\end{aligned}$$

It follows from the Davydov's inequality that $|\mathbb{E}(Q_{k_1}Q_{k_2})|\lesssim K_*^2\{\alpha_Q(|k_2-k_1|)\}^{1-2/s_2}$ and

$|\mathbb{E}(Q_{t_1}Q_{t_1+j}Q_{t_2}Q_{t_2+j})-\mathbb{E}(Q_{t_1}Q_{t_2})\mathbb{E}(Q_{t_1+j}Q_{t_2+j})|\lesssim K_*^4\{\alpha_Q(t_1+j-t_2)\}^{1-4/s_2}$, Thus,

$$\begin{aligned}\mathrm{I}(j)\lesssim{}&\frac{K_*^4}{m^2}\sum_{\tau=1}^{j}|m-j-\tau|_+\{\alpha_Q(j-\tau)\}^{1-4/s_2}\\&+\frac{K_*^4}{m^2}\sum_{\tau=1}^{j}|m-j-\tau|_+\{\alpha_Q(\tau)\}^{2(s_2-2)/s_2}\\&+\frac{K_*^4}{m^2}(m-j)\min(j,m-j)\{\alpha_Q(j)\}^{2(s_2-2)/s_2}\\\lesssim{}&\frac{K_*^4}{m}\sum_{\tau=0}^{j}\{\alpha_Q(\tau)\}^{1-4/s_2}\,,\end{aligned}$$

where the last inequality is based on the facts $\alpha_Q(j)\le\alpha_Q(\tau)$ for any $\tau\le j$ and $(m-j)\min(j,m-j)\le\sum_{\tau=1}^{j}m$.

If $j+1\le t_2-t_1\le m-j-1$, by the Davydov's inequality, we have $|\mathrm{Cov}(\zeta_{t_1,j},\zeta_{t_2,j})|\lesssim K_*^4\{\alpha_Q(t_2-t_1-j)\}^{1-4/s_2}$, which implies that

$$\mathrm{II}(j)\lesssim\frac{K_*^4}{m^2}\sum_{\tau=1}^{m-2j-1}|m-2j-\tau|_+\{\alpha_Q(\tau)\}^{1-4/s_2}\lesssim\frac{K_*^4}{m}\sum_{\tau=1}^{m-2j-1}\{\alpha_Q(\tau)\}^{1-4/s_2}\,.$$

Recall that $\alpha_Q(\tau)\le\alpha(|\tau-K_0|_+)$ and $\sum_{\tau=1}^{\infty}\{\alpha(\tau)\}^{1-4/s_2}<\infty$. Then $\mathrm{I}(j)+\mathrm{II}(j)\lesssim K_*^5m^{-1}$. Together with (37), we have $\mathrm{Var}(m^{-1}\sum_{t=1}^{m-j}\zeta_{t,j})\lesssim K_*^5m^{-1}$. Notice that

$$\begin{aligned}S(m):={}&\sum_{j=0}^{m-1}\mathcal{K}\bigg(\frac{j}{b_m}\bigg)\bigg[\frac{1}{m}\sum_{t=j+1}^{m}\{Q_tQ_{t-j}-\mathbb{E}(Q_tQ_{t-j})\}\bigg]\\={}&\sum_{j=0}^{m-1}\mathcal{K}\bigg(\frac{j}{b_m}\bigg)\bigg(\frac{1}{m}\sum_{t=1}^{m-j}\zeta_{t,j}\bigg)=:\sum_{j=0}^{m-1}\mathcal{K}\bigg(\frac{j}{b_m}\bigg)\eta_j\,.\end{aligned}$$

It follows from the Jensen's inequality that

$$\mathbb{E}\{|S(m)|^2\}\le\bigg\{\sum_{j=0}^{m-1}\bigg|\mathcal{K}\bigg(\frac{j}{b_m}\bigg)\bigg|\bigg\}\bigg\{\sum_{j=0}^{m-1}\bigg|\mathcal{K}\bigg(\frac{j}{b_m}\bigg)\bigg|\mathrm{Var}(\eta_j)\bigg\}\lesssim\frac{b_m^2K_*^5}{m}\,,$$

where the last step is based on the fact $\sum_{j=0}^{m-1}|\mathcal{K}(j/b_m)|\asymp b_m$. By the Markov inequality, we have

$$\bigg|\sum_{j=0}^{m-1}\mathcal{K}\bigg(\frac{j}{b_m}\bigg)\bigg[\frac{1}{m}\sum_{t=j+1}^{m}\{Q_tQ_{t-j}-\mathbb{E}(Q_tQ_{t-j})\}\bigg]\bigg|=O_{\mathrm{p}}\bigg(\frac{b_mK_*^{5/2}}{m^{1/2}}\bigg)\,.$$

We complete the proof of Lemma 4. □

4.5.2 Proof of Theorem 4

Let $m = 2N - K_0$. Define

$$\hat{V}_m = \sum_{j=-m+1}^{m-1} \mathcal{K}\bigg(\frac{j}{b_m}\bigg) G_j$$

with $G_j = m^{-1}\sum_{t=j+1}^m \mathbb{E}(Q_tQ_{t-j})$ if $j \geq 0$ and $G_j = m^{-1}\sum_{t=-j+1}^m \mathbb{E}(Q_{t+j}Q_t)$ otherwise. Our proof includes two steps: (i) to show $\tilde{V}_m - \hat{V}_m = o_{\rm p}(1)$, and (ii) to show $\hat{V}_m - V_m = o(1)$. It follows from these two results that $\tilde{V}_m - V_m = o_{\rm p}(1)$. Due to $B_\ell^2 = \ell V_\ell$ and $B_\ell^2 \geq c_0\ell$ for all $\ell \geq 1$ with some uniform positive constant c_0, we know V_m is uniformly bounded away from zero. Thus $\tilde{V}_m - V_m = o_{\rm p}(1)$ implies $\tilde{V}_m/V_m \xrightarrow{\rm P} 1$.

Notice that

$$\tilde{V}_m - \hat{V}_m = \underbrace{\sum_{j=0}^{m-1} \mathcal{K}\bigg(\frac{j}{b_m}\bigg)(\tilde{G}_j - G_j)}_{\rm I} + \underbrace{\sum_{j=-m+1}^{-1} \mathcal{K}\bigg(\frac{j}{b_m}\bigg)(\tilde{G}_j - G_j)}_{\rm II} \,.$$

To show $\tilde{V}_m - \hat{V}_m = o_{\rm p}(1)$, it suffices to show ${\rm I} = o_{\rm p}(1)$ and ${\rm II} = o_{\rm p}(1)$, respectively. Recall that $\tilde{G}_j = m^{-1}\sum_{t=j+1}^m \tilde{Q}_t\tilde{Q}_{t-j}$ if $j \geq 0$ and $\tilde{G}_j = m^{-1}\sum_{t=-j+1}^m \tilde{Q}_{t+j}\tilde{Q}_t$ otherwise. For any $j \geq 0$, it holds that

$$\begin{aligned}\tilde{G}_j = {} & \frac{1}{m}\sum_{t=j+1}^m Q_tQ_{t-j} + \frac{1}{m}\sum_{t=j+1}^m (\tilde{Q}_t - Q_t)Q_{t-j} \\ & + \frac{1}{m}\sum_{t=j+1}^m Q_t(\tilde{Q}_{t-j} - Q_{t-j}) + \frac{1}{m}\sum_{t=j+1}^m (\tilde{Q}_t - Q_t)(\tilde{Q}_{t-j} - Q_{t-j})\,,\end{aligned}$$

which implies that

$$\begin{aligned}{\rm I} = {} & \sum_{j=0}^{m-1} \mathcal{K}\bigg(\frac{j}{b_n}\bigg)\bigg[\frac{1}{m}\sum_{t=j+1}^m \{Q_tQ_{t-j} - \mathbb{E}(Q_tQ_{t-j})\}\bigg] \\ & + \sum_{j=0}^{m-1} \mathcal{K}\bigg(\frac{j}{b_m}\bigg)\bigg\{\frac{1}{m}\sum_{t=j+1}^m (\tilde{Q}_t - Q_t)Q_{t-j}\bigg\} \\ & + \sum_{j=0}^{m-1} \mathcal{K}\bigg(\frac{j}{b_m}\bigg)\bigg\{\frac{1}{m}\sum_{t=j+1}^m Q_t(\tilde{Q}_{t-j} - Q_{t-j})\bigg\} \\ & + \sum_{j=0}^{m-1} \mathcal{K}\bigg(\frac{j}{b_m}\bigg)\bigg\{\frac{1}{m}\sum_{t=j+1}^m (\tilde{Q}_t - Q_t)(\tilde{Q}_{t-j} - Q_{t-j})\bigg\}\,.\end{aligned} \tag{38}$$

Notice that

$$
\begin{aligned}
|\tilde{Q}_t - Q_t| \lesssim & \sum_{k=0}^{K_0} |(Y_t-\mu)(Y_{t+k}-\mu)||\hat{\gamma}(k)-\gamma(k)| \\
&+ |\bar{Y}-\mu|\sum_{k=0}^{K_0}|Y_{t+k}-\mu||\hat{\gamma}(k)| + |\bar{Y}-\mu||Y_t-\mu|\sum_{k=0}^{K_0}|\hat{\gamma}(k)| \\
&+ |\bar{Y}-\mu|^2\sum_{k=0}^{K_0}|\hat{\gamma}(k)| + \sum_{k=0}^{K_0}|\hat{\gamma}(k)-\gamma(k)||\hat{\gamma}(k)+\gamma(k)|\,.
\end{aligned} \tag{39}
$$

Define

$$
\mathcal{E}(\varepsilon) = \bigg\{\max_{0\le k\le K_0} |\hat{\gamma}(k)-\gamma(k)| \le \varepsilon\bigg\}.
$$

Restricted on $\mathcal{E}(\varepsilon)$, it holds that

$$
\begin{aligned}
\max_{1\le t\le m}|\tilde{Q}_t - Q_t| \lesssim\ & K_*\varepsilon \max_{1\le t\le n}|Y_t-\mu|^2 + |\bar{Y}-\mu|\max_{1\le t\le n}|Y_t-\mu| \\
&+ K_*\varepsilon|\bar{Y}-\mu|\max_{1\le t\le n}|Y_t-\mu| + |\bar{Y}-\mu|^2 + K_*\varepsilon|\bar{Y}-\mu|^2 \\
&+ K_*\varepsilon^2 + \varepsilon\,.
\end{aligned}
$$

Same as Lemma 3, we have

$$
\begin{aligned}
\mathbb{P}\{\mathcal{E}(\varepsilon)^c\} \le\ & \sum_{k=0}^{K_0}\mathbb{P}\big\{|\hat{\gamma}(k)-\gamma(k)|>\varepsilon\big\} \\
\lesssim\ & K_*\exp(-CK_*^{-1}n\varepsilon^2) + K_*^{s_2}n(\varepsilon n)^{-s_2} + K_*n(\varepsilon n)^{-(\beta_2+2)s_2/(\beta_2+s_2)}\,.
\end{aligned}
$$

If we select $\varepsilon = C_*(n^{-1}K_*\log K_*)^{1/2}$ for some sufficiently large $C_*>0$, we then have that $K_*\exp(-CK_*^{-1}n\varepsilon^2) \le K_*^{1-CC_*^2}$. If $K_* = o(n^{1-2/s_2})$, it holds that $K_*^{s_2}n(\varepsilon n)^{-s_2} + K_*n(\varepsilon n)^{-(\beta_2+2)s_2/(\beta_2+s_2)} \to 0$ which implies $\mathbb{P}\{\mathcal{E}(\varepsilon)^c\} = o(1)$. Same as (34), under Condition 5, it holds that $|\bar{Y}-\mu| = O_{\rm p}[n^{-(2s_2-1)\beta_2/\{2s_2(\beta_2+1)\}}]$. By the Markov inequality, Condition 5 implies that $\max_{1\le t\le n}|Y_t-\mu| = O_{\rm p}\{n^{1/(2s_2)}\}$. It follows from (40) that

$$
\max_{1\le t\le m}|\tilde{Q}_t - Q_t| = O_{\rm p}\bigg(\frac{K_*^{3/2}\log^{1/2}K_*}{n^{1/2-1/s_2}}\bigg)\,,
$$

which implies that

$$
\begin{aligned}
&\bigg|\sum_{j=0}^{m-1}\mathcal{K}\bigg(\frac{j}{b_m}\bigg)\bigg\{\frac{1}{m}\sum_{t=j+1}^{m}(\tilde{Q}_t-Q_t)Q_{t-j}\bigg\}\bigg| \\
&\qquad\le \bigg\{\sum_{j=0}^{m-1}\bigg|\mathcal{K}\bigg(\frac{j}{b_m}\bigg)\bigg|\bigg\}\bigg(\frac{1}{m}\sum_{t=1}^{m}|Q_t|\bigg)\cdot\max_{1\le t\le m}|\tilde{Q}_t-Q_t|
\end{aligned}
$$

$$= O_{\mathrm{p}}\bigg(\frac{b_m K_*^{5/2}\log^{1/2} K_*}{n^{1/2-1/s_2}}\bigg)\,.$$

Similarly, we have

$$\sum_{j=0}^{m-1}\mathcal{K}\bigg(\frac{j}{b_m}\bigg)\bigg\{\frac{1}{m}\sum_{t=j+1}^{m} Q_t(\tilde{Q}_{t-j}-Q_{t-j})\bigg\} = O_{\mathrm{p}}\bigg(\frac{b_m K_*^{5/2}\log^{1/2} K_*}{n^{1/2-1/s_2}}\bigg)\,,$$

$$\sum_{j=0}^{m-1}\mathcal{K}\bigg(\frac{j}{b_m}\bigg)\bigg\{\frac{1}{m}\sum_{t=j+1}^{m} (\tilde{Q}_t-Q_t)(\tilde{Q}_{t-j}-Q_{t-j})\bigg\} = O_{\mathrm{p}}\bigg(\frac{b_m K_*^{3}\log K_*}{n^{1-2/s_2}}\bigg)\,.$$

Due to $b_m^2 K_*^5 \log K_* = o(n^{1-2/s_2})$, by (38), we have

$$\mathrm{I} = \sum_{j=0}^{m-1}\mathcal{K}\bigg(\frac{j}{b_m}\bigg)\bigg[\frac{1}{m}\sum_{t=j+1}^{m}\{Q_tQ_{t-j}-\mathbb{E}(Q_tQ_{t-j})\}\bigg] + o_{\mathrm{p}}(1)\,.$$

As we will show in Lemma 4, if $b_m^2 K_*^5 = o(m)$, it holds that

$$\sum_{j=0}^{m-1}\mathcal{K}\bigg(\frac{j}{b_n}\bigg)\bigg[\frac{1}{m}\sum_{t=j+1}^{m}\{Q_tQ_{t-j}-\mathbb{E}(Q_tQ_{t-j})\}\bigg] = O_{\mathrm{p}}\bigg(\frac{b_m K_*^{5/2}}{m^{1/2}}\bigg) = o_{\mathrm{p}}(1)\,.$$

Thus, $\mathrm{I} = o_{\mathrm{p}}(1)$. Similarly, we have $\mathrm{II} = o_{\mathrm{p}}(1)$, which implies $\tilde{V}_m - \hat{V}_m = o_{\mathrm{p}}(1)$. We construct the result (i).

We begin to construct result (ii). Notice that

$$\begin{aligned}
V_m &= \frac{1}{m}\sum_{t=1}^{m}\mathbb{E}(Q_t^2) + \frac{2}{m}\sum_{t_1=1}^{m-1}\sum_{t_2=t_1+1}^{m}\mathbb{E}(Q_{t_1}Q_{t_2})\\
&= \frac{1}{m}\sum_{t=1}^{m}\mathbb{E}(Q_t^2) + \frac{2}{m}\sum_{t_1=1}^{m-1}\sum_{j=1}^{m-t_1}\mathbb{E}(Q_{t_1}Q_{t_1+j})\\
&= G_0 + 2\sum_{j=1}^{m-1} G_j\,.
\end{aligned}$$

Recall $\mathcal{K}(\cdot)$ is symmetric with $\mathcal{K}(0)=1$, and $G_{-j}=G_j$ for any $j>0$. It follows from the Davydov's inequality that $|G_j| \lesssim m^{-1}(m-j)K_*^2\{\alpha_Q(j)\}^{1-2/s_2}$ for any $j \geq 1$. Thus, by the triangle inequality,

$$\begin{aligned}
|\hat{V}_m - V_m| &\leq 2\sum_{j=1}^{m-1}\bigg|\mathcal{K}\bigg(\frac{j}{b_m}\bigg)-1\bigg||G_j|\\
&\lesssim K_*^2\sum_{j=1}^{m-1}\frac{j}{b_m}\frac{m-j}{m}\{\alpha_Q(j)\}^{1-2/s_2}
\end{aligned}$$

$$\lesssim \frac{K_*^2}{b_m}\Bigg[\sum_{j=1}^{K_0} j + \sum_{j=K_0+1}^{m-1} j\{\alpha(|j-K_0|_+)\}^{1-2/s_2}\Bigg]$$
$$\lesssim \frac{K_*^4}{b_m} + \frac{K_*^2}{b_m}\sum_{j=1}^{m} j^{1-\beta_2(s_2-2)/s_2} = o(1)$$

provided that $K_*^4/b_m \to 0$ and $\beta_2 > 2s_2/(s_2-2)$. We construct result (ii). Therefore, we have $\tilde{V}_m/V_m \xrightarrow{\mathrm{P}} 1$, which completes the proof of Theorem 4. □

More technical proofs and simulation results are provided in the supplementary material.

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Supplementary Material for "A Power One Test for Unit Roots Based on Sample Autocovariances" by Chang, Cheng and Yao.

A Proof of Lemma 1

Without loss of generality, we assume $\mu = 0$. Let $\mathcal{T} = \{1, \ldots, m-1\}$. Select $h \in \mathcal{T}$ and $q \in \mathcal{T}\backslash\{1\}$ satisfying $2qh \leq m+1$. We will specify h and q later. For any $t = 1, \ldots, m$, let $A_t = Q_t/B_m$. Write $Z_m = \sum_{t=1}^m A_t$. For any $j = 1, \ldots, q$, let $W_{t,j} = \sum_{p=t-jh+1}^{t+jh-1} A_p$, $Z_{t,j} = Z_m - W_{t,j}$. Here we adopt the convention $A_p = 0$ if $p \leq 0$ or $p \geq m+1$. We also write $Z_{t,0} = Z_m$. Denote by $i = \sqrt{-1}$ the unit imaginary number. For any $r = 2, \ldots, q$, let

$$\varphi_{t,r-1} = \mathbb{E}\bigg(A_t \prod_{l=1}^{r-1} \psi_{t,l}\bigg) \quad \text{and} \quad \eta_{t,r} = e^{-iuW_{t,r}} - 1$$

with $\psi_{t,l} = e^{iu(Z_{t,l-1}-Z_{t,l})} - 1$. Define $f_m(u) = \mathbb{E}(e^{iuZ_m})$. Notice that $f'_m(u) = i\sum_{t=1}^m \mathbb{E}(A_t e^{iuZ_{t,0}})$. Then it holds that

$$\begin{aligned} f'_m(u) &= i\sum_{t=1}^m \mathbb{E}\big(A_t e^{iuZ_{t,0}} - A_t e^{iuZ_{t,1}} + A_t e^{iuZ_{t,1}}\big) \\ &= i\sum_{t=1}^m \mathbb{E}\big[A_t\{e^{iu(Z_{t,0}-Z_{t,1})} - 1\}e^{iuZ_{t,1}}\big] + i\sum_{t=1}^m \mathbb{E}\big(A_t e^{iuZ_{t,1}}\big) \\ &= i\sum_{t=1}^m \mathbb{E}\big(A_t e^{iuZ_{t,1}}\big) + i\sum_{t=1}^m\sum_{r=2}^q \mathbb{E}\bigg(A_t\bigg[\prod_{l=1}^{r-1}\{e^{iu(Z_{t,l-1}-Z_{t,l})} - 1\}\bigg]e^{iuZ_{t,r}}\bigg) \\ &\quad + i\sum_{t=1}^m \mathbb{E}\bigg(A_t\bigg[\prod_{l=1}^{q}\{e^{iu(Z_{t,l-1}-Z_{t,l})} - 1\}\bigg]e^{iuZ_{t,q}}\bigg) \\ &= i\sum_{t=1}^m \mathbb{E}\big(A_t e^{iuZ_{t,1}}\big) + i\sum_{t=1}^m\sum_{r=2}^q \mathbb{E}\bigg(A_t e^{iuZ_{t,r}}\prod_{l=1}^{r-1}\psi_{t,l}\bigg) + i\sum_{t=1}^m \mathbb{E}\bigg(A_t e^{iuZ_{t,q}}\prod_{l=1}^{q}\psi_{t,l}\bigg). \end{aligned} \tag{E.1}$$

On the other hand, we know $\mathbb{E}(e^{iuZ_{t,r}}) = f_m(u)\mathbb{E}(\eta_{t,r}+1) + \mathbb{E}[\{\eta_{t,r} - \mathbb{E}(\eta_{t,r})\}e^{iuZ_m}]$ for any $r = 2, \ldots, q$. Therefore, by (E.1), we can reformulate $f'_m(u)$ as follows:

$$\begin{aligned} f'_m(u) &= i\bigg(\sum_{t=1}^m \varphi_{t,1}\bigg)f_m(u) + i\bigg\{\sum_{t=1}^m \varphi_{t,1}\mathbb{E}(\eta_{t,2}) + \sum_{t=1}^m\sum_{r=3}^q \varphi_{t,r-1}\mathbb{E}(\eta_{t,r}+1)\bigg\}f_m(u) \\ &\quad + i\sum_{t=1}^m\sum_{r=2}^q \varphi_{t,r-1}\mathbb{E}\big[\{\eta_{t,r} - \mathbb{E}(\eta_{t,r})\}e^{iuZ_m}\big] + i\sum_{t=1}^m \mathbb{E}\big(A_t e^{iuZ_{t,1}}\big) \\ &\quad + i\sum_{t=1}^m\sum_{r=2}^q \bigg\{\mathbb{E}\bigg(A_t e^{iuZ_{t,r}}\prod_{l=1}^{r-1}\psi_{t,l}\bigg) - \mathbb{E}\bigg(A_t\prod_{l=1}^{r-1}\psi_{t,l}\bigg)\mathbb{E}\big(e^{iuZ_{t,r}}\big)\bigg\} \end{aligned} \tag{E.2}$$

$$+\, i\sum_{t=1}^m \mathbb{E}\bigg(A_t e^{iuZ_{t,q}}\prod_{l=1}^q \psi_{t,l}\bigg)\,.$$

Recall $K_* = 1 + K_0$. Let

$$\tilde{\alpha} = \sum_{\tau=1}^m \{\alpha_Q(\tau)\}^{(s_1-2)/s_1} \ \text{ and } \ d = 4^{s_1} c_1 K_*^{s_1}\,, \tag{E.3}$$

where c_1 is specified in Condition 1. Since $\alpha_Q(\tau) \le \alpha(|\tau - K_0|_+)$, it follows from Condition 2 that $\tilde{\alpha} \lesssim K_*$. To construct Lemma 1, we need Lemmas L1–L5 as follows.

Lemma L1. *Under the null hypothesis H_0 with Conditions* 1 *and* 2 *being satisfied, it holds that*

$$i\sum_{t=1}^m \varphi_{t,1} = -u + \theta_1(u)$$

for any $u \in \mathbb{R}$, where

$$|\theta_1(u)| \lesssim \frac{|u|}{B_m^2} m^{3/2} d^{2/s_1} K_*^{1/2} \{\alpha_Q(h)\}^{(s_1-2)/(2s_1)} + \frac{|u|^{s_1-1}}{B_m^{s_1}} m d h^{s_1-1}$$

with d specified in (E.3).

Proof. Notice that $e^{iv} - 1 - iv = i\int_0^v (e^{it}-1)\,\mathrm{d}t$ for any $v \in \mathbb{R}$. By the mean value theorem, we have $e^{iv} - 1 - iv = i\{e^{i\zeta_1(v)} - 1\}v$ for some $\zeta_1(v) \in (0, v)$ if $v > 0$ and $\zeta_1(v) \in (v, 0)$ if $v < 0$. On the other hand, it follows from the Taylor expansion that $e^{iv} - 1 - iv = -2^{-1}e^{i\zeta_2(v)}v^2$ for some $\zeta_2(v) \in (0, v)$ if $v > 0$ and $\zeta_2(v) \in (v, 0)$ if $v < 0$. Thus,

$$\begin{aligned} e^{iv} - 1 - iv &= \big[i\{e^{i\zeta_1(v)} - 1\}v\big]^{3-s_1}\bigg\{-\frac{e^{i\zeta_2(v)}v^2}{2}\bigg\}^{s_1-2} \\ &= \frac{i^{3-s_1}\{e^{i\zeta_1(v)} - 1\}^{3-s_1} e^{i(s_1-2)\zeta_2(v)}}{(-1)^{s_1-2}2^{s_1-2}} \cdot v^{s_1-1} =: \vartheta(v)\cdot v^{s_1-1}\,, \end{aligned}$$

where $|\vartheta(v)| \le 2^{5-2s_1}$ for any $v \in \mathbb{R}$, which implies that $e^{iv} - 1 = iv + \vartheta(v)\cdot v^{s_1-1}$ for any $v \in \mathbb{R}$. Recall $\varphi_{t,1} = \mathbb{E}(A_t\psi_{t,1})$ with $\psi_{t,1} = e^{iu(Z_{t,0}-Z_{t,1})} - 1$, and $W_{t,1} = Z_{t,0} - Z_{t,1}$. Then

$$\begin{aligned} i\sum_{t=1}^m \varphi_{t,1} &= i\sum_{t=1}^m \mathbb{E}\big[A_t\{e^{iu(Z_{t,0}-Z_{t,1})} - 1\}\big] \\ &= -u\sum_{t=1}^m \mathbb{E}(A_tW_{t,1}) + i\sum_{t=1}^m \mathbb{E}\big\{A_tW_{t,1}^{s_1-1}\vartheta(uW_{t,1})\big\}u^{s_1-1}\,. \end{aligned} \tag{E.4}$$

Notice that $W_{t,1} = \sum_{p=t-h+1}^{t+h-1} A_p$. It follows from the Jensen's inequality that

$$\mathbb{E}(|W_{t,1}|^{s_1}) \le (2h-1)^{s_1-1} \sum_{p=t-h+1}^{t+h-1} \mathbb{E}(|A_p|^{s_1}) \le (2h-1)^{s_1} \max_{1\le t\le m} \mathbb{E}(|A_t|^{s_1})\,.$$

Applying the Hölder's inequality, we have

$$\begin{aligned}\sum_{t=1}^{m} \mathbb{E}\{|A_t W_{t,1}^{s_1-1}\vartheta(uW_{t,1})|\} &\le 2^{5-2s_1} \sum_{t=1}^{m} \{\mathbb{E}(|A_t|^{s_1})\}^{1/s_1} \{\mathbb{E}(|W_{t,1}|^{s_1})\}^{(s_1-1)/s_1} \\ &\le 2^{5-2s_1}(2h-1)^{s_1-1} m \max_{1\le t\le m} \mathbb{E}(|A_t|^{s_1})\,.\end{aligned}$$

Write $Z_{t,1} = \hat{Z}_{t,1} + \tilde{Z}_{t,1}$ with $\hat{Z}_{t,1} = \sum_{p\le t-h} A_p$ and $\tilde{Z}_{t,1} = \sum_{p\ge t+h} A_p$. It follows from the Davydov's inequality that

$$|\mathbb{E}(A_t\hat{Z}_{t,1})| \le 6\{\alpha_Q(h)\}^{(s_1-2)/2s_1} \{\mathbb{E}(|A_t|^{s_1})\}^{1/s_1} \{\mathbb{E}(\hat{Z}_{t,1}^2)\}^{1/2}\,.$$

Using the Davydov's inequality again, we have

$$\begin{aligned}\mathbb{E}(\hat{Z}_{t,1}^2) &\le 6 \sum_{p_1,p_2\le t-h} \{\mathbb{E}(|A_{p_1}|^{s_1})\}^{1/s_1} \{\mathbb{E}(|A_{p_2}|^{s_1})\}^{1/s_1} \{\alpha_Q(|p_1-p_2|)\}^{(s_1-2)/s_1} \\ &\le 6|t-h|_+(1+2\tilde{\alpha}) \Big\{ \max_{1\le t\le m} \mathbb{E}(|A_t|^{s_1}) \Big\}^{2/s_1}\,.\end{aligned}$$

Notice that $A_t = Q_t/B_m$ with $Q_t = \sum_{k=0}^{K_0} \xi_{t,k}$ where $\xi_{t,k} = y_{t,k}\mathrm{sgn}\{\gamma(k)\}$ and $y_{t,k} = 2\{Y_tY_{t+k} - \gamma(k)\}\mathrm{sgn}(k+t-N-1/2)$. It follows from the Jensen's inequality, the Cauchy-Schwarz inequality and Condition 1 that

$$\begin{aligned}\mathbb{E}(|A_t|^{s_1}) &\le \frac{2^{s_1}K_*^{s_1-1}}{B_m^{s_1}} \sum_{k=0}^{K_0} \mathbb{E}\{|Y_tY_{t+k} - \gamma(k)|^{s_1}\} \\ &\le \frac{4^{s_1}K_*^{s_1-1}}{B_m^{s_1}} \sum_{k=0}^{K_0} \mathbb{E}(|Y_tY_{t+k}|^{s_1}) \le \frac{4^{s_1}K_*^{s_1}c_1}{B_m^{s_1}} = \frac{d}{B_m^{s_1}}\,,\end{aligned} \tag{E.5}$$

which implies

$$\sum_{t=1}^{m} \mathbb{E}\{|A_t W_{t,1}^{s_1-1}\vartheta(uW_{t,1})|\} \le \frac{2^{4-s_1}h^{s_1-1}md}{B_m^{s_1}}$$

and

$$|\mathbb{E}(A_t\hat{Z}_{t,1})| \le \frac{6\sqrt{6}d^{2/s_1}}{B_m^2}|t-h|_+^{1/2}(1+2\tilde{\alpha})^{1/2}\{\alpha_Q(h)\}^{(s_1-2)/(2s_1)}$$

$$\lesssim \frac{d^{2/s_1}}{B_m^2} m^{1/2} K_*^{1/2} \{\alpha_Q(h)\}^{(s_1-2)/(2s_1)}$$

for any $t = 1, \ldots, m$. Analogously, we have

$$|\mathbb{E}(A_t \tilde{Z}_{t,1})| \lesssim \frac{d^{2/s_1}}{B_m^2} m^{1/2} K_*^{1/2} \{\alpha_Q(h)\}^{(s_1-2)/(2s_1)}$$

for any $t = 1, \ldots, m$. Thus,

$$\left| \sum_{t=1}^{m} \{\mathbb{E}(A_t \hat{Z}_{t,1}) + \mathbb{E}(A_t \tilde{Z}_{t,1})\} \right| \lesssim \frac{d^{2/s_1}}{B_m^2} m^{3/2} K_*^{1/2} \{\alpha_Q(h)\}^{(s_1-2)/(2s_1)} .$$

Since $\sum_{t=1}^m \mathbb{E}(A_t Z_m) = 1$, then $\sum_{t=1}^m \mathbb{E}(A_t W_{t,1}) = 1 - \sum_{t=1}^m \{\mathbb{E}(A_t \hat{Z}_{t,1}) + \mathbb{E}(A_t \tilde{Z}_{t,1})\}$. Together with (E.4), it holds that

$$\begin{aligned} i \sum_{t=1}^{m} \varphi_{t,1} &= -u + u \sum_{t=1}^{m} \{\mathbb{E}(A_t \hat{Z}_{t,1}) + \mathbb{E}(A_t \tilde{Z}_{t,1})\} + i \sum_{t=1}^{m} \mathbb{E}\{A_t W_{t,1}^{s_1-1} \vartheta(u W_{t,1})\} u^{s_1-1} \\ &=: -u + \theta_1(u) , \end{aligned}$$

where

$$|\theta_1(u)| \lesssim \frac{|u|}{B_m^2} m^{3/2} d^{2/s_1} K_*^{1/2} \{\alpha_Q(h)\}^{(s_1-2)/(2s_1)} + \frac{|u|^{s_1-1}}{B_m^{s_1}} m d h^{s_1-1}$$

for any $u \in \mathbb{R}$. We complete the proof of Lemma L1. □

Lemma L2. *Let $U_* = B_m/(32hd^{1/s_1})$. Under the null hypothesis H_0 with Conditions* 1 *and* 2 *being satisfied, if $|u| \le U_*$ and*

$$2^{3q} \{\alpha_Q(h)\}^{1/s_1} \le 1 , \tag{E.6}$$

then it holds that

$$|\varphi_{t,r-1}| \lesssim \frac{d^{1/s_1}}{B_m} \left[\frac{d^{2/s_1} h^2 u^2}{16^r B_m^2} + \frac{r}{2^r} \{\alpha_Q(h)\}^{(s_1-2)/s_1} \right] \quad (r = 3, \ldots, q)$$

and

$$|\varphi_{t,1}| \le \frac{2|u| h d^{2/s_1}}{B_m^2}$$

for any $t = 1, \ldots, m$, where d is specified in (E.3).

Proof. Let $\tilde{\psi}_{t,l} = \exp\{iu \sum_{p=t+(l-1)h}^{t+lh-1} A_p\} - 1$ and $\hat{\psi}_{t,l} = \exp\{iu \sum_{p=t-lh+1}^{t-(l-1)h} A_p\} - 1$. Recall $\psi_{t,l} = e^{iu(Z_{t,l-1} - Z_{t,l})} - 1$. Notice that $Z_{t,l-1} - Z_{t,l} = \sum_{p=t+(l-1)h}^{t+lh-1} A_p + \sum_{p=t-lh+1}^{t-(l-1)h} A_p$. By the inequality $|e^{a+b} - 1| \le |e^a||e^b - 1| + |e^a - 1|$ for any $a, b \in \mathbb{C}$, we have $|\psi_{t,l}| \le |\tilde{\psi}_{t,l}| + |\hat{\psi}_{t,l}|$, which implies that

$$|\varphi_{t,r-1}| \le \mathbb{E}\bigg(|A_t|\prod_{l=1}^{r-1}|\psi_{t,l}|\bigg) \le \sum_{\tau_1,\ldots,\tau_{r-1}\in\{0,1\}} \mathbb{E}\bigg(|A_t|\prod_{l=1}^{r-1}|\tilde{\psi}_{t,l}|^{\tau_l}|\hat{\psi}_{t,l}|^{1-\tau_l}\bigg)\,. \qquad \text{(E.7)}$$

For given $(\tau_1,\ldots,\tau_{r-1})$, define $\mathcal{L} := \mathcal{L}(\tau_1,\ldots,\tau_{r-1}) = \{1 \le l \le r-1 : \tau_l = 1\}$. Then

$$\mathbb{E}\bigg(|A_t|\prod_{l=1}^{r-1}|\tilde{\psi}_{t,l}|^{\tau_l}|\hat{\psi}_{t,l}|^{1-\tau_l}\bigg) = \mathbb{E}\bigg(|A_t|\prod_{l\in\mathcal{L}}|\tilde{\psi}_{t,l}|\prod_{l\in\mathcal{L}^c}|\hat{\psi}_{t,l}|\bigg)\,.$$

Pick $\delta = s_1/(s_1-1)$. It follows from the Hölder's inequality that

$$\begin{aligned}\mathbb{E}\bigg(|A_t|\prod_{l\in\mathcal{L}}|\tilde{\psi}_{t,l}|\prod_{l\in\mathcal{L}^c}|\hat{\psi}_{t,l}|\bigg) \le{}& \bigg\{\mathbb{E}\bigg(|A_t|^\delta\prod_{l\in\mathcal{L},\,l\,\mathrm{even}}|\tilde{\psi}_{t,l}|^\delta\prod_{l\in\mathcal{L}^c,\,l\,\mathrm{even}}|\hat{\psi}_{t,l}|^\delta\bigg)\bigg\}^{1/\delta} \\ &\times\bigg[\mathbb{E}\bigg\{\prod_{l\in\mathcal{L},\,l\,\mathrm{odd}}|\tilde{\psi}_{t,l}|^{\delta/(\delta-1)}\prod_{l\in\mathcal{L}^c,\,l\,\mathrm{odd}}|\hat{\psi}_{t,l}|^{\delta/(\delta-1)}\bigg\}\bigg]^{(\delta-1)/\delta}\,.\end{aligned}$$

Due to $|\tilde{\psi}_{t,l}| \le 2$ and $|\hat{\psi}_{t,l}| \le 2$ for any $l = 1,\ldots,r-1$, by the Davydov's inequality and the Hölder's inequality, we have

$$\begin{aligned}&\mathbb{E}\bigg(|A_t|^\delta\prod_{l\in\mathcal{L},\,l\,\mathrm{even}}|\tilde{\psi}_{t,l}|^\delta\prod_{l\in\mathcal{L}^c,\,l\,\mathrm{even}}|\hat{\psi}_{t,l}|^\delta\bigg) \\ &\quad\le \mathbb{E}(|A_t|^\delta)\mathbb{E}\bigg(\prod_{l\in\mathcal{L},\,l\,\mathrm{even}}|\tilde{\psi}_{t,l}|^\delta\prod_{l\in\mathcal{L}^c,\,l\,\mathrm{even}}|\hat{\psi}_{t,l}|^\delta\bigg) \\ &\qquad + 6\times 2^{\delta(r-1)/2}\{\mathbb{E}(|A_t|^{s_1})\}^{\delta/s_1}\{\alpha_Q(h)\}^{(s_1-\delta)/s_1} \\ &\quad\le \{\mathbb{E}(|A_t|^{s_1})\}^{\delta/s_1}\prod_{l\in\mathcal{L},\,l\,\mathrm{even}}\mathbb{E}(|\tilde{\psi}_{t,l}|^\delta)\prod_{l\in\mathcal{L}^c,\,l\,\mathrm{even}}\mathbb{E}(|\hat{\psi}_{t,l}|^\delta) \\ &\qquad + 3r\times 2^{\delta(r-1)/2}\{\mathbb{E}(|A_t|^{s_1})\}^{\delta/s_1}\{\alpha_Q(h)\}^{(s_1-\delta)/s_1}\end{aligned}$$

and

$$\begin{aligned}&\mathbb{E}\bigg\{\prod_{l\in\mathcal{L},\,l\,\mathrm{odd}}|\tilde{\psi}_{t,l}|^{\delta/(\delta-1)}\prod_{l\in\mathcal{L}^c,\,l\,\mathrm{odd}}|\hat{\psi}_{t,l}|^{\delta/(\delta-1)}\bigg\} \\ &\quad\le \prod_{l\in\mathcal{L},\,l\,\mathrm{odd}}\mathbb{E}\{|\tilde{\psi}_{t,l}|^{\delta/(\delta-1)}\}\prod_{l\in\mathcal{L}^c,\,l\,\mathrm{odd}}\mathbb{E}\{|\hat{\psi}_{t,l}|^{\delta/(\delta-1)}\} + 3r\times 2^{\delta r/\{2(\delta-1)\}}\alpha_Q(h)\,.\end{aligned}$$

By (E.5) and the inequality $(x+y)^\gamma \le x^\gamma + y^\gamma$ for any $x, y > 0$ if $\gamma \in (0,1)$, we have

$$
\begin{aligned}
&\mathbb{E}\bigg(|A_t| \prod_{l\in\mathcal{L}} |\tilde{\psi}_{t,l}| \prod_{l\in\mathcal{L}^c} |\hat{\psi}_{t,l}|\bigg) \\
&\le \frac{d^{1/s_1}}{B_m}\bigg[\prod_{l\in\mathcal{L},\, l\,\text{even}} \big\{\mathbb{E}\big(|\tilde{\psi}_{t,l}|^\delta\big)\big\}^{1/\delta} \prod_{l\in\mathcal{L}^c,\, l\,\text{even}} \big\{\mathbb{E}\big(|\hat{\psi}_{t,l}|^\delta\big)\big\}^{1/\delta} \\
&\qquad + (3r)^{1/\delta} 2^{r/2} \{\alpha_Q(h)\}^{(s_1-\delta)/(s_1\delta)}\bigg] \qquad (E.8) \\
&\quad \times \bigg(\prod_{l\in\mathcal{L},\, l\,\text{odd}} \big[\mathbb{E}\big\{|\tilde{\psi}_{t,l}|^{\delta/(\delta-1)}\big\}\big]^{(\delta-1)/\delta} \prod_{l\in\mathcal{L}^c,\, l\,\text{odd}} \big[\mathbb{E}\big\{|\hat{\psi}_{t,l}|^{\delta/(\delta-1)}\big\}\big]^{(\delta-1)/\delta} \\
&\qquad + (3r)^{(\delta-1)/\delta} 2^{r/2} \{\alpha_Q(h)\}^{(\delta-1)/\delta}\bigg).
\end{aligned}
$$

On the other hand, by the Taylor expansion, it holds that

$$
\tilde{\psi}_{t,l} = i \exp\bigg\{icu \sum_{p=t+(l-1)h}^{t+lh-1} A_p\bigg\}\bigg\{u \sum_{p=t+(l-1)h}^{t+lh-1} A_p\bigg\}
$$

for some $c \in (0,1)$. Recall $\delta = s_1/(s_1-1)$, then $\delta/(\delta-1) = s_1$. It follows from the Hölder inequality that $\{\mathbb{E}(|\tilde{\psi}_{t,l}|^\delta)\}^{1/\delta} \le \{\mathbb{E}(|\tilde{\psi}_{t,l}|^{s_1})\}^{1/s_1}$ and $\{\mathbb{E}(|\hat{\psi}_{t,l}|^\delta)\}^{1/\delta} \le \{\mathbb{E}(|\hat{\psi}_{t,l}|^{s_1})\}^{1/s_1}$. Together with the Jensen's inequality and (E.5), we have

$$
\mathbb{E}(|\tilde{\psi}_{t,l}|^{s_1}) \le |u|^{s_1} \mathbb{E}\bigg(\bigg|\sum_{p=t+(l-1)h}^{t+lh-1} A_p\bigg|^{s_1}\bigg) \le |u|^{s_1} h^{s_1-1} \sum_{p=t+(l-1)h}^{t+lh-1} \mathbb{E}(|A_p|^{s_1}) \le \frac{|u|^{s_1} h^{s_1} d}{B_m^{s_1}}, \qquad (E.9)
$$

which implies $\{\mathbb{E}(|\tilde{\psi}_{t,l}|^{s_1})\}^{1/s_1} \le |u| h d^{1/s_1} B_m^{-1}$. Analogously, we have $\{\mathbb{E}(|\hat{\psi}_{t,l}|^{s_1})\}^{1/s_1} \le |u| h d^{1/s_1} B_m^{-1}$. Notice that $s_1 \in (2,3]$ and $0 \le \alpha_Q(h) \le 1/4$ for any $h \ge 0$, then $(s_1-2)/s_1 \le 1/s_1 < (s_1-1)/s_1$ and $\{\alpha_Q(h)\}^{1/s_1} \le \{\alpha_Q(h)\}^{(s_1-2)/s_1}$. Hence, if $|u| \le U_*$, (E.8) implies that

$$
\begin{aligned}
\mathbb{E}\bigg(|A_t| \prod_{l\in\mathcal{L}} |\tilde{\psi}_{t,l}| \prod_{l\in\mathcal{L}^c} |\hat{\psi}_{t,l}|\bigg) &\le \frac{d^{1/s_1}}{B_m}\bigg[\bigg(\frac{|u|hd^{1/s_1}}{B_m}\bigg)^{(r-1)/2} + (3r)^{(s_1-1)/s_1} 2^{r/2} \{\alpha_Q(h)\}^{(s_1-2)/s_1}\bigg] \\
&\qquad \times \bigg[\bigg(\frac{|u|hd^{1/s_1}}{B_m}\bigg)^{(r-1)/2} + (3r)^{1/s_1} 2^{r/2} \{\alpha_Q(h)\}^{1/s_1}\bigg] \\
&\le \frac{d^{1/s_1}}{B_m}\bigg[\bigg(\frac{|u|hd^{1/s_1}}{B_m}\bigg)^{r-1} + (3r)2^r \{\alpha_Q(h)\}^{(s_1-1)/s_1} \\
&\qquad + (3r)^{(s_1-1)/s_1} 2^{(r+2)/2} \bigg(\frac{|u|hd^{1/s_1}}{B_m}\bigg)^{(r-1)/2} \{\alpha_Q(h)\}^{(s_1-2)/s_1}\bigg].
\end{aligned}
$$

Since there are 2^{r-1} different selections of $(\tau_1,\ldots,\tau_{r-1})$, then (E.7) implies that

$$|\varphi_{t,r-1}| \lesssim \frac{d^{1/s_1}}{B_m}\bigg[\bigg(\frac{2|u|hd^{1/s_1}}{B_m}\bigg)^{r-1} + r4^r\{\alpha_Q(h)\}^{(s_1-1)/s_1} + \bigg(\frac{2|u|hd^{1/s_1}}{B_m}\bigg)^{(r-1)/2} r^{(s_1-1)/s_1}2^r\{\alpha_Q(h)\}^{(s_1-2)/s_1}\bigg]$$

for any $|u|\le U_*$. Thus, if $|u|\le U_*$, we have

$$|\varphi_{t,r-1}| \lesssim \frac{d^{1/s_1}}{B_m}\bigg[\frac{d^{2/s_1}h^2u^2}{16^rB_m^2} + \frac{r^{(s_1-1)/s_1}}{2^r}\{\alpha_Q(h)\}^{(s_1-2)/s_1} + r4^r\{\alpha_Q(h)\}^{(s_1-1)/s_1}\bigg]$$

for any $r=3,\ldots,q$. Moreover, it follows from (E.6) that $r4^r\{\alpha_Q(h)\}^{(s_1-1)/s_1}\le r2^{-r}\{\alpha_Q(h)\}^{(s_1-2)/s_1}$ for any $r=3,\ldots,q$. Thus, it holds that

$$|\varphi_{t,r-1}| \lesssim \frac{d^{1/s_1}}{B_m}\bigg[\frac{d^{2/s_1}h^2u^2}{16^rB_m^2} + \frac{r}{2^r}\{\alpha_Q(h)\}^{(s_1-2)/s_1}\bigg]$$

for any $r=3,\ldots,q$. We have the first result of Lemma L2.

Recall $\varphi_{t,1}=\mathbb{E}(A_t\psi_{t,1})$ and $|\psi_{t,1}|\le|\tilde\psi_{t,1}|+|\hat\psi_{t,1}|$. By the Hölder's inequality, (E.5) and (E.9), we have

$$\begin{aligned}|\varphi_{t,1}| &\le \mathbb{E}(|A_t\tilde\psi_{t,1}|)+\mathbb{E}(|A_t\hat\psi_{t,1}|)\\ &\le \big\{\mathbb{E}\big(|A_t\tilde\psi_{t,1}|^{s_1/2}\big)\big\}^{2/s_1} + \big\{\mathbb{E}\big(|A_t\hat\psi_{t,1}|^{s_1/2}\big)\big\}^{2/s_1}\\ &\le \{\mathbb{E}(|A_t|^{s_1})\}^{1/s_1}\Big[\big\{\mathbb{E}\big(|\tilde\psi_{t,1}|^{s_1}\big)\big\}^{1/s_1}+\big\{\mathbb{E}\big(|\hat\psi_{t,1}|^{s_1}\big)\big\}^{1/s_1}\Big] \le \frac{2|u|hd^{2/s_1}}{B_m^2}.\end{aligned}$$

We have the second result of Lemma L2. □

Lemma L3. *Let $U_*=B_m/(32hd^{1/s_1})$. Under the null hypothesis H_0 with Conditions* 1 *and* 2 *being satisfied, if $|u|\le U_*$, $16^q\ge m^{1/2}$ and* (E.6) *is satisfied, then it holds that*

$$\bigg|\sum_{t=1}^m\varphi_{t,1}\mathbb{E}(\eta_{t,2})+\sum_{t=1}^m\sum_{r=3}^q\varphi_{t,r-1}\mathbb{E}(\eta_{t,r}+1)\bigg| \lesssim \frac{md^{1/s_1}}{B_m}\bigg[\frac{d^{2/s_1}h^2u^2}{B_m^2}+\{\alpha_Q(h)\}^{(s_1-2)/s_1}\bigg],$$

$$\sum_{r=2}^q\bigg|\sum_{t=1}^m\varphi_{t,r-1}\mathbb{E}\big[\{\eta_{t,r}-\mathbb{E}(\eta_{t,r})\}e^{iuZ_m}\big]\bigg| \lesssim \frac{d^{1/s_1}m^{1/2}}{B_m}(h^{1/2}+\tilde\alpha^{1/2})\bigg[\frac{d^{2/s_1}h^2u^2}{B_m^2}+\{\alpha_Q(h)\}^{(s_1-2)/s_1}\bigg],$$

and

$$\sum_{t=1}^m\bigg|\mathbb{E}\bigg(A_te^{iuZ_{t,q}}\prod_{l=1}^q\psi_{t,l}\bigg)\bigg| \lesssim \frac{m^{1/2}d^{1/s_1}}{B_m}\bigg[\frac{d^{2/s_1}h^2u^2}{B_m^2}+m^{1/2}\{\alpha_Q(h)\}^{(s_1-2)/s_1}\bigg].$$

where $\tilde{\alpha}$ *and* d *are specified in* (E.3).

Proof. Recall that $\eta_{t,r} = e^{-iuW_{t,r}} - 1$ with $W_{t,r} = \sum_{p=t-rh+1}^{t+rh-1} A_p$. By the triangle inequality, we have

$$\bigg|\sum_{t=1}^{m} \varphi_{t,1}\mathbb{E}(\eta_{t,2}) + \sum_{t=1}^{m}\sum_{r=3}^{q} \varphi_{t,r-1}\mathbb{E}(\eta_{t,r}+1)\bigg| \le \sum_{t=1}^{m} |\varphi_{t,1}|\mathbb{E}(|\eta_{t,2}|) + \sum_{t=1}^{m}\sum_{r=3}^{q} |\varphi_{t,r-1}|\,.$$

By the Taylor expansion, it holds that $\eta_{t,2} = -i\exp(-icu\sum_{p=t-2h+1}^{t+2h-1} A_p)(u\sum_{p=t-2h+1}^{t+2h-1} A_p)$ for some $c \in (0,1)$. Then (E.5) implies that

$$\mathbb{E}(|\eta_{t,2}|) \le |u| \sum_{p=t-2h+1}^{t+2h-1} \mathbb{E}(|A_p|) \le \frac{4h|u|d^{1/s_1}}{B_m}\,.$$

It follows from Lemma L2 that

$$\begin{aligned} &\bigg|\sum_{t=1}^{m} \varphi_{t,1}\mathbb{E}(\eta_{t,2}) + \sum_{t=1}^{m}\sum_{r=3}^{q} \varphi_{t,r-1}\mathbb{E}(\eta_{t,r}+1)\bigg| \\ &\qquad \lesssim \frac{mh^2d^{3/s_1}u^2}{B_m^3} + \frac{md^{1/s_1}}{B_m}\bigg[\frac{d^{2/s_1}h^2u^2}{B_m^2} + \{\alpha_Q(h)\}^{(s_1-2)/s_1}\bigg] \\ &\qquad \lesssim \frac{md^{1/s_1}}{B_m}\bigg[\frac{d^{2/s_1}h^2u^2}{B_m^2} + \{\alpha_Q(h)\}^{(s_1-2)/s_1}\bigg]\,. \end{aligned}$$

We have the first result of Lemma L3.

For any complex number $a \in \mathbb{C}$, we denote by $\bar{a}$ the complex conjugate of a. Notice that $|\eta_{t,r}| \le 2$ for any $t = 1,\ldots,m$ and $r = 2,\ldots,q$. For any $r = 3,\ldots,q$, by the Cauchy-Schwarz inequality, we have

$$\begin{aligned} &\bigg|\sum_{t=1}^{m} \varphi_{t,r-1}\mathbb{E}\big[\{\eta_{t,r} - \mathbb{E}(\eta_{t,r})\}e^{iuZ_m}\big]\bigg|^2 \\ &\quad \le \mathbb{E}\bigg[\bigg|\sum_{t=1}^{m} \varphi_{t,r-1}\{\eta_{t,r} - \mathbb{E}(\eta_{t,r})\}e^{iuZ_m}\bigg|^2\bigg] = \mathbb{E}\bigg[\bigg|\sum_{t=1}^{m} \varphi_{t,r-1}\{\eta_{t,r} - \mathbb{E}(\eta_{t,r})\}\bigg|^2\bigg] \\ &\quad = \sum_{t_1,t_2=1}^{m} \varphi_{t_1,r-1}\bar{\varphi}_{t_2,r-1}\mathbb{E}[\{\eta_{t_1,r} - \mathbb{E}(\eta_{t_1,r})\}\{\bar{\eta}_{t_2,r} - \mathbb{E}(\bar{\eta}_{t_2,r})\}] \\ &\quad \le \sum_{t_1,t_2=1}^{m} |\varphi_{t_1,r-1}||\varphi_{t_2,r-1}|\big|\mathbb{E}[\{\eta_{t_1,r} - \mathbb{E}(\eta_{t_1,r})\}\{\bar{\eta}_{t_2,r} - \mathbb{E}(\bar{\eta}_{t_2,r})\}]\big| \\ &\quad \lesssim \sum_{t=1}^{m} |\varphi_{t,r-1}|^2 + \sum_{t_1<t_2} |\varphi_{t_1,r-1}||\varphi_{t_2,r-1}|\big|\mathbb{E}[\{\eta_{t_1,r} - \mathbb{E}(\eta_{t_1,r})\}\{\bar{\eta}_{t_2,r} - \mathbb{E}(\bar{\eta}_{t_2,r})\}]\big|\,. \end{aligned}$$

It follows from the Davydov's inequality that

$$
\begin{aligned}
&\big|\mathbb{E}[\{\eta_{t_1,r}-\mathbb{E}(\eta_{t_1,r})\}\{\bar{\eta}_{t_2,r}-\mathbb{E}(\bar{\eta}_{t_2,r})\}]\big| \\
&\qquad \lesssim \{\mathbb{E}(|\eta_{t_1,r}|^{s_1})\}^{1/s_1}\{\mathbb{E}(|\eta_{t_2,r}|^{s_1})\}^{1/s_1}\{\alpha_Q(|t_2-t_1-2rh+2|_+)\}^{(s_1-2)/s_1} \\
&\qquad \lesssim \{\alpha_Q(|t_2-t_1-2rh+2|_+)\}^{(s_1-2)/s_1}
\end{aligned}
$$

for any $t_1 < t_2$, which implies that

$$
\begin{aligned}
&\sum_{t_1<t_2}|\varphi_{t_1,r-1}||\varphi_{t_2,r-1}|\big|\mathbb{E}[\{\eta_{t_1,r}-\mathbb{E}(\eta_{t_1,r})\}\{\bar{\eta}_{t_2,r}-\mathbb{E}(\bar{\eta}_{t_2,r})\}]\big| \\
&\qquad \lesssim \sum_{t_1<t_2}|\varphi_{t_1,r-1}||\varphi_{t_2,r-1}|\{\alpha_Q(|t_2-t_1-2rh+2|_+)\}^{(s_1-2)/s_1} \\
&\qquad \lesssim \bigg(\sum_{t_1<t_2}|\varphi_{t_1,r-1}||\varphi_{t_2,r-1}|\bigg)\bigg[\sum_{t_1<t_2}\{\alpha_Q(|t_2-t_1-2rh+2|_+)\}^{(s_1-2)/s_1}\bigg] \\
&\qquad \lesssim \bigg(\sum_{t_1<t_2}|\varphi_{t_1,r-1}||\varphi_{t_2,r-1}|\bigg)m(rh+\tilde{\alpha})\,.
\end{aligned}
$$

Therefore, we have that

$$
\bigg|\sum_{t=1}^{m}\varphi_{t,r-1}\mathbb{E}\big[\{\eta_{t,r}-\mathbb{E}(\eta_{t,r})\}e^{iuZ_m}\big]\bigg| \lesssim m^{1/2}(r^{1/2}h^{1/2}+\tilde{\alpha}^{1/2})\sum_{t=1}^{m}|\varphi_{t,r-1}|\,.
$$

It follows from Lemma L2 that

$$
\begin{aligned}
&\sum_{r=3}^{q}\bigg|\sum_{t=1}^{m}\varphi_{t,r-1}\mathbb{E}\big[\{\eta_{t,r}-\mathbb{E}(\eta_{t,r})\}e^{iuZ_m}\big]\bigg| \\
&\qquad \lesssim \frac{d^{1/s_1}m^{1/2}}{B_m}(h^{1/2}+\tilde{\alpha}^{1/2})\bigg[\frac{d^{2/s_1}h^2u^2}{B_m^2}+\{\alpha_Q(h)\}^{(s_1-2)/s_1}\bigg]\,. \qquad \text{(E.10)}
\end{aligned}
$$

For $r = 2$, applying the facts $\{\mathbb{E}(|\eta_{t,r}|^2)\}^{1/2} \leq \{\mathbb{E}(|\eta_{t,r}|)^{s_1}\}^{1/s_1} \leq 4h|u|d^{1/s_1}B_m^{-1}$ and $|\varphi_{t,1}| \leq 2|u|hd^{2/s_1}B_m^{-2}$ as stated in Lemma L2, we have

$$
\begin{aligned}
&\bigg|\sum_{t=1}^{m}\varphi_{t,1}\mathbb{E}\big[\{\eta_{t,2}-\mathbb{E}(\eta_{t,2})\}e^{iuZ_m}\big]\bigg|^2 \\
&\qquad \lesssim \frac{h^4u^4d^{6/s_1}}{B_m^6}\bigg[m+\sum_{t_1<t_2}\{\alpha_Q(|t_2-t_1-4h+2|_+)\}^{(s_1-2)/s_1}\bigg] \\
&\qquad \lesssim \frac{h^4u^4d^{6/s_1}}{B_m^6}m(h+\tilde{\alpha})\,.
\end{aligned}
$$

Hence, together with (E.10), it holds that

$$\sum_{r=2}^{q}\bigg|\sum_{t=1}^{m}\varphi_{t,r-1}\mathbb{E}\big[\{\eta_{t,r}-\mathbb{E}(\eta_{t,r})\}e^{iuZ_m}\big]\bigg| \lesssim \frac{d^{1/s_1}m^{1/2}}{B_m}(h^{1/2}+\tilde{\alpha}^{1/2})\bigg[\frac{d^{2/s_1}h^2u^2}{B_m^2}+\{\alpha_Q(h)\}^{(s_1-2)/s_1}\bigg].$$

We have the second result of Lemma L3.

Notice that

$$\bigg|\mathbb{E}\bigg(A_t e^{iuZ_{t,q}}\prod_{l=1}^{q}\psi_{t,l}\bigg)\bigg| \le \mathbb{E}\bigg(|A_t|\prod_{l=1}^{q}|\psi_{t,l}|\bigg).$$

Applying the technique to bound $\mathbb{E}(|A_t|\prod_{l=1}^{r-1}|\psi_{t,l}|)$, the upper bound of $|\varphi_{t,r-1}|$, stated in (E.7), with the restrictions (E.6) and $16^q \ge m^{1/2}$, it holds that

$$\begin{aligned}\bigg|\mathbb{E}\bigg(A_t e^{iuZ_{t,q}}\prod_{l=1}^{q}\psi_{t,l}\bigg)\bigg| &\lesssim \frac{d^{1/s_1}}{B_m}\bigg[\frac{d^{2/s_1}h^2u^2}{16^qB_m^2}+\frac{q}{2^q}\{\alpha_Q(h)\}^{(s_1-2)/s_1}\bigg]\\ &\lesssim \frac{d^{1/s_1}}{B_m}\bigg[\frac{d^{2/s_1}h^2u^2}{m^{1/2}B_m^2}+\{\alpha_Q(h)\}^{(s_1-2)/s_1}\bigg]\end{aligned}$$

for any $|u|\le U_*$, which implies that

$$\sum_{t=1}^{m}\bigg|\mathbb{E}\bigg(A_t e^{iuZ_{t,q}}\prod_{l=1}^{q}\psi_{t,l}\bigg)\bigg| \lesssim \frac{m^{1/2}d^{1/s_1}}{B_m}\bigg[\frac{d^{2/s_1}h^2u^2}{B_m^2}+m^{1/2}\{\alpha_Q(h)\}^{(s_1-2)/s_1}\bigg].$$

Then we have the third result of Lemma L3. □

Lemma L4. *Let* $U_* = B_m/(32hd^{1/s_1})$. *Under the null hypothesis* H_0 *with Conditions* 1 *and* 2 *being satisfied, if* $|u|\le U_*$ *and* (E.6) *is satisfied, then it holds that*

$$\sum_{t=1}^{m}\big|\mathbb{E}\big(A_t e^{iuZ_{t,1}}\big)\big| \lesssim \frac{md^{1/s_1}}{B_m}\{\alpha_Q(h)\}^{(s_1-1)/s_1}$$

and

$$\sum_{r=2}^{q}\sum_{t=1}^{m}\bigg|\mathbb{E}\bigg(A_t e^{iuZ_{t,r}}\prod_{l=1}^{r-1}\psi_{t,l}\bigg)-\mathbb{E}\bigg(A_t\prod_{l=1}^{r-1}\psi_{t,l}\bigg)\mathbb{E}\big(e^{iuZ_{t,r}}\big)\bigg| \lesssim \frac{md^{1/s_1}}{B_m}\{\alpha_Q(h)\}^{(s_1-2)/s_1}.$$

Proof. Recall that $Z_{t,1}=\hat{Z}_{t,1}+\tilde{Z}_{t,1}$ with $\hat{Z}_{t,1}=\sum_{p=1}^{t-h}A_p$ and $\tilde{Z}_{t,1}=\sum_{p=t+h}^{m}A_p$. It follows from the Davydov's inequality that

$$\begin{aligned}\big|\mathbb{E}\big(A_t e^{iuZ_{t,1}}\big)\big| &\lesssim \big\{\mathbb{E}\big(\big|A_t e^{iu\hat{Z}_{t,1}}\big|^{s_1}\big)\big\}^{1/s_1}\{\alpha_Q(h)\}^{(s_1-1)/s_1}\\ &= \{\mathbb{E}(|A_t|^{s_1})\}^{1/s_1}\{\alpha_Q(h)\}^{(s_1-1)/s_1}.\end{aligned}$$

By (E.5), we have the first result of Lemma L4.

Let $\hat{Z}_{t,r} = \sum_{p\le t-rh} A_p$ and $\tilde{Z}_{t,r} = \sum_{p\ge t+rh} A_p$. Then $Z_{t,r} = \hat{Z}_{t,r} + \tilde{Z}_{t,r}$. Without loss of generality we assume that $\hat{Z}_{t,r} \neq 0$ and $\tilde{Z}_{t,r} \neq 0$. It follows from the triangle inequality that

$$\begin{aligned}
&\bigg|\mathbb{E}\bigg(A_t e^{iuZ_{t,r}}\prod_{l=1}^{r-1}\psi_{t,l}\bigg) - \mathbb{E}\bigg(A_t\prod_{l=1}^{r-1}\psi_{t,l}\bigg)\mathbb{E}\big(e^{iuZ_{t,r}}\big)\bigg| \\
&\quad\le \bigg|\mathbb{E}\bigg(A_t e^{iuZ_{t,r}}\prod_{l=1}^{r-1}\psi_{t,l}\bigg) - \mathbb{E}\bigg(A_t e^{iu\hat{Z}_{t,r}}\prod_{l=1}^{r-1}\psi_{t,l}\bigg)\mathbb{E}\big(e^{iu\tilde{Z}_{t,r}}\big)\bigg| \\
&\qquad + \bigg|\mathbb{E}\bigg(A_t e^{iu\hat{Z}_{t,r}}\prod_{l=1}^{r-1}\psi_{t,l}\bigg) - \mathbb{E}\bigg(A_t\prod_{l=1}^{r-1}\psi_{t,l}\bigg)\mathbb{E}\big(e^{iu\hat{Z}_{t,r}}\big)\bigg|\big|\mathbb{E}\big(e^{iu\tilde{Z}_{t,r}}\big)\big| \\
&\qquad + \big|\mathbb{E}\big(e^{iuZ_{t,r}}\big) - \mathbb{E}\big(e^{iu\hat{Z}_{t,r}}\big)\mathbb{E}\big(e^{iu\tilde{Z}_{t,r}}\big)\big|\bigg|\mathbb{E}\bigg(A_t\prod_{l=1}^{r-1}\psi_{t,l}\bigg)\bigg| \\
&\quad\le \bigg|\mathbb{E}\bigg(A_t e^{iuZ_{t,r}}\prod_{l=1}^{r-1}\psi_{t,l}\bigg) - \mathbb{E}\bigg(A_t e^{iu\hat{Z}_{t,r}}\prod_{l=1}^{r-1}\psi_{t,l}\bigg)\mathbb{E}\big(e^{iu\tilde{Z}_{t,r}}\big)\bigg| \\
&\qquad + \bigg|\mathbb{E}\bigg(A_t e^{iu\hat{Z}_{t,r}}\prod_{l=1}^{r-1}\psi_{t,l}\bigg) - \mathbb{E}\bigg(A_t\prod_{l=1}^{r-1}\psi_{t,l}\bigg)\mathbb{E}\big(e^{iu\hat{Z}_{t,r}}\big)\bigg| \\
&\qquad + \big|\mathbb{E}\big(e^{iuZ_{t,r}}\big) - \mathbb{E}\big(e^{iu\hat{Z}_{t,r}}\big)\mathbb{E}\big(e^{iu\tilde{Z}_{t,r}}\big)\big||\varphi_{t,r-1}|\,.
\end{aligned}$$

By the Davydov's inequality, it holds that $|\mathbb{E}(e^{iuZ_{t,r}}) - \mathbb{E}(e^{iu\hat{Z}_{t,r}})\mathbb{E}(e^{iu\tilde{Z}_{t,r}})| \lesssim \alpha_Q(2rh)$,

$$\begin{aligned}
&\bigg|\mathbb{E}\bigg(A_t e^{iuZ_{t,r}}\prod_{l=1}^{r-1}\psi_{t,l}\bigg) - \mathbb{E}\bigg(A_t e^{iu\hat{Z}_{t,r}}\prod_{l=1}^{r-1}\psi_{t,l}\bigg)\mathbb{E}\big(e^{iu\tilde{Z}_{t,r}}\big)\bigg| \\
&\qquad \lesssim \{\alpha_Q(h)\}^{(s_1-1)/s_1}\bigg\{\mathbb{E}\bigg(|A_t|^{s_1}\prod_{l=1}^{r-1}|\psi_{t,l}|^{s_1}\bigg)\bigg\}^{1/s_1}
\end{aligned}$$

and

$$\begin{aligned}
&\bigg|\mathbb{E}\bigg(A_t e^{iu\hat{Z}_{t,r}}\prod_{l=1}^{r-1}\psi_{t,l}\bigg) - \mathbb{E}\bigg(A_t\prod_{l=1}^{r-1}\psi_{t,l}\bigg)\mathbb{E}\big(e^{iu\hat{Z}_{t,r}}\big)\bigg| \\
&\qquad \lesssim \{\alpha_Q(h)\}^{(s_1-1)/s_1}\bigg\{\mathbb{E}\bigg(|A_t|^{s_1}\prod_{l=1}^{r-1}|\psi_{t,l}|^{s_1}\bigg)\bigg\}^{1/s_1}\,.
\end{aligned}$$

Recall $\psi_{t,l} = e^{iu(Z_{t,l-1}-Z_{t,l})} - 1$. Then (E.5) yields that

$$\mathbb{E}\bigg(|A_t|^{s_1}\prod_{l=1}^{r-1}|\psi_{t,l}|^{s_1}\bigg) \le 2^{s_1(r-1)}\mathbb{E}(|A_t|^{s_1}) \le \frac{2^{s_1(r-1)}d}{B_m^{s_1}}\,,$$

which implies that

$$\left|\mathbb{E}\left(A_t e^{iuZ_{t,r}}\prod_{l=1}^{r-1}\psi_{t,l}\right)-\mathbb{E}\left(A_t e^{iu\hat{Z}_{t,r}}\prod_{l=1}^{r-1}\psi_{t,l}\right)\mathbb{E}\left(e^{iu\tilde{Z}_{t,r}}\right)\right|\lesssim\frac{2^{r-1}d^{1/s_1}}{B_m}\{\alpha_Q(h)\}^{(s_1-1)/s_1}$$

and

$$\left|\mathbb{E}\left(A_t e^{iu\hat{Z}_{t,r}}\prod_{l=1}^{r-1}\psi_{t,l}\right)-\mathbb{E}\left(A_t\prod_{l=1}^{r-1}\psi_{t,l}\right)\mathbb{E}\left(e^{iu\hat{Z}_{t,r}}\right)\right|\lesssim\frac{2^{r-1}d^{1/s_1}}{B_m}\{\alpha_Q(h)\}^{(s_1-1)/s_1}\,.$$

On the other hand, it follows from Lemma L2 that

$$|\varphi_{t,r-1}|\lesssim\frac{d^{1/s_1}}{B_m}\left[\frac{d^{2/s_1}h^2u^2}{16^r B_m^2}+\frac{r}{2^r}\{\alpha_Q(h)\}^{(s_1-2)/s_1}\right]\qquad(r=3,\ldots,q)$$

and

$$|\varphi_{t,1}|\leq\frac{2|u|hd^{2/s_1}}{B_m^2}$$

for any $t=1,\ldots,m$. Thus,

$$\sum_{r=2}^{q}\sum_{t=1}^{m}\left|\mathbb{E}\left(A_t e^{iuZ_{t,r}}\prod_{l=1}^{r-1}\psi_{t,l}\right)-\mathbb{E}\left(A_t\prod_{l=1}^{r-1}\psi_{t,l}\right)\mathbb{E}\left(e^{iuZ_{t,r}}\right)\right|\lesssim\frac{2^q md^{1/s_1}}{B_m}\{\alpha_Q(h)\}^{(s_1-1)/s_1}\,.$$

By (E.6), we have $2^q\{\alpha_Q(h)\}^{1/s_1}\leq 2^{-2q}<1$, which implies

$$\sum_{r=2}^{q}\sum_{t=1}^{m}\left|\mathbb{E}\left(A_t e^{iuZ_{t,r}}\prod_{l=1}^{r-1}\psi_{t,l}\right)-\mathbb{E}\left(A_t\prod_{l=1}^{r-1}\psi_{t,l}\right)\mathbb{E}\left(e^{iuZ_{t,r}}\right)\right|\lesssim\frac{md^{1/s_1}}{B_m}\{\alpha_Q(h)\}^{(s_1-2)/s_1}\,.$$

We complete the proof of Lemma L4. □

Lemma L5. *Let*

$$U_{**}=\min\left\{\left(\frac{12C}{B_m^{s_1}}mdh^{s_1-1}\right)^{-1/(s_1-2)},\frac{1}{12a_3},\frac{1}{a_1}\right\}$$

for some sufficiently large $C>0$, where a_1,a_2,a_3,a_4,b_1 and b_2 will be defined in the proof below. Under the null hypothesis H_0 with Conditions 1 *and* 2 *being satisfied, if $16^q\geq m^{1/2}$, $K_*=O(h)$, and* (E.6) *is satisfied, then it holds that*

$$\Delta_m\lesssim a_1+a_2+a_3+a_4+b_1+b_2U_{**}+U_{**}^{-1}$$

provided that $B_m^{-2}m^{3/2}d^{2/s_1}K_*^{1/2}\{\alpha_Q(h)\}^{(s_1-2)/(2s_1)}$ is sufficiently small, where d is specified in (E.3).

Proof. Let

$$\theta_2(u) = i\bigg\{\sum_{t=1}^{m}\varphi_{t,1}\mathbb{E}(\eta_{t,2}) + \sum_{t=1}^{m}\sum_{r=3}^{q}\varphi_{t,r-1}\mathbb{E}(\eta_{t,r}+1)\bigg\}\,.$$

It follows from Lemma L3 that

$$|\theta_2(u)| \lesssim \frac{d^{1/s_1}m}{B_m}\bigg[\frac{d^{2/s_1}h^2u^2}{B_m^2} + \{\alpha_Q(h)\}^{(s_1-2)/s_1}\bigg]\,.$$

Based on (E.2), it follows from Lemmas L3 and L4 that

$$f_m'(u) = \bigg\{i\sum_{t=1}^{m}\varphi_{t,1} + \theta_2(u)\bigg\}f_m(u) + R(u)$$

for any $|u| \leq U_*$, where

$$|R(u)| \lesssim \frac{d^{3/s_1}m^{1/2}h^2u^2}{B_m^3}(h^{1/2}+\tilde{\alpha}^{1/2}) + \frac{d^{1/s_1}m}{B_m}\{\alpha_Q(h)\}^{(s_1-2)/s_1}$$

for any $|u| \leq U_*$. Together with Lemma L1, we have

$$f_m'(u) = \{-u + \theta_1(u) + \theta_2(u)\}f_m(u) + R(u) \tag{E.11}$$

with

$$|\theta_1(u)| \lesssim \frac{|u|}{B_m^2}m^{3/2}d^{2/s_1}K_*^{1/2}\{\alpha_Q(h)\}^{(s_1-2)/(2s_1)} + \frac{|u|^{s_1-1}}{B_m^{s_1}}mdh^{s_1-1}\,.$$

Let $\theta(u) = \theta_1(u) + \theta_2(u)$. By solving the linear differential equation (E.11), we have that

$$f_m(u) = \exp\bigg\{-\frac{u^2}{2} + \int_0^u \theta(w)\,\mathrm{d}w\bigg\}\cdot\bigg[1 + \int_0^u R(w)\exp\bigg\{\frac{w^2}{2} - \int_0^w \theta(v)\,\mathrm{d}v\bigg\}\,\mathrm{d}w\bigg]$$

for any $|u| \leq U_*$, which implies

$$\begin{aligned}
\big|f_m(u) - e^{-u^2/2}\big| &\leq \bigg|\exp\bigg\{-\frac{u^2}{2} + \int_0^u \theta(w)\,\mathrm{d}w\bigg\} - e^{-u^2/2}\bigg| \\
&\quad + \bigg|e^{-u^2/2}\exp\bigg\{\int_0^u \theta(w)\,\mathrm{d}w\bigg\}\int_0^u R(w)\exp\bigg\{\frac{w^2}{2} - \int_0^w \theta(v)\,\mathrm{d}v\bigg\}\,\mathrm{d}w\bigg| \\
&\leq \exp\bigg\{-\frac{u^2}{2} + \bigg|\int_0^u \theta(w)\,\mathrm{d}w\bigg|\bigg\} \times \bigg|\int_0^u \theta(w)\,\mathrm{d}w\bigg| \\
&\quad + \bigg|\int_0^u R(w)\exp\bigg\{-\frac{u^2}{2} + \frac{w^2}{2} + \int_w^u \theta(v)\,\mathrm{d}v\bigg\}\,\mathrm{d}w\bigg|
\end{aligned}$$

for any $|u| \leq U_*$. We will bound the two terms on the right-hand side of above inequality, respectively.

Clearly, if $|u| \le U_*$, it holds that

$$
\begin{aligned}
\bigg|\int_0^u \theta(w)\,\mathrm{d}w\bigg| &\le \int_{-|u|}^{|u|} |\theta(w)|\,\mathrm{d}w \\
&\le \frac{C_* u^2}{B_m^2} m^{3/2} d^{2/s_1} K_*^{1/2} \{\alpha_Q(h)\}^{(s_1-2)/(2s_1)} + \frac{C_*|u|^{s_1}}{B_m^{s_1}} m d h^{s_1-1} \\
&\quad + \frac{C_* m d^{1/s_1}}{B_m}\bigg[\frac{d^{2/s_1} h^2 |u|^3}{B_m^2} + \{\alpha_Q(h)\}^{(s_1-2)/s_1}|u|\bigg] \\
&=: a_1|u| + a_2 u^2 + a_3|u|^3 + a_4|u|^{s_1}
\end{aligned}
$$

for some sufficiently large $C_* > 0$. Since $B_m^{-2} m^{3/2} d^{2/s_1} K_*^{1/2}\{\alpha_Q(h)\}^{(s_1-2)/(2s_1)}$ is sufficiently small, we can have

$$
a_2 = \frac{C_*}{B_m^2} m^{3/2} d^{2/s_1} K_*^{1/2} \{\alpha_Q(h)\}^{(s_1-2)/(2s_1)} \le \frac{1}{12}\,. \tag{E.12}
$$

Let

$$
|u| \le U_{**} = \min\bigg\{\bigg(\frac{12C}{B_m^{s_1}} m d h^{s_1-1}\bigg)^{-1/(s_1-2)}, \frac{1}{12a_3}, \frac{1}{a_1}\bigg\} \tag{E.13}
$$

for some sufficiently large $C > 0$. By the Davydov's inequality,

$$
\begin{aligned}
B_m^2 = m\mathrm{Var}\bigg(\frac{1}{\sqrt{m}}\sum_{t=1}^m Q_t\bigg) &= \sum_{t=1}^m \mathbb{E}(|Q_t|^2) + \sum_{t_1\ne t_2} \mathbb{E}(Q_{t_1}Q_{t_2}) \\
\lesssim \sum_{t=1}^m \mathbb{E}(|Q_t|^2) &+ \sum_{t_1\ne t_2} \{\mathbb{E}(|Q_{t_1}|^{s_1})\}^{1/s_1}\{\mathbb{E}(|Q_{t_2}|^{s_1})\}^{1/s_1}\{\alpha_Q(|t_1-t_2|)\}^{1-2/s_1}\,.
\end{aligned}
$$

Recall $Q_t = B_m A_t$ and $d = 4^{s_1} c_1 K_*^{s_1}$. It follows from (E.5) that

$$
\begin{aligned}
B_m^2 &\lesssim m d^{2/s_1} + d^{2/s_1}\sum_{t_1\ne t_2}\{\alpha_Q(|t_1-t_2|)\}^{1-2/s_1} \\
&= m d^{2/s_1} + 2d^{2/s_1}\sum_{\tau=1}^{m-1}(m-\tau)\{\alpha_Q(\tau)\}^{1-2/s_1} \\
&\le m d^{2/s_1} + 2m d^{2/s_1}\tilde{\alpha} \lesssim m K_*^3\,.
\end{aligned}
$$

Recall $U_* = B_m/(32hd^{1/s_1})$ and $K_* = O(h)$. Then $U_{**} \le U_*$ for sufficiently large $C > 0$ specified in (E.13). Under (E.12) and (E.13), we have that $|\int_0^u \theta(w)\,\mathrm{d}w| \le u^2/4 + 1$ for any $|u| \le U_{**}$. More generally, we have $|\int_w^u \theta(w)\,\mathrm{d}w| \le (u^2-w^2)/4 + 1$ for any $|u| \le U_{**}$. Therefore,

$$
\big|f_m(u) - e^{-u^2/2}\big| \lesssim (a_1|u| + a_2u^2 + a_3|u|^3 + a_4|u|^{s_1})e^{-u^2/4}
$$

$$+\bigg|\int_0^u |R(w)|\exp\bigg(-\frac{u^2}{4}+\frac{w^2}{4}\bigg)\,\mathrm{d}w\bigg| \tag{E.14}$$

for any $|u|\leq U_{**}$. It follows from the facts $\tilde{\alpha}\lesssim K_*$ and $K_*=O(h)$ that

$$\begin{aligned}|R(u)| &\leq \frac{C_{**}d^{3/s_1}m^{1/2}h^{5/2}u^2}{B_m^3}+\frac{C_{**}d^{1/s_1}m}{B_m}\{\alpha_Q(h)\}^{(s_1-2)/s_1}\\ &=: b_1+b_2u^2\end{aligned}$$

for any $|u|\leq U_{**}$, where $C_{**}>0$ is a sufficiently large constant. According to integration by parts, it holds that

$$\int_0^{|u|} w^2e^{w^2/4}\,\mathrm{d}w\leq 2|u|e^{u^2/4},\qquad \int_0^{|u|} e^{w^2/4}\,\mathrm{d}w\leq \min(2|u|^{-1},|u|)\cdot e^{u^2/4}$$

and

$$\int_{-|u|}^0 w^2e^{w^2/4}\,\mathrm{d}w\leq 2|u|e^{u^2/4},\qquad \int_{-|u|}^0 e^{w^2/4}\,\mathrm{d}w\leq \min(2|u|^{-1},|u|)\cdot e^{u^2/4}\,,$$

which implies that

$$\bigg|\int_0^u |R(w)|\exp\bigg(-\frac{u^2}{4}+\frac{w^2}{4}\bigg)\,\mathrm{d}w\bigg|\leq b_1\min(2|u|^{-1},|u|)+2b_2|u|$$

for any $|u|\leq U_{**}$. Therefore, (E.14) leads to

$$\begin{aligned}\big|f_m(u)-e^{-u^2/2}\big| &\leq (a_1|u|+a_2u^2+a_3|u|^3+a_4|u|^s)e^{-u^2/4}\\ &\quad+b_1\min(2|u|^{-1},|u|)+2b_2|u|\end{aligned} \tag{E.15}$$

for any $|u|\leq U_{**}$.

Denote by $F_m(x)$ the distribution function of $B_m^{-1}\sum_{t=1}^m Q_t$. By the Essen's inequality [Theorem 1.5.2 of Ibragimov and Linik (1971)], we have

$$\Delta_m=\sup_{-\infty<x<\infty}|F_m(x)-\Phi(x)|\leq\frac{1}{\pi}\int_{-U_{**}}^{U_{**}}\bigg|\frac{f_m(u)-e^{-u^2/2}}{u}\bigg|\,\mathrm{d}u+\frac{C}{U_{**}}\,.$$

It follows from (E.15) that

$$\begin{aligned}\Delta_m &\leq \frac{1}{\pi}\int_{-U_{**}}^{U_{**}}(a_1+a_2|u|+a_3|u|^2+a_4|u|^{s_1-1})e^{-u^2/4}\,\mathrm{d}u\\ &\quad+\frac{1}{\pi}\int_{-U_{**}}^{U_{**}}\{b_1\min(2|u|^{-2},1)+2b_2\}\,\mathrm{d}u+\frac{C}{U_{**}}\\ &\lesssim a_1+a_2+a_3+a_4+b_1+b_2U_{**}+U_{**}^{-1}\,.\end{aligned}$$

We complete the proof. □

Now, we begin to simplify the upper bound for Δ_m specified in Lemma L5. Recall that $a_1 \asymp B_m^{-1} m d^{1/s_1} \{\alpha_Q(h)\}^{(s_1-2)/s_1}$, $a_2 \asymp B_m^{-2} m^{3/2} d^{2/s_1} K_*^{1/2} \{\alpha_Q(h)\}^{(s_1-2)/(2s_1)}$, $a_3 \asymp B_m^{-3} m d^{3/s_1} h^2$, $a_4 \asymp B_m^{-s_1} m d h^{s_1-1}$, $b_1 \asymp a_1$ and $b_2 \asymp B_m^{-3} d^{3/s_1} m^{1/2} h^{5/2}$ with d specified in (E.3). Notice that $B_m^2 \geq c_0 m$, $d = 4^{s_1} c_1 K_*^{s_1}$ and $\alpha_Q(\tau) \leq \alpha(|\tau - K_0|_+)$ with $\alpha(\tau) \leq c_2 \tau^{-\beta_1}$, then $a_1 \lesssim m^{1/2} K_* |h - K_0|_+^{-\beta_1(s_1-2)/s_1}$, $a_2 \lesssim m^{1/2} K_*^{5/2} |h - K_0|_+^{-\beta_1(s_1-2)/(2s_1)}$, $a_3 \lesssim m^{-1/2} K_*^3 h^2$, $a_4 \lesssim m^{-(s_1-2)/2} K_*^{s_1} h^{s_1-1}$ and $b_1 \lesssim m^{1/2} K_* |h - K_0|_+^{-\beta_1(s_1-2)/s_1}$. Recall $2qh \leq m+1$ and $q \geq 2$. Thus,

$$\Delta_m \lesssim m^{1/2} K_*^{5/2} |h - K_0|_+^{-\beta_1(s_1-2)/(2s_1)} + m^{-1/2} K_*^3 h^2 + m^{-(s_1-2)/2} K_*^{s_1} h^{s_1-1} + b_2 U_{**} + U_{**}^{-1} .$$

To make $\Delta_m \to 0$, it suffices to require $h - K_0 \to \infty$ and $m^{-(s_1-2)/2} K_*^{s_1} h^{s_1-1} = o(1)$. Due to $K_* h = o(m^{1/2})$ and $s_1 \in (2, 3]$, we know $(m^{-1/2} K_*^3 h^2)/\{m^{-(s_1-2)/2} K_*^{s_1} h^{s_1-1}\} \lesssim 1$. Then,

$$\Delta_m \lesssim m^{1/2} K_*^{5/2} |h - K_0|_+^{-\beta_1(s_1-2)/(2s_1)} + m^{-(s_1-2)/2} K_*^{s_1} h^{s_1-1} + b_2 U_{**} + U_{**}^{-1} . \tag{E.16}$$

Notice that

$$U_{**} = \min\left\{ \left(\frac{12C}{B_m^{s_1}} m d h^{s_1-1} \right)^{-1/(s_1-2)}, \frac{1}{12a_3}, \frac{1}{a_1} \right\}$$

for some sufficiently large $C > 0$, $a_1 \asymp B_m^{-1} m d^{1/s_1} \{\alpha_Q(h)\}^{(s_1-2)/s_1}$ and $a_3 \asymp B_m^{-3} m d^{3/s_1} h^2$. Recall $U_* = B_m/(32 h d^{1/s_1})$. As we have shown in the proof of Lemma L5 that $U_{**} \leq U_*$, we have

$$\begin{aligned} b_2 U_{**} + U_{**}^{-1} &\lesssim \frac{m^{1/2} h^{5/2} d^{3/s_1}}{B_m^3} \cdot \frac{B_m}{h d^{1/s_1}} + \left(\frac{B_m^{s_1}}{m d h^{s_1-1}} \right)^{-1/(s_1-2)} + a_1 + a_3 \\ &= \frac{m^{1/2} h^{3/2} d^{2/s_1}}{B_m^2} + \left(\frac{B_m^{s_1}}{m d h^{s_1-1}} \right)^{-1/(s_1-2)} + a_1 + a_3 \\ &\lesssim m^{-1/2} h^{3/2} K_*^2 + \left(\frac{B_m^{s_1}}{m K_*^{s_1} h^{s_1-1}} \right)^{-1/(s_1-2)} + a_1 + a_3 \end{aligned}$$

The last step is based on the facts $B_m^2 \geq c_0 m$ and $d \asymp K_*^{s_1}$. Since $B_m^2 \geq c_0 m$, it holds that $B_m^{-s_1} m K_*^{s_1} h^{s_1-1} = O\{m^{-(s_1-2)/2} K_*^{s_1} h^{s_1-1}\}$. Due to $s_1 \in (2, 3]$ and $m^{-(s_1-2)/2} K_*^{s_1} h^{s_1-1} = o(1)$, then $B_m^{-s_1/(s_1-2)} m^{1/(s_1-2)} K_*^{s_1/(s_1-2)} h^{(s_1-1)/(s_1-2)} \lesssim B_m^{-s_1} m K_*^{s_1} h^{s_1-1} \lesssim m^{-(s_1-2)/2} K_*^{s_1} h^{s_1-1}$. Since $h - K_0 \to \infty$ and $(m^{-1/2} K_*^3 h^2)/\{m^{-(s_1-2)/2} K_*^{s_1} h^{s_1-1}\} \lesssim 1$, it holds that $m^{-1/2} h^{3/2} K_*^2 = o\{m^{-(s_1-2)/2} K_*^{s_1} h^{s_1-1}\}$. Together with (E.16), we have

$$\Delta_m \lesssim m^{1/2} K_*^{5/2} |h - K_0|_+^{-\beta_1(s_1-2)/(2s_1)} + m^{-(s_1-2)/2} K_*^{s_1} h^{s_1-1} . \tag{E.17}$$

We set $h = K_0 + \lfloor m^\zeta \rfloor$ for some $\zeta > 0$. Then (E.17) implies that

$$\Delta_m \lesssim m^{1/2} K_*^{5/2} m^{-\beta_1 \zeta (s_1-2)/(2s_1)} + m^{-(s_1-2)/2} K_*^{2s_1-1} + m^{-(s_1-2)/2} K_*^{s_1} m^{(s_1-1)\zeta} .$$

Write $\beta_1 = 2\beta(s_1-1)s_1/(s_1-2)^2$ for some $\beta > 1$. Then

$$\Delta_m \lesssim m^{1/2}K_*^{5/2}m^{-\beta\zeta(s_1-1)/(s_1-2)} + m^{-(s_1-2)/2}K_*^{2s_1-1} + m^{-(s_1-2)/2}K_*^{s_1}m^{(s_1-1)\zeta}\,.$$

Choosing $\zeta = (s_1-2)/\{(\beta+1)(s_1-1)\}$, we have

$$\Delta_m \lesssim K_*^{5/2}m^{-(\beta-1)/(2\beta+2)} + K_*^{2s_1-1}m^{-(s_1-2)/2} + K_*^{s_1}m^{-(\beta-1)(s_1-2)/(2\beta+2)}\,.$$

To make $\Delta_m \to 0$, we need to require $K_*^{s_1}m^{-(\beta-1)(s_1-2)/(2\beta+2)} = o(1)$. Together with the fact $s_1 \in (2,3]$, we know $K_*^{5/2}m^{-(\beta-1)/(2\beta+2)} \lesssim K_*^{s_1}m^{-(\beta-1)(s_1-2)/(2\beta+2)}$. Therefore,

$$\Delta_m \lesssim K_*^{2s_1-1}m^{-(s_1-2)/2} + K_*^{s_1}m^{-(\beta-1)(s_1-2)/(2\beta+2)}\,. \tag{E.18}$$

Recall Lemma L5 requires $16^q \geq m^{1/2}$, $2^{3q}\{\alpha_Q(h)\}^{1/s_1} \leq 1$, and $B_m^{-2}m^{3/2}d^{2/s_1}K_*^{1/2}\{\alpha_Q(h)\}^{(s_1-2)/(2s_1)}$ is sufficiently small. Notice that $B_m^{-2}m^{3/2}d^{2/s_1}K_*^{1/2}\{\alpha_Q(h)\}^{(s_1-2)/(2s_1)} \lesssim K_*^{5/2}m^{-(\beta-1)/(2\beta+2)}$. Thus $B_m^{-2}m^{3/2}d^{2/s_1}K_*^{1/2}\{\alpha_Q(h)\}^{(s_1-2)/(2s_1)}$ is sufficiently small provided that $K_*^{s_1}m^{-(\beta-1)(s_1-2)/(2\beta+2)} = o(1)$. Select $q = (\log_2 m)/8$. Then $16^q \geq m^{1/2}$ holds automatically, and $2^{3q}\{\alpha_Q(h)\}^{1/s_1} \lesssim m^{3/8-\zeta\beta_1/s_1}$. Due to $3/8-\zeta\beta_1/s_1 = 3/8-2\beta/\{(\beta+1)(s_1-2)\} < 0$, we know $2^{3q}\{\alpha_Q(h)\}^{1/s_1} \leq 1$ also holds for such selected q and h. Hence, (E.18) holds provided that $K_*^{s_1}m^{-(\beta-1)(s_1-2)/(2\beta+2)} = o(1)$. We complete the proof of Lemma 1. □

References

Ibragimov, I. A. and Linnik, Yu. V. (1971) Independent and stationary sequences of random variables. Groningen : Wolters-Noordhoff.

Table S1: Empirical sizes ($\times 10^2$) of the proposed test T_n defined as (6) for $K_0 = 1$ with the untruncated critical value ($c_\kappa = \infty$) and the truncated critical values defined as (18) with $c_\kappa = 0.45, 0.55, 0.65$, and the KPSS test in a simulation with 2000 replications. Constant c_κ determines the level of truncation for the critical values of T_n. The innovations $\epsilon_t \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \sigma_\epsilon^2)$. The nominal size of the tests is 5%.

			$\sigma_\epsilon^2 = 1$					$\sigma_\epsilon^2 = 2$				
	Setting	N	∞	0.45	0.55	0.65	KPSS	∞	0.45	0.55	0.65	KPSS
Model 1	0.5	40	6.9	6.9	6.9	6.9	10.2	6.0	6.0	6.0	6.0	11.5
		70	6.1	6.1	6.1	6.1	10.0	6.5	6.5	6.5	6.5	9.8
		100	6.7	6.7	6.7	6.7	9.5	4.5	4.5	4.5	4.5	9.0
	0.9	40	9.3	42.4	29.6	21.3	51.7	8.9	39.9	27.3	19.1	50.0
		70	9.2	24.4	15.2	11.6	48.6	9.8	22.6	14.3	11.4	47.0
		100	8.4	13.1	9.6	8.8	51.5	8.9	14.2	10.1	9.1	52.8
	−0.5	40	7.3	7.3	7.3	7.3	1.8	7.5	7.5	7.5	7.5	1.9
		70	6.2	6.2	6.2	6.2	2.1	7.0	7.0	7.0	7.0	1.8
		100	6.6	6.6	6.6	6.6	2.0	6.5	6.5	6.5	6.5	1.8
Model 2	(0.8, 0.3)	40	6.9	6.9	6.9	6.9	6.9	7.5	7.5	7.5	7.5	7.8
		70	6.3	6.3	6.3	6.3	6.6	5.8	5.8	5.8	5.8	7.9
		100	6.8	6.8	6.8	6.8	6.2	7.1	7.1	7.1	7.1	6.6
	(0.9, 0.5)	40	6.5	6.5	6.5	6.5	7.6	7.0	7.0	7.0	7.0	8.4
		70	6.8	6.8	6.8	6.8	8.2	6.3	6.3	6.3	6.3	7.3
		100	6.0	6.0	6.0	6.0	7.5	6.1	6.1	6.1	6.1	7.9
	(0.95, 0.9)	40	7.8	7.8	7.8	7.8	9.8	6.6	6.6	6.6	6.6	7.8
		70	6.5	6.5	6.5	6.5	7.0	6.8	6.8	6.8	6.8	7.1
		100	6.8	6.8	6.8	6.8	7.1	6.6	6.6	6.6	6.6	7.4
Model 3	(0.4, 0.2)	40	8.3	9.2	8.6	8.4	20.8	8.8	9.9	8.9	8.8	20.8
		70	7.0	7.0	7.0	7.0	17.1	7.3	7.3	7.3	7.3	18.3
		100	7.3	7.3	7.3	7.3	15.8	6.9	6.9	6.9	6.9	18.5
	(0.5, 0.1)	40	7.9	8.2	8.0	7.9	18.5	9.2	9.5	9.2	9.2	19.9
		70	7.5	7.5	7.5	7.5	17.8	7.6	7.6	7.6	7.6	15.8
		100	6.3	6.3	6.3	6.3	17.1	8.3	8.3	8.3	8.3	14.6
	(0.6, 0.1)	40	9.8	13.0	10.2	10.0	27.9	11.5	15.4	12.6	11.8	24.3
		70	8.9	8.9	8.9	8.9	23.5	8.8	8.9	8.8	8.8	20.1
		100	7.8	7.8	7.8	7.8	21.1	7.1	7.1	7.1	7.1	20.2
Model 4	0.4	40	7.8	8.2	7.9	7.8	19.9	8.5	8.8	8.5	8.5	18.3
		70	7.8	7.8	7.8	7.8	15.7	7.4	7.4	7.4	7.4	15.2
		100	8.0	8.0	8.0	8.0	14.0	6.5	6.5	6.5	6.5	15.8
	0.5	40	8.3	9.6	8.5	8.3	19.9	7.5	8.5	7.6	7.5	17.5
		70	6.7	6.7	6.7	6.7	14.6	6.4	6.4	6.4	6.4	15.7
		100	7.5	7.5	7.5	7.5	15.2	7.0	7.0	7.0	7.0	16.1
	0.6	40	8.2	9.6	8.6	8.2	19.2	8.0	9.2	8.1	8.0	20.4
		70	7.1	7.1	7.1	7.1	14.1	7.7	7.7	7.7	7.7	16.0
		100	7.7	7.7	7.7	7.7	14.4	6.0	6.0	6.0	6.0	16.2

Table S2: Empirical sizes ($\times 10^2$) of the proposed test T_n defined as (6) for $K_0 = 2$ with the untruncated critical value ($c_\kappa = \infty$) and the truncated critical values defined as (18) with $c_\kappa = 0.45, 0.55, 0.65$, and the KPSS test in a simulation with 2000 replications. Constant c_κ determines the level of truncation for the critical values of T_n. The innovations $\epsilon_t \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \sigma_\epsilon^2)$. The nominal size of the tests is 5%.

			$\sigma_\epsilon^2 = 1$					$\sigma_\epsilon^2 = 2$				
	Setting	N	∞	0.45	0.55	0.65	KPSS	∞	0.45	0.55	0.65	KPSS
Model 1	0.5	40	6.9	6.9	6.9	6.9	10.3	5.9	5.9	5.9	5.9	11.1
		70	6.2	6.2	6.2	6.2	10.3	5.9	5.9	5.9	5.9	9.0
		100	6.4	6.4	6.4	6.4	9.6	5.7	5.7	5.7	5.7	8.5
	0.9	40	9.5	39.6	28.2	20.3	47.6	11.1	42.0	29.8	22.1	51.0
		70	9.7	24.6	16.1	11.7	48.5	9.4	26.1	15.8	11.8	49.6
		100	8.9	14.5	10.2	9.3	50.2	8.8	14.7	10.2	9.1	50.3
	−0.5	40	7.6	7.6	7.6	7.6	1.8	8.2	8.2	8.2	8.2	1.8
		70	6.5	6.5	6.5	6.5	2.2	6.4	6.4	6.4	6.4	2.4
		100	7.0	7.0	7.0	7.0	2.8	6.6	6.6	6.6	6.6	1.7
Model 2	(0.8, 0.3)	40	6.4	6.4	6.4	6.4	7.9	6.3	6.3	6.3	6.3	6.3
		70	6.2	6.2	6.2	6.2	7.8	6.0	6.0	6.0	6.0	7.1
		100	5.9	5.9	5.9	5.9	7.0	6.6	6.6	6.6	6.6	6.2
	(0.9, 0.5)	40	7.1	7.1	7.1	7.1	7.0	6.6	6.6	6.6	6.6	7.7
		70	5.8	5.8	5.8	5.8	7.1	7.0	7.0	7.0	7.0	6.6
		100	5.9	5.9	5.9	5.9	7.0	5.7	5.7	5.7	5.7	7.1
	(0.95, 0.9)	40	8.3	8.3	8.3	8.3	8.3	6.6	6.6	6.6	6.6	9.8
		70	6.8	6.8	6.8	6.8	8.8	7.8	7.8	7.8	7.8	8.0
		100	6.6	6.6	6.6	6.6	7.7	5.7	5.7	5.7	5.7	6.0
Model 3	(0.4, 0.2)	40	10.2	10.8	10.3	10.2	20.9	9.1	10.2	9.3	9.2	20.1
		70	6.7	6.7	6.7	6.7	19.7	7.0	7.0	7.0	7.0	18.6
		100	6.7	6.7	6.7	6.7	18.6	7.1	7.1	7.1	7.1	17.4
	(0.5, 0.1)	40	8.7	9.3	8.8	8.7	18.6	9.0	9.3	9.0	9.0	20.8
		70	8.3	8.3	8.3	8.3	17.1	7.4	7.4	7.4	7.4	16.2
		100	7.5	7.5	7.5	7.5	17.0	6.8	6.8	6.8	6.8	17.0
	(0.6, 0.1)	40	11.1	14.7	11.8	11.1	26.6	10.3	13.7	11.3	10.5	25.9
		70	9.2	9.3	9.2	9.2	21.1	9.0	9.1	9.0	9.0	21.0
		100	7.4	7.4	7.4	7.4	20.8	9.5	9.5	9.5	9.5	23.2
Model 4	0.4	40	8.0	8.5	8.0	8.0	19.4	7.9	8.6	8.1	8.0	18.2
		70	6.9	6.9	6.9	6.9	14.9	7.6	7.6	7.6	7.6	15.3
		100	6.7	6.7	6.7	6.7	14.4	7.5	7.5	7.5	7.5	15.2
	0.5	40	7.6	8.5	7.8	7.6	20.1	7.2	8.6	7.4	7.2	19.1
		70	7.0	7.0	7.0	7.0	15.3	7.2	7.2	7.2	7.2	14.8
		100	6.6	6.6	6.6	6.6	16.2	6.8	6.8	6.8	6.8	16.9
	0.6	40	8.2	9.8	8.5	8.2	20.2	8.2	9.5	8.5	8.3	18.3
		70	8.0	8.0	8.0	8.0	15.6	7.2	7.2	7.2	7.2	15.0
		100	7.0	7.0	7.0	7.0	14.1	6.9	6.9	6.9	6.9	15.6

Table S3: Empirical sizes ($\times 10^2$) of the proposed test T_n defined as (6) for $K_0 = 3$ with the untruncated critical value ($c_\kappa = \infty$) and the truncated critical values defined as (18) with $c_\kappa = 0.45, 0.55, 0.65$, and the KPSS test in a simulation with 2000 replications. Constant c_κ determines the level of truncation for the critical values of T_n. The innovations $\epsilon_t \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \sigma_\epsilon^2)$. The nominal size of the tests is 5%.

			$\sigma_\epsilon^2 = 1$					$\sigma_\epsilon^2 = 2$				
	Setting	N	∞	0.45	0.55	0.65	KPSS	∞	0.45	0.55	0.65	KPSS
Model 1	0.5	40	6.5	6.5	6.5	6.5	11.7	6.2	6.2	6.2	6.2	9.8
		70	5.4	5.4	5.4	5.4	9.5	5.3	5.3	5.3	5.3	9.0
		100	5.3	5.3	5.3	5.3	10.0	5.8	5.8	5.8	5.8	10.1
	0.9	40	9.6	41.7	29.8	21.3	52.8	11.2	42.4	30.5	21.9	50.5
		70	10.2	27.0	16.9	12.1	47.5	10.6	25.7	16.1	12.7	48.8
		100	9.8	14.5	10.8	10.0	49.6	10	15.4	11.5	10.2	50.7
	−0.5	40	7.5	7.5	7.5	7.5	1.5	7.2	7.2	7.2	7.2	1.4
		70	5.9	5.9	5.9	5.9	2.4	5.3	5.3	5.3	5.3	2.1
		100	6.3	6.3	6.3	6.3	2.1	6.8	6.8	6.8	6.8	2.2
Model 2	(0.8, 0.3)	40	7.6	7.6	7.6	7.6	6.3	6.5	6.5	6.5	6.5	8.4
		70	5.8	5.8	5.8	5.8	6.3	6.2	6.2	6.2	6.2	6.3
		100	6.0	6.0	6.0	6.0	7.2	5.9	5.9	5.9	5.9	6.7
	(0.9, 0.5)	40	7.4	7.4	7.4	7.4	7.0	6.4	6.4	6.4	6.4	8.3
		70	6.8	6.8	6.8	6.8	7.3	7.0	7.0	7.0	7.0	7.9
		100	5.7	5.7	5.7	5.7	7.0	6.2	6.2	6.2	6.2	8.0
	(0.95, 0.9)	40	7.8	7.8	7.8	7.8	9.2	7.5	7.5	7.5	7.5	8.5
		70	6.6	6.6	6.6	6.6	7.8	6.2	6.2	6.2	6.2	8.7
		100	6.0	6.0	6.0	6.0	7.8	6.2	6.2	6.2	6.2	7.5
Model 3	(0.4, 0.2)	40	9.2	9.8	9.2	9.2	22.1	8.8	9.9	9.2	8.9	21.4
		70	8.0	8.1	8.0	8.0	16.9	7.5	7.5	7.5	7.5	18.1
		100	7.1	7.1	7.1	7.1	18.7	7.5	7.5	7.5	7.5	17.5
	(0.5, 0.1)	40	10.1	10.4	10.2	10.1	20.9	9.2	9.8	9.2	9.2	18.9
		70	8.6	8.6	8.6	8.6	15.3	7.2	7.2	7.2	7.2	16.1
		100	7.7	7.7	7.7	7.7	15.6	8.2	8.2	8.2	8.2	17.4
	(0.6, 0.1)	40	11.5	15.3	12.4	11.8	25.6	9.6	12.4	10.2	9.8	24.0
		70	8.7	8.7	8.7	8.7	21.9	9.4	9.5	9.4	9.4	20.6
		100	8.5	8.5	8.5	8.5	21.9	8.3	8.3	8.3	8.3	22.7
Model 4	0.4	40	8.6	9.1	8.6	8.6	17.8	7.6	8.1	7.7	7.6	18.1
		70	7.6	7.6	7.6	7.6	14.5	8.0	8.1	8.0	8.0	16.1
		100	6.9	6.9	6.9	6.9	14.3	5.2	5.2	5.2	5.2	16.0
	0.5	40	8.1	9.6	8.3	8.2	20.3	7.5	8.3	7.8	7.6	19.1
		70	8.1	8.1	8.1	8.1	14.7	7.0	7.0	7.0	7.0	15.3
		100	6.2	6.2	6.2	6.2	14.2	6.0	6.0	6.0	6.0	15.6
	0.6	40	8.0	9.5	8.1	8.0	18.9	8.0	9.9	8.6	8.1	19.2
		70	6.2	6.2	6.2	6.2	14.6	7.8	7.8	7.8	7.8	13.8
		100	7.6	7.6	7.6	7.6	13.2	6.7	6.7	6.7	6.7	16.0

Table S4: Empirical sizes ($\times 10^2$) of the proposed test T_n defined as (6) for $K_0 = 4$ with the untruncated critical value ($c_\kappa = \infty$) and the truncated critical values defined as (18) with $c_\kappa = 0.45, 0.55, 0.65$, and the KPSS test in a simulation with 2000 replications. Constant c_κ determines the level of truncation for the critical values of T_n. The innovations $\epsilon_t \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \sigma_\epsilon^2)$. The nominal size of the tests is 5%.

					$\sigma_\epsilon^2 = 1$					$\sigma_\epsilon^2 = 2$		
	Setting	N	∞	0.45	0.55	0.65	KPSS	∞	0.45	0.55	0.65	KPSS
Model 1	0.5	40	6.4	6.4	6.4	6.4	9.7	7.1	7.1	7.1	7.1	12.0
		70	6.8	6.8	6.8	6.8	8.1	7.0	7.0	7.0	7.0	10.0
		100	5.5	5.5	5.5	5.5	8.1	5.6	5.6	5.6	5.6	9.7
	0.9	40	9.8	44.2	30.2	21.6	52.7	11.3	41.9	28.7	22.0	47.9
		70	10.4	25.1	16.1	12.8	48.4	10.3	26.0	16.0	12.5	47.8
		100	8.6	14.4	10.1	9.0	49.4	9.1	14.7	10.4	9.4	50.8
	-0.5	40	8.8	8.8	8.8	8.8	1.8	7.4	7.4	7.4	7.4	1.3
		70	6.8	6.8	6.8	6.8	1.8	7.4	7.4	7.4	7.4	2.5
		100	6.2	6.2	6.2	6.2	2.1	7.0	7.0	7.0	7.0	2.5
Model 2	(0.8,0.3)	40	7.6	7.6	7.6	7.6	8.2	7.8	7.8	7.8	7.8	7.6
		70	5.8	5.8	5.8	5.8	6.7	5.7	5.7	5.7	5.7	7.6
		100	5.9	5.9	5.9	5.9	7.1	5.7	5.7	5.7	5.7	6.4
	(0.9,0.5)	40	7.4	7.4	7.4	7.4	8.2	7.3	7.3	7.3	7.3	7.4
		70	5.9	5.9	5.9	5.9	7.1	6.1	6.1	6.1	6.1	7.5
		100	5.5	5.5	5.5	5.5	7.0	5.5	5.5	5.5	5.5	7.3
	(0.95, 0.9)	40	6.9	6.9	6.9	6.9	8.0	7.0	7.0	7.0	7.0	8.8
		70	7.2	7.2	7.2	7.2	7.8	6.8	6.8	6.8	6.8	7.5
		100	6.4	6.4	6.4	6.4	8.6	5.4	5.4	5.4	5.4	7.8
Model 3	(0.4,0.2)	40	8.8	9.6	8.9	8.8	22.9	9.4	10.4	9.6	9.4	21.8
		70	7.0	7.0	7.0	7.0	18.4	7.5	7.5	7.5	7.5	18.0
		100	8.2	8.2	8.2	8.2	20.2	6.6	6.6	6.6	6.6	18.1
	(0.5,0.1)	40	9.7	10.2	9.8	9.7	17.5	8.5	9.2	8.5	8.5	20.3
		70	8.9	8.9	8.9	8.9	15.8	8.2	8.2	8.2	8.2	16.5
		100	7.6	7.6	7.6	7.6	17.8	6.8	6.8	6.8	6.8	17.2
	(0.6,0.1)	40	11.5	15.3	12.7	11.8	26.1	10.9	14.9	11.5	11.1	26.2
		70	8.9	9.1	8.9	8.9	21.7	8.0	8.1	8.0	8.0	21.1
		100	7.0	7.0	7.0	7.0	23.0	7.1	7.1	7.1	7.1	22.7
Model 4	0.4	40	8.2	8.9	8.2	8.2	19.2	7.3	7.8	7.4	7.3	20.8
		70	6.6	6.6	6.6	6.6	15.8	7.2	7.2	7.2	7.2	14.0
		100	6.7	6.7	6.7	6.7	16.0	6.4	6.4	6.4	6.4	14.2
	0.5	40	7.9	8.6	8.1	7.9	19.4	7.8	8.8	8.0	7.9	19.8
		70	7.2	7.2	7.2	7.2	14.6	6.2	6.3	6.2	6.2	15.6
		100	6.8	6.8	6.8	6.8	14.3	6.5	6.5	6.5	6.5	15.2
	0.6	40	9.2	10.7	9.6	9.3	19.7	8.8	10.3	8.9	8.8	20.0
		70	7.2	7.2	7.2	7.2	14.1	6.7	6.7	6.7	6.7	17.4
		100	7.6	7.6	7.6	7.6	14.9	6.5	6.5	6.5	6.5	16.2

Table S5: Empirical powers ($\times 10^2$) of the proposed test T_n defined as (6) for $K_0 = 1$ with the untruncated critical value ($c_\kappa = \infty$) and the truncated critical values defined as (18) with $c_\kappa = 0.45, 0.55, 0.65$, and the KPSS test in a simulation with 2000 replications. Constant c_κ determines the level of truncation for the critical values of T_n. The innovations $\epsilon_t \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \sigma_\epsilon^2)$. The nominal size of the tests is 5%.

			$\sigma_\epsilon^2 = 1$					$\sigma_\epsilon^2 = 2$				
	Setting	N	∞	0.45	0.55	0.65	KPSS	∞	0.45	0.55	0.65	KPSS
Model 5	0.5	40	14.0	94.0	89.0	84.0	83.7	12.8	93.8	89.0	83.4	84.4
		70	14.0	96.9	93.8	89.4	89.5	14.1	96.5	93.5	89.2	90.0
		100	15.3	98.2	95.9	93.0	95.2	14.0	97.9	95.0	92.0	95.1
	0.9	40	14.1	99.1	97.9	95.5	92.7	14.5	99.6	98.7	96.7	92.6
		70	16.2	99.7	99.1	98.5	95.5	17.0	99.8	99.6	99.3	94.3
		100	17.5	100.0	99.7	99.2	97.7	19.0	100.0	99.9	99.8	96.9
	−0.5	40	6.0	83.8	75.8	68.5	82.8	5.9	84.5	76.8	69.0	81.8
		70	7.3	92.2	86.4	80.0	89.8	7.1	90.7	84.7	78.0	89.8
		100	6.4	95.1	90.8	85.8	94.6	6.4	95.0	90.6	85.5	94.7
Model 6	(0.8, 0.3)	40	12.0	94.2	89.1	83.3	82.7	11.8	93.4	88.1	82.2	83.1
		70	12.3	97.5	93.3	88.8	90.2	12.8	97.0	93.8	89.9	90.0
		100	12.2	98.0	95.5	91.9	95.0	12.7	98.6	96.0	92.2	95.8
	(0.9, 0.5)	40	14.4	95.6	90.4	85.0	82.3	13.8	95.6	90.6	84.1	82.3
		70	13.4	97.8	93.5	88.5	90.0	12.8	97.4	93.3	88.2	89.1
		100	12.3	98.6	96.4	92.5	95.5	12.2	98.0	95.4	91.2	94.3
	(0.95, 0.9)	40	14.2	96.0	91.5	85.8	83.1	13.8	95.3	89.8	83.2	82.9
		70	13.6	97.5	93.8	89.5	90.7	11.9	97.3	93.6	88.8	90.1
		100	12.7	98.8	96.5	94.0	95.3	12.8	98.0	95.3	92.6	95.0
Model 7	(0.4, 0.2)	40	14.6	98.2	94.9	91.2	86.2	15.0	98.3	95.4	91.1	86.7
		70	15.7	99.0	97.0	93.5	91.1	17.0	99.4	97.9	95.7	92.2
		100	16.0	99.6	98.2	95.3	95.0	17.5	99.6	98.9	97.8	95.6
	(0.5, 0.1)	40	15.9	98.6	96.3	92.5	84.9	14.1	98.8	95.8	92.2	85.8
		70	17.6	99.2	97.5	94.8	90.8	15.7	99.4	97.9	96.2	91.8
		100	18.1	99.6	98.4	96.1	96.3	17.4	99.7	99.2	97.9	95.1
	(0.6, 0.1)	40	15.6	99.1	97.0	93.7	87.7	15.9	99.5	97.8	94.7	88.0
		70	16.8	99.6	98.5	96.7	92.0	17.2	99.8	99.6	98.3	92.0
		100	17.5	99.6	99.2	98.0	96.2	17.7	100.0	99.6	99.1	95.3
Model 8	(0.8, 0.3)	40	5.5	100.0	100.0	100.0	99.2	6.9	100.0	100.0	100.0	98.3
		70	6.1	100.0	100.0	100.0	99.6	6.3	100.0	100.0	100.0	99.1
		100	6.0	100.0	100.0	100.0	99.9	5.9	100.0	100.0	100.0	100.0
	(0.9, 0.5)	40	5.9	100.0	100.0	100.0	99.9	6.2	100.0	100.0	100.0	99.8
		70	5.5	100.0	100.0	100.0	98.4	6.8	100.0	100.0	100.0	98.8
		100	6.9	100.0	100.0	100.0	99.2	5.9	100.0	100.0	100.0	99.2
	(0.95, 0.9)	40	7.5	100.0	100.0	100.0	98.0	7.0	100.0	100.0	100.0	98.2
		70	5.8	100.0	100.0	100.0	99.3	5.9	100.0	100.0	100.0	99.4
		100	6.9	100.0	100.0	100.0	99.9	6.1	100.0	100.0	100.0	99.7

Table S6: Empirical powers ($\times 10^2$) of the proposed test T_n defined as (6) for $K_0 = 2$ with the untruncated critical value ($c_\kappa = \infty$) and the truncated critical values defined as (18) with $c_\kappa = 0.45, 0.55, 0.65$, and the KPSS test in a simulation with 2000 replications. Constant c_κ determines the level of truncation for the critical values of T_n. The innovations $\epsilon_t \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \sigma_\epsilon^2)$. The nominal size of the tests is 5%.

			$\sigma_\epsilon^2 = 1$					$\sigma_\epsilon^2 = 2$				
	Setting	N	∞	0.45	0.55	0.65	KPSS	∞	0.45	0.55	0.65	KPSS
Model 5	0.5	40	13.7	94.4	89.5	85.7	84.4	14.1	94.6	90.4	84.7	84.9
		70	14.4	97.1	94.1	90.5	90.8	14.2	97.5	94.8	90.9	90.5
		100	15.0	98.6	96.7	93.8	95.2	14.3	98.3	95.4	92.1	95.2
	0.9	40	14.4	99.1	97.6	95.3	92.5	16.2	99.6	98.2	97.0	93.4
		70	18.2	99.9	99.4	98.4	95.5	18.9	100.0	99.8	99.2	95.2
		100	17.8	100.0	99.7	99.6	97.9	17.6	100.0	100.0	99.9	97.8
	-0.5	40	6.9	82.9	75.3	68.2	81.5	6.8	82.4	75.5	68.9	81.0
		70	7.3	91.7	84.9	78.8	90.1	7.4	90.8	85.0	79.2	89.8
		100	6.7	95.5	90.8	86.1	94.1	7.8	95.0	90.8	86.7	94.3
Model 6	(0.8, 0.3)	40	15.2	95.2	89.8	83.0	82.9	14.8	93.5	87.7	81.5	82.8
		70	13.2	97.0	93.6	89.0	90.3	14.8	97.4	93.3	89.1	90.8
		100	13.3	98.7	96.2	93.5	95.6	12.8	98.0	95.3	92.3	94.8
	(0.9, 0.5)	40	14.3	95.9	90.7	84.7	83.8	14.6	95.8	90.3	84.2	83.5
		70	14.4	97.3	94.1	90.0	91.0	14.1	97.7	94.1	89.2	89.7
		100	14.2	98.0	95.2	91.5	94.4	12.9	98.5	95.4	91.8	94.0
	(0.95, 0.9)	40	16.3	96.2	91.3	86.7	83.5	14.1	96.0	91.3	85.8	83.9
		70	13.9	97.7	94.3	90.8	90.5	14.5	97.8	93.8	89.8	90.2
		100	13.5	98.5	96.3	93.2	95.3	13.5	98.2	95.5	92.3	95.4
Model 7	(0.4, 0.2)	40	17.9	98.8	95.0	90.9	85.3	15.5	98.0	95.5	91.3	86.1
		70	17.2	99.1	97.1	93.8	90.5	16.2	99.5	98.2	96.0	92.0
		100	18.6	99.6	98.5	96.9	95.9	17.1	99.7	98.9	97.6	94.8
	(0.5, 0.1)	40	16.5	98.6	96.2	91.2	86.7	16.6	99.0	97.0	93.3	86.6
		70	17.0	99.5	97.5	94.2	91.0	16.6	99.9	98.5	96.4	90.7
		100	16.7	99.8	98.2	96.4	96.2	17.8	99.8	99.1	98.5	95.2
	(0.6, 0.1)	40	17.2	99.2	97.3	93.9	87.5	18.4	99.2	97.2	94.8	87.2
		70	17.1	99.8	98.6	96.6	91.5	16.4	99.5	99.0	97.5	92.2
		100	19.2	99.7	99.0	97.2	95.5	18.9	100	99.7	99.5	95.9
Model 8	(0.8, 0.3)	40	6.8	100.0	100.0	100.0	97.7	6.7	100.0	100.0	100.0	98.9
		70	6.3	100.0	100.0	100.0	99.4	5.3	100.0	100.0	100.0	99.2
		100	6.2	100.0	100.0	100.0	99.8	5.8	100.0	100.0	100.0	99.8
	(0.9, 0.5)	40	6.3	100.0	100.0	100.0	98.2	6.4	100.0	100.0	100.0	98.3
		70	6.7	100.0	100.0	100.0	99.2	6.9	100.0	100.0	100.0	99.4
		100	5.9	100.0	100.0	100.0	99.9	6.2	100.0	100.0	100.0	99.9
	(0.95, 0.9)	40	8.2	100.0	100.0	100.0	98.0	6.4	100.0	100.0	100.0	98.6
		70	6.0	100.0	100.0	100.0	99.4	6.8	100.0	100.0	100.0	99.5
		100	5.7	100.0	100.0	100.0	99.7	6.9	100.0	100.0	100.0	99.8

Table S7: Empirical powers ($\times 10^2$) of the proposed test T_n defined as (6) for $K_0 = 3$ with the untruncated critical value ($c_\kappa = \infty$) and the truncated critical values defined as (18) with $c_\kappa = 0.45, 0.55, 0.65$, and the KPSS test in a simulation with 2000 replications. Constant c_κ determines the level of truncation for the critical values of T_n. The innovations $\epsilon_t \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \sigma_\epsilon^2)$. The nominal size of the tests is 5%.

			$\sigma_\epsilon^2 = 1$					$\sigma_\epsilon^2 = 2$				
	Setting	N	∞	0.45	0.55	0.65	KPSS	∞	0.45	0.55	0.65	KPSS
Model 5	0.5	40	15.8	94.6	89.5	85.0	83.2	14.7	94.4	90.0	84.9	83.7
		70	13.7	97.3	94.1	90.9	90.8	15.9	97.2	94.1	90.2	89.8
		100	15.8	98.3	95.5	93.1	95.0	14.9	98.7	96	93.6	95.3
	0.9	40	13.8	99.1	97.5	95.5	92.2	14.3	99.6	98.5	96.9	92.2
		70	17.9	100.0	99.6	98.9	94.8	16.4	99.9	99.7	99.4	95.0
		100	18.6	99.9	99.7	99.2	97.4	19.8	100.0	100.0	99.9	97.4
	−0.5	40	7.9	83.5	75.9	68.9	81.5	8.0	83.4	75.8	68.3	82.5
		70	6.7	92.0	85.7	79.8	89.7	7.8	91.3	84.8	79.5	89.3
		100	8.2	95.0	90.0	85.2	93.9	7.8	94.8	90.2	85.2	94.5
Model 6	(0.8, 0.3)	40	14.9	94.7	89.6	82.7	82.4	14.0	94.5	89.4	83.0	83.9
		70	15.3	97.2	93.9	89.2	89.8	15.2	96.8	93.5	89.5	90.0
		100	15.8	97.8	94.9	91.4	94.5	15.0	98.0	95.1	90.9	94.7
	(0.9, 0.5)	40	14.0	95.0	90.2	85.0	82.0	14.5	96.0	90.5	84.2	82.6
		70	14.2	97.2	94.2	90.1	91.5	14.1	97.4	93.5	88.6	89.5
		100	15.4	98.4	96.0	92.9	94.2	15.9	98.2	96.3	92.8	94.8
	(0.95, 0.9)	40	12.9	96.5	91.2	84.8	82.0	16.0	95.5	91.3	85.8	83.7
		70	16.0	97.2	94.0	90.1	89.2	17.8	97.9	93.9	90.8	90.2
		100	15.3	98.6	96.0	92.8	94.8	16.8	98.3	95.8	93.2	95.0
Model 7	(0.4,0.2)	40	17.2	98.8	95.8	91.2	85.2	17.9	98.6	96.3	92.8	85.4
		70	18.8	99.2	97.1	93.7	90.5	17.6	99.1	97.7	96.0	90.4
		100	16.9	99.6	98.2	96.2	95.0	18.4	99.7	99.2	98.2	95.3
	(0.5,0.1)	40	15.8	98.9	95.7	91.5	86.2	17.6	98.8	96.0	92.3	86.2
		70	17.6	99.5	97.8	94.3	91.0	17.9	99.4	98.7	97.4	91.1
		100	18.5	99.8	99.0	97.3	96.2	19.4	99.7	99.0	98.5	95.1
	(0.6,0.1)	40	16.2	99.1	97.2	93.0	86.4	15.9	99.2	97.2	94.5	86.8
		70	19.1	99.8	98.8	97.2	91.8	18.2	99.9	99.5	98.7	92.6
		100	19.9	99.9	99.2	97.9	96.8	18.4	100.0	99.8	99.5	95.2
Model 8	(0.8, 0.3)	40	6.2	100.0	100.0	100.0	98.2	5.5	100.0	100.0	100.0	98.6
		70	6.9	100.0	100.0	100.0	99.5	6.2	100.0	100.0	100.0	99.4
		100	6.2	100.0	100.0	100.0	99.7	6.1	100.0	100.0	100.0	99.9
	(0.9, 0.5)	40	6.2	100.0	100.0	100.0	98.2	6.8	100.0	100.0	100.0	98.6
		70	6.7	100.0	100.0	100.0	99.2	6.4	100.0	100.0	100.0	98.9
		100	5.8	100.0	100.0	100.0	100.0	6.3	100.0	100.0	100.0	99.9
	(0.95, 0.9)	40	7.3	100.0	100.0	100.0	98.6	6.9	100.0	100.0	100.0	98.3
		70	6.1	100.0	100.0	100.0	99.6	6.6	100.0	100.0	100.0	99.5
		100	5.8	100.0	100.0	100.0	99.9	6.6	100.0	100.0	100.0	99.7

Table S8: Empirical powers ($\times 10^2$) of the proposed test T_n defined as (6) for $K_0 = 4$ with the untruncated critical value ($c_\kappa = \infty$) and the truncated critical values defined as (18) with $c_\kappa = 0.45, 0.55, 0.65$, and the KPSS test in a simulation with 2000 replications. Constant c_κ determines the level of truncation for the critical values of T_n. The innovations $\epsilon_t \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \sigma_\epsilon^2)$. The nominal size of the tests is 5%.

			$\sigma_\epsilon^2 = 1$					$\sigma_\epsilon^2 = 2$				
	Setting	N	∞	0.45	0.55	0.65	KPSS	∞	0.45	0.55	0.65	KPSS
Model 5	0.5	40	15.8	93.5	90.1	85.0	85.7	14.6	94.8	90.7	85.9	84.9
		70	15.1	97.0	94.0	89.8	90.2	16.6	96.5	93.8	90.0	90.0
		100	15.3	98.2	96.0	93.4	95.4	16.8	98.4	96.0	92.8	94.5
	0.9	40	15.8	99.5	98.0	96.0	92.8	14.0	99.5	98.1	96.7	91.5
		70	16.1	99.8	99.2	98.4	94.7	16.4	99.9	99.7	99.5	95.0
		100	18.9	100.0	99.7	99.5	97.2	17.8	99.9	99.9	99.8	97.5
	-0.5	40	7.3	83.3	75.0	67.7	81.7	6.9	84.0	76.0	67.6	81.5
		70	8.8	91.8	86.1	79.7	89.6	7.8	93.0	87.4	80.8	90.6
		100	6.6	94.5	89.7	85.3	94.2	8.2	94.4	89.6	85.6	94.1
Model 6	(0.8, 0.3)	40	14.2	94.5	89.1	82.8	83.1	14.2	95.0	88.8	83.0	82.8
		70	15.1	96.8	93.2	90.0	89.0	15.2	97.1	93.0	88.5	90.0
		100	14.1	98.4	95.4	91.3	94.7	15.1	98.4	95.7	92.0	95.2
	(0.9, 0.5)	40	15.0	95.0	89.5	83.7	82.9	16.5	95.5	91.0	84.3	84.0
		70	13.8	97.5	93.9	89.2	89.5	16.4	97.7	94.7	90.3	90.0
		100	14.9	98.7	96.4	93.8	95.2	16.8	98.7	96.5	93.5	95.0
	(0.95, 0.9)	40	13.9	94.8	89.8	83.7	83.2	17.1	95.2	90.1	85.0	83.0
		70	16.0	97.8	93.5	88.8	89.9	14.9	96.5	93.0	89.6	90.4
		100	16.4	98.4	95.2	91.5	94.3	16.2	98.5	96.0	93.0	95.3
Model 7	(0.4, 0.2)	40	17.9	98.5	95.0	90.5	85.9	17.1	98.7	96.0	92.0	86.4
		70	17.2	99.2	97.5	95.1	91.3	18.1	99.5	97.9	95.9	91.4
		100	17.5	99.8	98.4	96.5	96.3	17.5	99.7	99.0	98.3	95.5
	(0.5, 0.1)	40	17.9	98.7	96.2	92.3	85.0	16.4	98.8	96.3	93.2	86.0
		70	18.2	99.0	96.8	94.5	91.3	20.8	99.4	98.5	96.9	91.1
		100	17.2	99.6	98.7	96.3	94.8	18.5	99.8	99.4	98.8	95.4
	(0.6, 0.1)	40	15.0	99.2	97.3	94.0	87.7	18.3	99.5	98.0	95.8	86.8
		70	19.2	99.8	98.8	97.5	92.6	19.3	99.8	99.4	98.8	92.5
		100	18.8	99.9	99.5	98.5	95.5	18.9	100.0	99.7	99.5	95.7
Model 8	(0.8, 0.3)	40	7.8	100.0	100.0	100.0	98.8	6.8	100.0	100.0	100.0	98.0
		70	5.7	100.0	100.0	100.0	99.2	7.3	100.0	100.0	100.0	99.6
		100	6.5	100.0	100.0	100.0	99.9	5.5	100.0	100.0	100.0	99.9
	(0.9, 0.5)	40	6.7	100.0	100.0	100.0	98.0	7.1	100.0	100.0	100.0	97.9
		70	6.3	100.0	100.0	100.0	99.1	5.9	100.0	100.0	100.0	99.6
		100	6.7	100.0	100.0	100.0	99.9	6.5	100.0	100.0	100.0	99.9
	(0.95, 0.9)	40	7.3	100.0	100.0	100.0	98.2	6.8	100.0	100.0	100.0	98.1
		70	5.9	100.0	100.0	100.0	99.5	6.9	100.0	100.0	100.0	99.2
		100	5.2	100.0	100.0	100.0	99.9	5.9	100.0	100.0	100.0	99.9

Table S9: Empirical sizes ($\times 10^2$) of the proposed test T_n defined as (6) for $K_0 = 0$ with the untruncated critical value ($c_\kappa = \infty$) and the truncated critical values defined as (18) with $c_\kappa = 0.45, 0.55, 0.65$, and the KPSS test in a simulation with 2000 replications. Constant c_κ determines the level of truncation for the critical values of T_n. The innovations $\epsilon_t \overset{\text{i.i.d.}}{\sim} t(df)$. The nominal size of the tests is 5%.

			$df = 2$					$df = 5$				
	Setting	N	∞	0.45	0.55	0.65	KPSS	∞	0.45	0.55	0.65	KPSS
Model 1	0.5	40	5.3	5.6	5.6	5.6	9.7	6.2	6.2	6.2	6.2	10.9
		70	4.9	5.1	5.1	5.1	8.6	6.2	6.2	6.2	6.2	9.7
		100	3.8	4.1	4.1	4.1	9.7	5.5	5.5	5.5	5.5	9.2
	0.9	40	9.2	44.0	30.5	23.2	55.1	8.2	41.1	28.2	19.1	51.0
		70	6.6	22.6	14.0	10.2	50.9	8.1	23.2	14.1	10.6	47.2
		100	6.2	12.0	8.9	8.1	52.0	8.0	13.7	9.6	8.3	49.9
	−0.5	40	5.9	6.6	6.6	6.6	1.5	7.5	7.5	7.5	7.5	1.7
		70	5.5	6.1	6.1	6.1	1.5	5.8	5.8	5.8	5.8	1.9
		100	4.5	4.8	4.8	4.8	2.0	6.5	6.5	6.5	6.5	2.5
Model 2	(0.8, 0.3)	40	6.5	7.1	7.1	7.1	7.7	6.2	6.2	6.2	6.2	7.0
		70	4.2	4.8	4.8	4.8	6.5	5.8	5.8	5.8	5.8	7.5
		100	4.1	4.3	4.3	4.3	7.1	6.6	6.6	6.6	6.6	6.9
	(0.9, 0.5)	40	6.2	6.8	6.8	6.8	6.9	7.1	7.1	7.1	7.1	7.8
		70	4.0	4.9	4.9	4.9	6.7	6.5	6.5	6.5	6.5	7.0
		100	4.6	5.2	5.2	5.2	7.5	5.5	5.5	5.5	5.5	8.8
	(0.95, 0.9)	40	6.0	6.8	6.8	6.8	7.6	7.4	7.4	7.4	7.4	9.6
		70	6.3	7.6	7.6	7.6	7.3	6.2	6.2	6.2	6.2	7.1
		100	5.7	6.7	6.7	6.7	8.0	6.4	6.4	6.4	6.4	7.3
Model 3	(0.4, 0.2)	40	6.9	8.5	7.8	7.7	20.1	7.2	8.3	7.5	7.3	22.1
		70	5.3	6.3	6.3	6.3	17.2	7.9	7.9	7.9	7.9	15.2
		100	4.3	5.2	5.2	5.2	18.1	7.6	7.6	7.6	7.6	18.2
	(0.5, 0.1)	40	8.2	9.3	9.0	8.9	20.1	8.4	9.4	8.5	8.4	19.7
		70	6.2	7.3	7.3	7.3	16.4	7.8	7.8	7.8	7.8	15.5
		100	5.6	6.2	6.2	6.2	17.0	6.2	6.2	6.2	6.2	15.7
	(0.6, 0.1)	40	8.1	12.1	10.1	9.7	25.9	9.2	12.8	10.1	9.4	25.4
		70	7.0	8.1	8.1	8.1	22.1	7.2	7.2	7.2	7.2	20.8
		100	7.0	8.2	8.2	8.2	21.9	6.6	6.6	6.6	6.6	20.6
Model 4	0.4	40	10.8	11.6	11.5	11.3	20.6	8.8	9.2	8.8	8.8	20.9
		70	9.6	10.2	10.2	10.2	16.2	9.3	9.3	9.3	9.3	16.2
		100	9.5	10.1	10.1	10.1	14.1	9.0	9.0	9.0	9.0	16.0
	0.5	40	10.9	12.5	11.9	11.8	20.8	9.8	10.8	10.0	9.8	21.2
		70	8.6	9.4	9.4	9.4	15.2	8.8	8.8	8.8	8.8	15.5
		100	6.8	7.9	7.9	7.9	16.0	7.0	7.0	7.0	7.0	15.1
	0.6	40	9.8	11.8	10.6	10.5	20.3	9.3	10.8	9.4	9.3	19.7
		70	6.9	8.1	8.1	8.1	15.6	7.6	7.6	7.6	7.6	15.5
		100	6.2	7.0	7.0	7.0	14.9	6.6	6.6	6.6	6.6	15.6

Table S10: Empirical sizes ($\times 10^2$) of the proposed test T_n defined as (6) for $K_0 = 1$ with the untruncated critical value ($c_\kappa = \infty$) and the truncated critical values defined as (18) with $c_\kappa = 0.45, 0.55, 0.65$, and the KPSS test in a simulation with 2000 replications. Constant c_κ determines the level of truncation for the critical values of T_n. The innovations $\epsilon_t \overset{\text{i.i.d.}}{\sim} t(df)$. The nominal size of the tests is 5%.

			$df = 2$					$df = 5$				
	Setting	N	∞	0.45	0.55	0.65	KPSS	∞	0.45	0.55	0.65	KPSS
Model 1	0.5	40	5.7	6.2	6.2	6.2	11.0	6.9	6.9	6.9	6.9	10.5
		70	4.0	4.3	4.3	4.3	9.4	5.7	5.7	5.7	5.7	11.0
		100	4.2	4.5	4.5	4.5	9.9	5.8	5.8	5.8	5.8	10.5
	0.9	40	8.6	42.2	29.6	21.5	52.1	10.2	41.4	28.3	21.3	51.7
		70	7.5	25.1	15.4	12.0	48.4	9.3	25.6	15.8	12.0	50.0
		100	7.2	14.4	10.8	10.0	53.5	8.2	13.4	9.3	8.6	49.3
	−0.5	40	6.5	7.5	7.5	7.5	1.4	6.8	6.9	6.9	6.9	2.0
		70	5.0	5.9	5.9	5.9	1.9	6.9	6.9	6.9	6.9	1.5
		100	4.5	5.1	5.1	5.1	1.3	5.9	5.9	5.9	5.9	2.5
Model 2	(0.8, 0.3)	40	5.7	6.1	6.1	6.1	7.9	7.1	7.1	7.1	7.1	6.8
		70	4.7	5.1	5.1	5.1	6.0	6.4	6.4	6.4	6.4	6.8
		100	3.8	4.7	4.7	4.7	6.3	5.2	5.2	5.2	5.2	7.3
	(0.9, 0.5)	40	6.8	7.8	7.8	7.8	6.2	6.1	6.1	6.1	6.1	8.1
		70	4.9	5.9	5.9	5.9	6.0	5.7	5.7	5.7	5.7	7.1
		100	4.5	5.2	5.2	5.2	7.3	6.1	6.1	6.1	6.1	6.5
	(0.95, 0.9)	40	6.5	7.5	7.5	7.5	7.1	6.8	6.8	6.8	6.8	7.4
		70	5.1	6.1	6.1	6.1	6.9	6.7	6.7	6.7	6.7	8.0
		100	4.8	5.9	5.9	5.9	7.6	5.8	5.8	5.8	5.8	7.4
Model 3	(0.4, 0.2)	40	7.6	9.7	8.8	8.6	21.6	8.0	9.0	8.2	8.0	21.7
		70	5.9	6.8	6.8	6.8	16.8	7.4	7.4	7.4	7.4	18.0
		100	5.1	6.5	6.5	6.5	19.0	6.5	6.5	6.5	6.5	17.4
	(0.5, 0.1)	40	7.9	9.7	9.2	9.1	17.5	8.2	8.8	8.3	8.2	20.0
		70	5.1	6.7	6.7	6.7	15.4	8.2	8.2	8.2	8.2	14.1
		100	4.9	6.2	6.2	6.2	15.7	6.6	6.6	6.6	6.6	17.4
	(0.6, 0.1)	40	7.8	11.8	10.0	9.5	27.6	10.2	14.5	11.3	10.6	26.8
		70	6.8	8.8	8.7	8.7	23.4	8.2	8.2	8.2	8.2	21.0
		100	5.7	7.6	7.6	7.6	21.6	7.5	7.5	7.5	7.5	21.0
Model 4	0.4	40	10.1	10.8	10.5	10.5	20.2	8.7	9.2	8.7	8.7	19.5
		70	9.2	9.6	9.6	9.6	14.8	7.2	7.2	7.2	7.2	14.5
		100	7.9	8.7	8.7	8.7	15.4	8.6	8.6	8.6	8.6	15.6
	0.5	40	8.8	10.3	9.6	9.5	20	8.5	8.9	8.6	8.5	18.9
		70	7.4	8.4	8.4	8.4	14.8	6.9	6.9	6.9	6.9	13.4
		100	6.5	7.2	7.2	7.2	15.8	7.2	7.2	7.2	7.2	15.5
	0.6	40	9.2	11.9	10.6	10.4	20.2	8.3	10.1	8.8	8.3	18.9
		70	7.4	8.4	8.4	8.4	14.6	7.6	7.6	7.6	7.6	15.2
		100	6.5	7.6	7.6	7.6	15.2	6.5	6.5	6.5	6.5	15.0

Table S11: Empirical sizes ($\times 10^2$) of the proposed test T_n defined as (6) for $K_0 = 2$ with the untruncated critical value ($c_\kappa = \infty$) and the truncated critical values defined as (18) with $c_\kappa = 0.45, 0.55, 0.65$, and the KPSS test in a simulation with 2000 replications. Constant c_κ determines the level of truncation for the critical values of T_n. The innovations $\epsilon_t \overset{\text{i.i.d.}}{\sim} t(df)$. The nominal size of the tests is 5%.

			$df = 2$					$df = 5$				
	Setting	N	∞	0.45	0.55	0.65	KPSS	∞	0.45	0.55	0.65	KPSS
Model 1	0.5	40	5.5	6.0	6.0	6.0	10.4	5.5	5.5	5.5	5.5	10.5
		70	5.5	5.9	5.9	5.9	9.2	5.8	5.8	5.8	5.8	9.2
		100	4.6	5.1	5.1	5.1	10.3	5.0	5.0	5.0	5.0	10.3
	0.9	40	10.0	44.4	31.1	23.8	53.4	10.9	42.4	29.6	21.1	49.5
		70	7.0	24.4	14.6	11.9	48.5	9.6	23.9	15.2	11.9	48.4
		100	6.3	12.6	9.7	9.0	51.3	8.4	12.6	9.4	8.6	50.3
	−0.5	40	7.2	8.1	8.1	8.1	1.5	7.5	7.5	7.5	7.5	1.8
		70	4.3	4.8	4.8	4.8	1.5	6.2	6.2	6.2	6.2	2.1
		100	4.2	4.5	4.5	4.5	1.6	6.6	6.6	6.6	6.6	2.4
Model 2	(0.8, 0.3)	40	6.6	7.4	7.4	7.4	6.5	5.9	5.9	5.9	5.9	7.2
		70	4.9	5.3	5.3	5.3	5.9	6.6	6.6	6.6	6.6	6.9
		100	4.5	5.1	5.1	5.1	7.3	6.3	6.3	6.3	6.3	7.2
	(0.9, 0.5)	40	5.8	6.6	6.6	6.6	7.4	8.5	8.5	8.5	8.5	7.4
		70	4.7	5.9	5.9	5.9	6.9	5.9	5.9	5.9	5.9	7.0
		100	5.5	6.2	6.2	6.2	6.8	5.9	5.9	5.9	5.9	7.4
	(0.95, 0.9)	40	6.6	7.6	7.6	7.6	7.8	8.1	8.1	8.1	8.1	8.2
		70	6.0	7.2	7.2	7.2	7.3	6.1	6.1	6.1	6.1	8.0
		100	5.2	6.0	6.0	6.0	7.6	6.1	6.1	6.1	6.1	7.9
Model 3	(0.4, 0.2)	40	6.7	8.5	7.8	7.8	21.3	8.7	9.6	8.9	8.7	20.5
		70	6.5	7.8	7.8	7.8	18.9	7.5	7.6	7.5	7.5	17.0
		100	5.1	6.5	6.5	6.5	18.3	7.4	7.4	7.4	7.4	19.0
	(0.5, 0.1)	40	8.2	10.5	10.0	10.0	20.6	9.7	10.2	9.8	9.8	20.2
		70	6.3	7.9	7.9	7.9	16.7	8.6	8.6	8.6	8.6	16.4
		100	5.3	7.1	7.1	7.1	16.6	6.9	6.9	6.9	6.9	16.9
	(0.6, 0.1)	40	9.7	14.9	12.3	11.9	26.0	9.8	13.8	10.7	10.2	25.9
		70	6.8	8.9	8.8	8.8	21.9	8.2	8.2	8.2	8.2	21.3
		100	6.0	8.6	8.6	8.6	21.4	7.3	7.3	7.3	7.3	22.9
Model 4	0.4	40	8.8	9.6	9.4	9.4	18.9	8.3	8.8	8.4	8.3	21.1
		70	8.9	9.8	9.8	9.8	15.2	7.5	7.5	7.5	7.5	14.2
		100	7.7	8.5	8.5	8.5	16.2	7.2	7.2	7.2	7.2	14.1
	0.5	40	9.2	11.2	10.1	10.1	21.3	7.7	9.2	7.8	7.8	20.4
		70	8.1	9.2	9.2	9.2	15.8	7.6	7.7	7.6	7.6	16.6
		100	6.5	7.8	7.8	7.8	14.6	6.8	6.8	6.8	6.8	15.0
	0.6	40	8.0	9.9	8.9	8.7	20	8.5	9.4	8.6	8.5	19.8
		70	8.5	9.6	9.6	9.6	15.2	7.1	7.1	7.1	7.1	14.5
		100	5.3	6.7	6.7	6.7	15.5	7.5	7.5	7.5	7.5	14.3

Table S12: Empirical sizes ($\times 10^2$) of the proposed test T_n defined as (6) for $K_0 = 3$ with the untruncated critical value ($c_\kappa = \infty$) and the truncated critical values defined as (18) with $c_\kappa = 0.45, 0.55, 0.65$, and the KPSS test in a simulation with 2000 replications. Constant c_κ determines the level of truncation for the critical values of T_n. The innovations $\epsilon_t \overset{\text{i.i.d.}}{\sim} t(df)$. The nominal size of the tests is 5%.

					$df = 2$					$df = 5$		
	Setting	N	∞	0.45	0.55	0.65	KPSS	∞	0.45	0.55	0.65	KPSS
Model 1	0.5	40	6.0	6.6	6.6	6.6	10.2	6.7	6.7	6.7	6.7	9.8
		70	4.8	5.4	5.4	5.4	8.8	6.8	6.8	6.8	6.8	9.7
		100	5.0	5.3	5.3	5.3	10.1	5.7	5.7	5.7	5.7	9.4
	0.9	40	9.8	43.9	32.1	23.6	54.3	11.0	42.4	29.2	20.9	49.4
		70	7.4	22.6	14.4	11.4	49.8	9.0	23.9	14.4	10.9	49.6
		100	8.5	15.2	12.0	11.1	54.9	8.6	14.1	9.5	8.9	51.3
	−0.5	40	7.1	7.6	7.6	7.6	2.0	7.0	7.0	7.0	7.0	1.4
		70	5.3	6.0	6.0	6.0	1.6	5.8	5.8	5.8	5.8	1.8
		100	4.8	5.8	5.8	5.8	1.7	6.6	6.6	6.6	6.6	2.1
Model 2	(0.8, 0.3)	40	6.5	7.2	7.2	7.2	7.2	7.2	7.2	7.2	7.2	9.2
		70	5.4	6.2	6.2	6.2	6.1	6.8	6.8	6.8	6.8	6.8
		100	4.1	4.5	4.5	4.5	6.3	6.1	6.1	6.1	6.1	7.1
	(0.9, 0.5)	40	6.8	7.3	7.3	7.3	8.0	7.3	7.3	7.3	7.3	8.2
		70	4.7	5.7	5.7	5.7	6.7	6.2	6.2	6.2	6.2	8.6
		100	4.8	5.4	5.4	5.4	7.4	5.7	5.7	5.7	5.7	6.5
	(0.95, 0.9)	40	7.7	8.6	8.6	8.6	8.5	7.0	7.0	7.0	7.0	8.9
		70	6.1	7.3	7.3	7.3	8.7	6.2	6.2	6.2	6.2	8.9
		100	5.8	7.0	7.0	7.0	7.4	6.5	6.5	6.5	6.5	7.6
Model 3	(0.4, 0.2)	40	8.4	10.2	9.8	9.7	21.3	7.8	8.8	8.1	7.8	20.8
		70	6.1	8.0	8.0	8.0	17.6	6.7	6.7	6.7	6.7	18.8
		100	6.2	7.4	7.4	7.4	19.5	7.4	7.4	7.4	7.4	17.4
	(0.5, 0.1)	40	7.4	9.2	8.8	8.7	20.5	8.7	9.3	8.9	8.7	19.0
		70	7.0	8.8	8.8	8.8	16.1	9.2	9.2	9.2	9.2	15.7
		100	6.2	7.4	7.4	7.4	17.3	7.1	7.1	7.1	7.1	15.7
	(0.6, 0.1)	40	8.2	13.4	10.8	10.5	26.0	10.9	13.9	11.7	10.9	24.4
		70	7.5	9.5	9.5	9.5	22.4	9.2	9.2	9.2	9.2	19.9
		100	6.6	8.9	8.9	8.8	23.0	6.8	6.8	6.8	6.8	21.4
Model 4	0.4	40	9.8	11.0	10.7	10.7	20.2	7.9	8.3	8.0	7.9	20.6
		70	8.1	8.6	8.6	8.6	12.8	7.2	7.2	7.2	7.2	16.3
		100	5.9	7.0	7.0	7.0	16.1	6.5	6.5	6.5	6.5	15.8
	0.5	40	9.0	10.7	10.0	10.0	19.6	9.0	10.1	9.0	9.0	18.9
		70	6.0	7.0	7.0	7.0	14.8	7.8	7.8	7.8	7.8	14.9
		100	5.2	6.5	6.5	6.5	16.4	6.3	6.3	6.3	6.3	15.1
	0.6	40	8.5	10.3	9.3	9.2	20.2	7.8	9.2	8.3	7.9	20.3
		70	6.2	7.4	7.4	7.4	14.6	7.7	7.7	7.7	7.7	15.8
		100	5.1	6.2	6.2	6.2	14.2	6.7	6.7	6.7	6.7	14.7

Table S13: Empirical sizes ($\times 10^2$) of the proposed test T_n defined as (6) for $K_0 = 4$ with the untruncated critical value ($c_\kappa = \infty$) and the truncated critical values defined as (18) with $c_\kappa = 0.45, 0.55, 0.65$, and the KPSS test in a simulation with 2000 replications. Constant c_κ determines the level of truncation for the critical values of T_n. The innovations $\epsilon_t \overset{\text{i.i.d.}}{\sim} t(df)$. The nominal size of the tests is 5%.

			$df = 2$					$df = 5$				
	Setting	N	∞	0.45	0.55	0.65	KPSS	∞	0.45	0.55	0.65	KPSS
Model 1	0.5	40	7.3	7.5	7.5	7.5	10.6	6.5	6.5	6.5	6.5	10.4
		70	4.5	5.0	5.0	5.0	9.3	5.9	5.9	5.9	5.9	8.5
		100	5.9	6.2	6.2	6.2	10.2	5.3	5.3	5.3	5.3	10.1
	0.9	40	9.6	44.9	31.9	23.8	51.9	10.8	44	31.4	23.2	51.7
		70	8.8	25.3	17.2	13.8	49.9	9.8	24.9	15.2	11.6	47.9
		100	7.5	14.4	11.2	10.2	51.0	10.5	15.2	12.0	11.1	50.2
	−0.5	40	7.5	8.5	8.5	8.5	1.2	7.9	7.9	7.9	7.9	2.1
		70	7.2	7.8	7.8	7.8	1.3	7.1	7.1	7.1	7.1	1.8
		100	5.3	5.8	5.8	5.8	2.1	6.2	6.2	6.2	6.2	2.2
Model 2	(0.8, 0.3)	40	6.8	7.1	7.1	7.1	6.9	7.0	7.0	7.0	7.0	7.8
		70	5.1	5.7	5.7	5.7	6.2	5.8	5.8	5.8	5.8	7.1
		100	3.9	4.7	4.7	4.7	7.1	6.3	6.3	6.3	6.3	7.4
	(0.9, 0.5)	40	7.8	8.6	8.6	8.6	7.3	8.0	8.0	8.0	8.0	8.6
		70	5.4	5.9	5.9	5.9	7.0	5.6	5.6	5.6	5.6	7.5
		100	4.7	5.5	5.5	5.5	6.2	5.7	5.7	5.7	5.7	6.8
	(0.95, 0.9)	40	7.1	8.2	8.2	8.2	8.1	6.9	6.9	6.9	6.9	8.5
		70	5.0	6.3	6.3	6.3	6.8	6.2	6.2	6.2	6.2	7.8
		100	4.7	5.5	5.5	5.5	7.2	6.7	6.7	6.7	6.7	9.2
Model 3	(0.4, 0.2)	40	7.7	9.8	8.9	8.5	20.9	8.8	9.8	8.9	8.8	21.3
		70	5.8	7.0	7.0	7.0	17.1	8.1	8.1	8.1	8.1	17.7
		100	7.0	8.6	8.6	8.6	19.4	7.0	7.0	7.0	7.0	17.4
	(0.5, 0.1)	40	9.4	11.5	11.0	10.8	19.7	8.8	9.2	8.8	8.8	18.5
		70	5.7	6.8	6.8	6.8	17.5	7.0	7.0	7.0	7.0	16.2
		100	5.1	6.5	6.5	6.5	15.6	6.6	6.6	6.6	6.6	15.4
	(0.6, 0.1)	40	8.9	13.4	11.4	10.8	26.7	9.8	13.7	10.5	10.0	26.3
		70	7.0	9.3	9.3	9.2	21.6	8.3	8.3	8.3	8.3	23.3
		100	6.2	8.8	8.8	8.8	21.9	8.2	8.2	8.2	8.2	21.4
Model 4	0.4	40	8.6	9.4	9.3	9.3	20.1	8.4	8.9	8.5	8.4	19.6
		70	7.5	8.2	8.2	8.2	14.1	5.7	5.7	5.7	5.7	16.1
		100	5.9	6.6	6.6	6.6	15.2	6.9	6.9	6.9	6.9	15.4
	0.5	40	8.8	10.5	10.1	10.1	19.5	8.2	9.4	8.5	8.3	18.9
		70	7.9	8.8	8.8	8.8	14.8	6.8	6.8	6.8	6.8	15.2
		100	6.3	7.4	7.4	7.4	15.0	5.7	5.7	5.7	5.7	15.2
	0.6	40	9.8	11.5	10.9	10.7	19.0	9.2	10.4	9.3	9.2	20.5
		70	6.7	8.0	8.0	8.0	14.2	6.4	6.4	6.4	6.4	15.6
		100	5.1	6.3	6.2	6.2	13.9	6.5	6.5	6.5	6.5	13.6

Table S14: Empirical powers ($\times 10^2$) of the proposed test T_n defined as (6) for $K_0 = 0$ with the untruncated critical value ($c_\kappa = \infty$) and the truncated critical values defined as (18) with $c_\kappa = 0.45, 0.55, 0.65$, and the KPSS test in a simulation with 2000 replications. Constant c_κ determines the level of truncation for the critical values of T_n. The innovations $\epsilon_t \overset{\text{i.i.d.}}{\sim} t(df)$. The nominal size of the tests is 5%.

			$df = 2$					$df = 5$				
	Setting	N	∞	0.45	0.55	0.65	KPSS	∞	0.45	0.55	0.65	KPSS
Model 5	0.5	40	16.8	95.0	91.5	87.8	85.4	13.7	93.5	89.4	84.4	83.9
		70	15.4	96.9	94.8	91.0	91.0	11.7	97.0	93.7	89.6	90.3
		100	15.4	98.1	96.4	93.8	95.3	13.4	97.9	96.4	92.8	95.0
	0.9	40	14.5	99.4	98.7	97.8	93.0	12.8	99.2	97.4	95.2	92.7
		70	18.4	99.9	99.7	99.5	95.4	15.3	99.9	99.5	98.4	95.2
		100	17.6	100.0	100.0	99.9	98.0	17.5	99.9	99.7	99.5	97.7
	−0.5	40	9.7	84.6	78.3	72.2	84.0	7.6	84.0	76.6	69.1	80.7
		70	9.1	92.8	88.1	82.5	91.0	6.2	91.1	85.9	79.2	89.7
		100	7.2	94.7	90.5	86.2	95.2	5.3	95.3	91.3	86.6	94.9
Model 6	(0.8, 0.3)	40	14.0	94.0	90.4	84.9	84.3	12.3	95.0	90.0	83.2	82.2
		70	14.1	97.7	94.7	91.2	90.7	11.2	97.2	93.0	88.2	90.5
		100	14.2	98.2	96.2	93.5	94.7	11.3	97.5	94.9	91.8	94.5
	(0.9, 0.5)	40	15.0	95.3	91.1	86.1	83.2	11.4	95.5	90.1	83.9	83.2
		70	15.3	97.2	94.5	91.0	91.5	12.8	97.3	93.7	88.4	89.5
		100	15.1	98.6	96.4	93.6	94.6	12.7	98.3	95.8	92.5	94.5
	(0.95, 0.9)	40	15.1	95.5	90.8	85.8	83.0	12.7	95.2	90.3	84.0	83.3
		70	15.5	97.9	95.0	92.2	90.8	11.5	97.4	94.2	90.8	91.5
		100	13.8	98.8	97.0	94.8	94.9	11.6	98.6	96.2	92.7	94.5
Model 7	(0.4, 0.2)	40	16.9	98.5	96.0	93.2	86.6	15.0	98.0	94.9	91.0	87.7
		70	16.4	99.3	98.7	97.4	92.0	14.8	99.1	97.0	94.0	92.2
		100	17.2	99.9	99.6	98.9	95.2	16.6	99.4	97.8	95.5	95.0
	(0.5, 0.1)	40	16.6	98.3	95.3	91.8	86.0	14.4	98.4	95.1	91.7	86.8
		70	18.0	99.6	98.7	97.9	92.0	16.7	99.2	97.1	93.3	90.8
		100	16.2	99.8	99.3	98.9	95.8	15.0	99.8	98.4	96.4	95.3
	(0.6, 0.1)	40	17.0	99.2	98.0	96.8	87.9	14.6	99.2	97.7	94.7	87.4
		70	16.2	99.8	99.6	99.2	93.4	15.2	99.8	98.5	96.9	91.6
		100	18.8	100	99.9	99.7	96.8	16.4	99.9	99.4	98.4	95.0
Model 8	(0.8, 0.3)	40	6.7	100.0	100.0	100.0	98.6	7.1	100.0	100.0	100.0	97.7
		70	6.4	100.0	100.0	100.0	99.4	6.4	100.0	100.0	100.0	99.7
		100	5.1	100.0	100.0	100.0	100.0	5.5	100.0	100.0	100.0	99.8
	(0.9, 0.5)	40	8.2	100.0	100.0	100.0	97.9	8.1	100.0	100.0	100.0	98.4
		70	6.8	100.0	100.0	100.0	99.5	6.9	100.0	100.0	100.0	99.4
		100	5.1	100.0	100.0	100.0	99.9	6.5	100.0	100.0	100.0	99.8
	(0.95, 0.9)	40	6.9	100.0	100.0	100.0	98.4	6.9	100.0	100.0	100.0	98.6
		70	6.2	100.0	100.0	100.0	99.4	7.1	100.0	100.0	100.0	99.4
		100	6.5	100.0	100.0	100.0	99.7	5.9	100.0	100.0	100.0	99.7

Table S15: Empirical powers ($\times 10^2$) of the proposed test T_n defined as (6) for $K_0 = 1$ with the untruncated critical value ($c_\kappa = \infty$) and the truncated critical values defined as (18) with $c_\kappa = 0.45, 0.55, 0.65$, and the KPSS test in a simulation with 2000 replications. Constant c_κ determines the level of truncation for the critical values of T_n. The innovations $\epsilon_t \overset{\text{i.i.d.}}{\sim} t(df)$. The nominal size of the tests is 5%.

			$df = 2$					$df = 5$				
	Setting	N	∞	0.45	0.55	0.65	KPSS	∞	0.45	0.55	0.65	KPSS
Model 5	0.5	40	16.7	94.3	89.8	85.5	84.2	13.6	94.0	90.0	83.7	83.0
		70	17.3	97.5	94.8	91.4	91.1	13.6	96.5	93.3	89.1	89.8
		100	15.6	97.9	96.1	93.5	95.9	12.7	97.9	95.0	92.3	94.9
	0.9	40	15.8	99.7	98.9	98.2	93.3	14.1	99.1	97.8	96.4	92.3
		70	18.6	100.0	100.0	100.0	96.1	18.4	99.9	99.4	99.2	94.9
		100	19.5	100.0	100.0	100.0	98.3	20.0	100.0	100.0	99.8	97.9
	−0.5	40	9.7	84.8	78.4	71.8	83.8	7.0	83.4	75.8	68.6	82.8
		70	7.8	91.9	87.2	81.6	91.6	6.5	91.8	86.7	80.3	89.3
		100	8.2	94.9	90.5	85.5	94.3	6.9	94.7	89.8	85.2	93.5
Model 6	(0.8, 0.3)	40	16.9	94.8	90.5	84.5	82.4	14.0	95.0	88.1	82.8	81.8
		70	16.4	97.0	94.0	91.1	90.2	13.5	96.7	93.2	88.6	89.0
		100	16.6	98.5	96.9	94.6	94.9	11.3	98.4	95.8	92.5	95.3
	(0.9, 0.5)	40	17.5	95.6	91.5	86.5	83.7	13.2	94.7	89.6	83.7	82.8
		70	16.6	97.7	94.9	91.4	90.5	14.8	98.0	94.5	90.8	89.8
		100	15.7	98.8	97.0	95.8	95.5	13.0	98.6	95.8	93.0	95.4
	(0.95, 0.9)	40	13.8	95.7	92.2	87.1	83.9	15.1	95.8	91.8	85.0	84.9
		70	15.2	98.2	96.0	94.0	92.0	15.7	98.2	94.6	90.6	90.3
		100	15.3	99.1	98.0	96.9	94.7	13.5	98.3	96.2	92.2	94.3
Model 7	(0.4, 0.2)	40	19.6	98.7	97.3	95.3	87.8	16.6	98.5	95.0	90.6	85.5
		70	17.4	99.5	99.0	98.2	91.8	16.4	99.0	97.1	94.4	91.3
		100	17.6	100.0	99.9	99.6	95.9	16.4	99.6	98.4	96.9	95.2
	(0.5, 0.1)	40	19.9	98.8	97.2	95.2	85.8	14.9	98.9	96.6	91.9	85.4
		70	16.9	99.5	99.1	98.5	92.0	16.4	99.2	97.8	95.3	91.9
		100	19.8	99.9	99.7	99.5	96.2	17.1	99.6	98.2	96.6	95.2
	(0.6, 0.1)	40	17.2	99.2	98.2	97.3	88.6	15.8	99.4	98.4	95.6	87.4
		70	18.5	100.0	99.8	99.6	92.5	17.6	99.7	98.8	97.5	91.3
		100	20.4	100.0	99.9	99.8	96.0	16.6	100.0	99.7	99.0	95.2
Model 8	(0.8, 0.3)	40	7.6	100.0	100.0	100.0	98.5	5.5	100.0	100.0	100.0	98.9
		70	7.2	100.0	100.0	100.0	99.2	5.1	100.0	100.0	100.0	99.5
		100	6.7	100.0	100.0	100.0	99.8	6.0	100.0	100.0	100.0	99.9
	(0.9, 0.5)	40	8.2	100.0	100.0	100.0	98.0	6.0	100.0	100.0	100.0	98.8
		70	6.6	100.0	100.0	100.0	99.2	6.5	100.0	100.0	100.0	99.6
		100	6.3	100.0	100.0	100.0	100.0	6.3	100.0	100.0	100.0	99.8
	(0.95, 0.9)	40	7.6	100.0	100.0	100.0	98.2	7.4	100.0	100.0	100.0	98.5
		70	6.2	100.0	100.0	100.0	99.5	5.9	100.0	100.0	100.0	99.5
		100	6.0	100.0	100.0	100.0	99.8	6.6	100.0	100.0	100.0	100.0

Table S16: Empirical powers ($\times 10^2$) of the proposed test T_n defined as (6) for $K_0 = 2$ with the untruncated critical value ($c_\kappa = \infty$) and the truncated critical values defined as (18) with $c_\kappa = 0.45, 0.55, 0.65$, and the KPSS test in a simulation with 2000 replications. Constant c_κ determines the level of truncation for the critical values of T_n. The innovations $\epsilon_t \overset{\text{i.i.d.}}{\sim} t(df)$. The nominal size of the tests is 5%.

			$df = 2$					$df = 5$				
	Setting	N	∞	0.45	0.55	0.65	KPSS	∞	0.45	0.55	0.65	KPSS
Model 5	0.5	40	17.1	95.2	90.8	85.8	85.0	16.0	94.6	89.7	84.2	83.0
		70	16.8	97.9	95.7	92.7	90.8	16.4	97.0	94.4	91.1	91.1
		100	17.0	98.2	96.7	94.7	94.7	14.9	98.3	96.2	92.7	94.3
	0.9	40	16.0	99.7	99.2	98.5	93.3	14.9	99.0	97.8	96.1	92.2
		70	17.5	100.0	99.8	99.8	95.0	16.1	99.9	99.5	99.2	95.5
		100	18.4	100.0	100.0	100.0	97.5	16.7	100.0	99.9	99.9	97.5
	−0.5	40	8.9	85.4	78.5	72.3	83.3	7.5	83.7	75.3	68.2	82.2
		70	8.8	93.0	88.3	82.2	91.3	6.5	91.8	86.2	80.4	91.0
		100	7.5	95.9	92.8	88.2	95.2	7.3	95.0	90.4	85.6	95.2
Model 6	(0.8, 0.3)	40	18.3	95.2	91.0	87.2	84.1	13.6	93.9	88.2	82.8	81.6
		70	17.2	97.7	94.9	91.7	91.0	13.8	96.3	93.0	89.3	91.1
		100	16.5	98.8	97.3	95.4	95.3	13.7	98.2	95.7	92.6	95.9
	(0.9, 0.5)	40	17.6	95.3	91.2	87.2	85.0	13.7	95.8	90.6	85.4	83.4
		70	16.6	97.4	95.0	91.9	90.4	13.2	97.9	93.9	89.8	90.1
		100	19.2	99.2	97.8	96.2	95.5	15.5	98.5	97.0	93.4	95.0
	(0.95, 0.9)	40	16.6	96.1	92.4	88.0	85.2	15.2	95.2	90.3	84.8	82.7
		70	15.4	98.5	96.7	94.4	91.6	15.5	97.7	94.2	89.8	90.1
		100	17.6	98.9	98.1	97.4	95.3	15.1	98.9	96.8	93.5	95.2
Model 7	(0.4, 0.2)	40	17.6	98.8	97.0	95.0	87.0	16.3	98.2	95.5	91.6	85.7
		70	20.0	99.8	99.1	98.4	92.0	17.7	99.3	97.2	94.5	92.1
		100	19.8	99.8	99.7	99.4	96.0	18.2	99.5	98.6	96.7	94.9
	(0.5, 0.1)	40	19.1	98.6	96.8	94.7	85.9	15.5	99.0	95.8	90.8	85.3
		70	19.2	99.6	99.2	98.6	91.5	17.2	99.6	98.4	96.8	91.5
		100	20.5	100.0	99.9	99.7	95.0	17.3	99.5	98.8	97.0	95.0
	(0.6, 0.1)	40	17.9	99.6	99.0	98.1	88.5	18.2	99.6	97.1	95.0	87.7
		70	18.1	100.0	99.8	99.5	93.2	18.7	99.7	99.1	97.7	91.6
		100	21.3	100.0	100.0	100.0	96.5	17.0	99.9	99.5	99.1	95.5
Model 8	(0.8, 0.3)	40	7.1	100.0	100.0	100.0	98.2	7.0	100.0	100.0	100.0	98.0
		70	7.4	100.0	100.0	100.0	99.5	5.7	100.0	100.0	100.0	99.5
		100	6.9	100.0	100.0	100.0	100.0	6.0	100.0	100.0	100.0	99.9
	(0.9, 0.5)	40	6.8	100.0	100.0	100.0	98.9	6.6	100.0	100.0	100.0	98.6
		70	6.2	100.0	100.0	100.0	99.5	5.5	100.0	100.0	100.0	99.2
		100	7.6	100.0	100.0	100.0	99.8	5.2	100.0	100.0	100.0	99.9
	(0.95, 0.9)	40	7.2	100.0	100.0	100.0	98.3	5.7	100.0	100.0	100.0	98.0
		70	7.2	100.0	100.0	100.0	99.6	5.8	100.0	100.0	100.0	99.4
		100	6.2	100.0	100.0	100.0	100.0	6.5	100.0	100.0	100.0	99.9

Table S17: Empirical powers ($\times 10^2$) of the proposed test T_n defined as (6) for $K_0 = 3$ with the untruncated critical value ($c_\kappa = \infty$) and the truncated critical values defined as (18) with $c_\kappa = 0.45, 0.55, 0.65$, and the KPSS test in a simulation with 2000 replications. Constant c_κ determines the level of truncation for the critical values of T_n. The innovations $\epsilon_t \overset{\text{i.i.d.}}{\sim} t(df)$. The nominal size of the tests is 5%.

			$df = 2$					$df = 5$				
	Setting	N	∞	0.45	0.55	0.65	KPSS	∞	0.45	0.55	0.65	KPSS
Model 5	0.5	40	18.4	93.8	89.3	85.5	84.7	16.0	94.0	89.5	85.2	83.8
		70	18.0	97.4	95.0	92.5	91.6	17.2	97.1	94.1	90.0	90.8
		100	17.8	98.8	97.5	95.5	95.2	16.2	98.2	96.0	93.0	95.2
	0.9	40	16.0	99.8	99.4	98.7	92.5	12.7	99.4	97.9	96.0	92.2
		70	18.4	100.0	100.0	100.0	95.8	17.8	100.0	99.8	99.3	95.0
		100	20.8	100.0	100.0	100.0	97.8	19.1	100.0	100.0	100.0	97.4
	-0.5	40	9.2	85.5	77.9	70.8	82.1	7.4	83.4	75.2	68.8	81.8
		70	8.0	92.3	87.8	81.8	91.4	7.2	92.1	87.0	80.9	89.2
		100	8.3	95.5	91.9	87.5	95.4	7.5	95.0	90.6	86.6	94.8
Model 6	(0.8, 0.3)	40	18.0	95.5	90.8	86.6	84.9	13.6	94.0	89.0	83.4	82.8
		70	17.9	97.6	94.5	91.9	90.7	15.2	97.3	93.5	89.2	89.8
		100	18.5	99.2	97.2	95.4	95.8	14.8	98.4	96.0	92.8	95.4
	(0.9, 0.5)	40	18.9	95.5	91.8	87.2	83.4	16.5	95.8	90.8	85.5	83.0
		70	17.0	98.4	95.8	93.5	91.8	15.0	98.0	94.5	91.0	90.4
		100	19.1	98.8	98.0	96.4	96.0	15.3	98.4	96.0	92.7	94.7
	(0.95, 0.9)	40	17.2	96.2	92.5	89.1	84.1	16.3	94.7	89.5	84.0	81.1
		70	16.5	98.9	97.1	94.8	91.7	15.1	97.5	93.6	89.9	89.9
		100	18.8	99.2	98.0	97.0	95.0	15.6	98.5	96.1	92.8	94.7
Model 7	(0.4, 0.2)	40	19.4	98.2	96.8	95.1	87.3	16.5	98.7	95.2	90.9	85.3
		70	20.7	99.8	99.6	99.2	92.7	18.2	99.1	97.3	95.1	91.3
		100	20.2	100.0	99.9	99.8	96.2	19.0	99.8	98.9	97.7	95.8
	(0.5, 0.1)	40	18.6	98.9	97.4	95.5	85.8	17.6	98.3	95.3	91.5	84.2
		70	20.9	99.7	99.5	98.9	92.3	16.0	99.3	98.0	96.4	91.0
		100	21.1	99.8	99.6	99.5	95.5	18.4	99.7	99.2	97.9	95.9
	(0.6, 0.1)	40	19.2	99.5	99.0	98.4	88.5	17.2	99.6	98.2	96.3	89.0
		70	19.1	99.9	99.7	99.5	91.4	15.3	99.7	99.0	98.1	91.9
		100	22.1	100.0	100.0	100.0	96.3	19.3	99.9	99.9	99.6	95.3
Model 8	(0.8, 0.3)	40	7.4	100.0	100.0	100.0	98.5	6.0	100.0	100.0	100.0	98.8
		40	6.5	100.0	100.0	100.0	99.2	5.7	100.0	100.0	100.0	99.6
		100	6.6	100.0	100.0	100.0	99.9	6.9	100.0	100.0	100.0	99.9
	(0.9, 0.5)	40	8.2	100.0	100.0	100.0	98.7	6.6	100.0	100.0	100.0	98.0
		70	7.5	100.0	100.0	100.0	99.2	6.6	100.0	100.0	100.0	99.5
		100	6.8	100.0	100.0	100.0	99.7	7.6	100.0	100.0	100.0	99.7
	(0.95, 0.9)	40	7.4	100.0	100.0	100.0	98.6	7.6	100.0	100.0	100.0	98.3
		70	6.6	100.0	100.0	100.0	99.3	6.0	100.0	100.0	100.0	99.5
		100	7.2	100.0	100.0	100.0	99.8	6.3	100.0	100.0	100.0	99.7

Table S18: Empirical powers ($\times 10^2$) of the proposed test T_n defined as (6) for $K_0 = 4$ with the untruncated critical value ($c_\kappa = \infty$) and the truncated critical values defined as (18) with $c_\kappa = 0.45, 0.55, 0.65$, and the KPSS test in a simulation with 2000 replications. Constant c_κ determines the level of truncation for the critical values of T_n. The innovations $\epsilon_t \overset{\text{i.i.d.}}{\sim} t(df)$. The nominal size of the tests is 5%.

			$df = 2$					$df = 5$				
	Setting	N	∞	0.45	0.55	0.65	KPSS	∞	0.45	0.55	0.65	KPSS
Model 5	0.5	40	17.9	95.2	91.2	87.1	84.2	16.8	94.1	90.3	86.0	83.8
		70	19.1	97.7	95.3	92.8	91.6	14.2	97.2	93.7	90.0	90.0
		100	17.8	98.6	97.0	95.5	94.8	15.6	98.7	96.0	92.8	94.7
	0.9	40	14.5	99.7	99.4	98.9	93.8	14.7	99.5	98.4	97.0	93.2
		70	20.0	100.0	100.0	99.9	95.9	18.6	99.9	99.6	99.4	94.9
		100	19.6	100.0	100.0	100.0	97.3	17.9	100.0	100.0	100.0	97.5
	-0.5	40	10.0	86.0	78.5	72.5	81.9	7.0	83.6	76.3	69.5	82.0
		70	7.8	90.3	85.7	79.9	90.4	6.9	91.6	85.6	79.1	88.6
		100	9.4	95.3	91.6	87.5	95.5	8.8	94.7	89.8	85.1	94.9
Model 6	(0.8, 0.3)	40	19.1	95.3	90.8	85.9	83.4	17.0	94.6	89.3	83.8	82.7
		70	17.5	97.1	94.3	91.3	90.6	14.8	97.2	94.0	90.0	90.5
		100	18.6	98.6	97.1	95.3	95.1	16.1	98.3	95.7	92.0	94.3
	(0.9, 0.5)	40	17.6	95.7	91.4	87.6	83.2	15.9	95.2	90.1	83.7	82.0
		70	18.4	98.2	96.3	93.0	90.2	15.0	97.6	93.8	89.6	90.1
		100	18.3	99.0	97.9	96.9	94.0	16.6	98.8	96.2	92.6	95.5
	(0.95, 0.9)	40	18.1	96.8	92.2	88.2	83.0	16.1	96.2	91.0	85.2	83.3
		70	17.8	98.8	97.5	96.0	91.6	17.3	97.7	93.5	89.8	89.5
		100	18.1	99.4	98.4	97.8	96.4	16.5	98.6	96.1	93.3	95.2
Model 7	(0.4, 0.2)	40	20.1	98.4	96.5	94.3	86.4	17.1	98.8	95.5	92.0	85.2
		70	22.2	99.7	99.2	98.7	92.2	17.7	99.2	97.4	95.4	91.6
		100	20.8	99.9	99.9	99.8	95.8	18.6	99.5	98.9	97.9	95.0
	(0.5, 0.1)	40	19.7	98.7	97.2	95.4	85.9	18.4	99.1	96.2	92.4	86.6
		70	20.0	99.9	99.6	99.2	92.0	17.9	99.4	98.0	96.2	90.6
		100	21.0	100.0	100.0	99.9	96.0	19.1	99.1	98.4	96.9	95.5
	(0.6, 0.1)	40	20.0	99.2	98.6	97.5	87.4	18.1	99.6	97.5	95.2	87.4
		70	19.4	100.0	99.8	99.4	92.0	18.3	99.7	99.2	98.5	93.0
		100	20.3	100.0	100.0	100.0	96.5	18.1	100.0	99.9	99.4	95.2
Model 8	(0.8, 0.3)	40	8.0	100.0	100.0	100.0	97.9	7.0	100.0	100.0	100.0	98.0
		70	6.8	100.0	100.0	100.0	99.6	7.2	100.0	100.0	100.0	99.6
		100	6.8	100.0	100.0	100.0	99.9	5.5	100.0	100.0	100.0	99.7
	(0.9, 0.5)	40	8.0	100.0	100.0	100.0	98.4	7.8	100.0	100.0	100.0	98.2
		70	7.0	100.0	100.0	100.0	99.3	7.0	100.0	100.0	100.0	99.3
		100	7.3	100.0	100.0	100.0	100.0	6.8	100.0	100.0	100.0	99.8
	(0.95, 0.9)	40	8.0	100.0	100.0	100.0	98.2	7.3	100.0	100.0	100.0	98.7
		70	7.4	100.0	100.0	100.0	99.2	7.0	100.0	100.0	100.0	99.4
		100	6.7	100.0	100.0	100.0	99.7	5.9	100.0	100.0	100.0	99.9